# The FMP for HAs

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# Locally finite reducts

Although Heyting algebras are not locally finite, they have locally finite reducts.

Heyting algebras $(A, \wedge, \vee, \to, 0, 1)$.

$\vee$-free reducts $(A, \wedge, \to, 0, 1)$: implicative semilattices.

$\to$-free reducts $(A, \wedge, \vee, 0, 1)$: distributive lattices.

**Theorem.**

- (Diego, 1966). The variety of implicative semilattices is locally finite.
- (Folklore). The variety of distributive lattices is locally finite.

# FMP

**Proof idea.** Suppose $B \not\models \varphi$.

Then there exist elements $a_1, \ldots, a_n \in B$ on which $\varphi$ is refuted.

We generate the implicative semilattice $(A, \wedge, \to, 0)$ of $B$ by the subpolynomials $\Sigma$ of $\varphi(a_1, \ldots, a_n)$.

By Diego's theorem $(A, \wedge, \to, 0)$ is finite.

# FMP

We define a "fake" $\dot{\vee}$ on $A$ by $a \dot{\vee} b = \bigwedge\{s \in A : s \geq a, b\}$. Then $(A, \wedge, \dot{\vee}, 0, \to)$ is a finite Heyting algebra. Also for $a, b \in A$ we have

$$a \vee b \leq a \dot{\vee} b.$$

Moreover, if $a \vee b \in \Sigma$ then

$$a \vee b = a \dot{\vee} b.$$

This implies that the algebra $(A, \wedge, \dot{\vee}, \to, 0)$ refutes $\varphi$.

# Open problems discussed at the lecture

- Characterize locally finite varieties of Heyting algebras.

  Conjecture: A variety $\mathbf{V}$ of Heyting algebras is locally finite iff $F_{\mathbf{V}}(2)$ is finite.

- Is every variety of Heyting algebras generated by a class of Heyting algebras of the form $\mathrm{Op}(X)$ for some topological space $X$ (Kuznetsov, 1975).

  **Heyt** is generated by $\mathrm{Op}(\mathbb{R})$ (McKinsey and Tarski, 1946).