Duality for Heyting algebras

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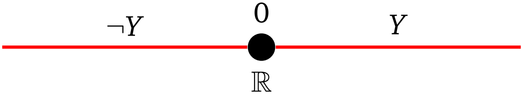
First typical example of a Heyting algebra

Open sets of any topological space X form a Heyting algebra, where for open $Y, Z \subseteq X$:

$$Y \to Z = \mathrm{Int}(Y^c \cup Z), \quad \neg Y = \mathrm{Int}(Y^c).$$



$$Y \vee \neg Y \neq \mathbb{R}$$

Stone Representation

Theorem (Stone, 1937). Every Heyting algebra can be embedded into the Heyting algebra of open sets of some topological space.

Stone representation

For every Heyting algebra A let X_A be the set of prime filters of A.

The Stone map $\varphi : A \to \mathcal{P}(X_A)$ is given by

$$\varphi(a) = \{x \in X_A : a \in x\}.$$

Let Ω_A be the topology generated by the basis $\{\varphi(a) : a \in A\}$.

Theorem. $\varphi : A \to \Omega_A$ is a Heyting algebra embedding.

Second typical example of a Heyting algebra

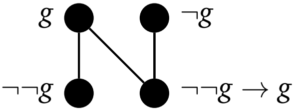
Up-sets of any poset $(X, \leq)$ form a Heyting algebra where for up-sets $U, V \subseteq X$:

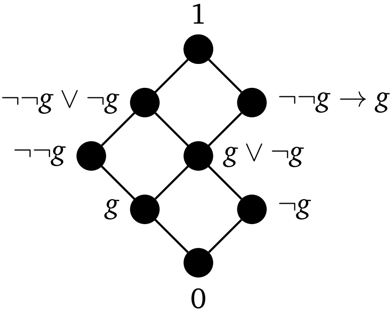
$$U \to V = X - {\downarrow}(U - V), \quad \neg U = X - {\downarrow}U$$

Here U is an up-set if $x \in U$ and $x \leq y$ imply $y \in U$ and

${\downarrow}U = \{x \in X : \exists y \in U \text{ with } x \leq y\}$.

Second typical example of a Heyting algebra





Kripke Representation

Theorem (Kripke, 1965). Every Heyting algebra can be embedded into the Heyting algebra of up-sets of some poset.

Kripke representation

For every Heyting algebra A, order the set X_A of prime filters of A by set-theoretic inclusion.

For a poset X let $\mathrm{Up}(X)$ be the Heyting algebra of up-sets of X.

Theorem. The Stone map $\varphi : A \to \mathrm{Up}(X_A)$ is a Heyting algebra embedding.

We want to characterize the φ-image of A.

For this we will define a topology on X_A and characterize this image in order-topological terms.

This topology will be the so-called patch topology of Ω_A.

Esakia duality

This approach was developed by Esakia in the 1970's.

Esakia duality

An Esakia space is a pair $(X, \leq)$, where:

1. X is a Stone space (compact, Hausdorff, zero-dimensional).
2. $(X, \leq)$ is a poset.
3. $\uparrow x$ is closed for each $x \in X$. Here $\uparrow x = \{y \in X : x \leq y\}$.
4. If U is clopen (closed and open), then so is $\downarrow U$. Recall that $\downarrow U = \{x \in X : \exists y \in U \text{ with } x \leq y\}$.

Esakia duality

Given an Esakia space $(X, \leq)$ we take the Heyting algebra $(\mathrm{CpUp}(X), \cap, \cup, \rightarrow, \emptyset, X)$ of all clopen up-sets of X, where for $U, V \in \mathrm{CpUp}(X)$:

$$U \rightarrow V = X - \downarrow(U - V).$$

For each Heyting algebra A we take the set X_A of prime filters of A ordered by inclusion and topologized by the subbasis

$$\{\varphi(a) : a \in A\} \cup \{\varphi(a)^c : a \in A\}.$$

Alternatively we can take $\{\varphi(a) - \varphi(b) : a, b \in A\}$ as a basis for the topology.

Esakia Duality

Theorem.

1. For each Heyting algebra A the map $\varphi : A \to \mathrm{CpUp}(X_A)$ is a Heyting algebra isomorphism.
2. For each Esakia space X, there is an order-hemeomorphism between X and $X_{\mathrm{CpUp}(X)}$.

This is the object part of the duality between the category of Heyting algebras and Heyting algebra homomorphisms and the category of Esakia spaces and Esakia morphisms.

Priestley spaces

Order-topological representation of bounded distributive lattices was developed by Priestley in the 1970s.

Priestley spaces

In each Esakia space the following Priestley separation holds:

$x \not\leq y$ implies there is a clopen up-set U such that $x \in U$ and $y \notin U$.

Thus, every Esakia space is a Priestley space, but not vice versa.

It follows that Esakia duality is a restricted version of Priestley duality.

Filters and congruences

As in Boolean algebras, the lattice of filters of a Heyting algebra is isomorphic to the lattice of congruences.

To each filter F corresponds the congruence θ_F defined by

$$a\theta_F b \text{ if } a \leftrightarrow b \in F.$$

To each congruence θ corresponds the filter

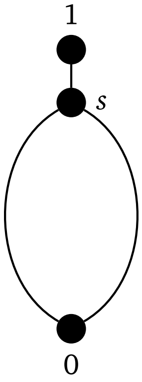
$$F_\theta = \{a \in A : a\theta 1\}.$$

Consequently, the variety of Heyting algebras is congruence distributive and has the congruence extension property.

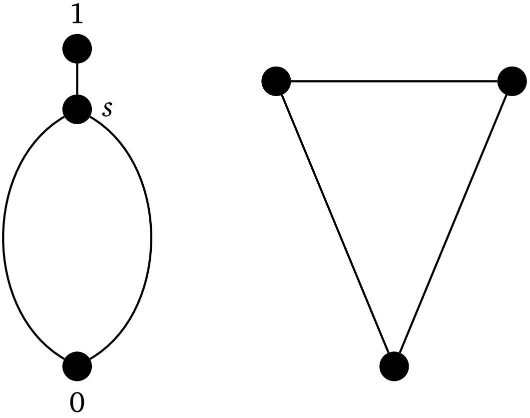
Subdirectly irreducible Heyting algebras

By another theorem of Birkhoff, every variety of algebras is generated by its subdirectly irreducible members.

Theorem (Jankov, 1963). A Heyting algebra is subdirectly irreducible (s.i. for short) if it has a second largest element.



Esakia duals of s.i. Heyting algebras



If a Heyting algebra A is s.i., then the dual of A has a least element, a root.

If an Esakia space is rooted and the root is an isolated point, then its dual Heyting algebra is s.i.

Locally finite varieties

A variety **V** is locally finite if every finitely generated **V**-algebra is finite.

Theorem (Rieger, 1949, Nishimura, 1960). The 1-generated free Heyting algebra, also called the Rieger-Nishimura lattice, is infinite.

Corollary. The variety of Heyting algebras is not locally finite.

The Rieger-Nishimura Lattice