MATHEMATICAL STRUCTURES IN LOGIC 2017
HOMEWORK 4

- Deadline: April 3 — at the **beginning** of class.
- In exceptional cases homework can be submitted electronically (in a single pdf-file!) to Frederik Lauridsen `f.m.lauridsen@uva.nl`
- Grading is from 0 to 100 points.
- Success!

(1) (40pt) The aim of this exercise is to understand the connection between filters and congruences of a Boolean algebra.

Given a Boolean algebra A and a filter $F \subseteq A$, define a relation $\sim_F$ by

$$a \sim_F b \text{ iff } a \leftrightarrow b \in F.$$

Conversely, given a congruence $\sim$ on A define $F_\sim$ by

$$F_\sim = [1]_\sim.$$

(a) $\sim_F$ is a congruence. Show the $\vee$-clause of this statement. That is, show that if $a \sim_F b$ and $c \sim_F d$, then $a \vee c \sim_F b \vee d$.

(b) Show that $\sim$ is equal to $\sim_{F_\sim}$.

(c) Show that F is equal to $F_{\sim_F}$.

(d) Show that $F \subseteq F'$ implies that $\sim_F$ is a subset of $\sim_{F'}$.

(2) (30pt) Let A be a Boolean algebra. A filter F of the form $\uparrow a$ for some $a \in A$ is called a *principal filter*. Let FinCofin($\mathbb{N}$) be the Booelan algebra of all finite and cofinite subsets of $\mathbb{N}$.

(a) Show that for each $n \in \mathbb{N}$, the set $\{U \in \text{FinCofin}(\mathbb{N}) : n \in U\}$ is a principal filter. Show also that it is a maximal filter.

(b) Show that there is a unique (!) maximal non-principle filter in FinCofin($\mathbb{N}$).

(3) (30pt)

(a) Let A be a finite Boolean algebra. We order the set of all filters of A by inclusion. Show that A has a least non-unital filter iff A is isomorphic to a two-element Boolean algebra. Note that a least non-unital filter is a filter $F \subseteq A$ such that $F \neq \{1\}$ and for each filter $F' \neq \{1\}$ we have $F \subseteq F'$.

(b) Deduce that the two-element Boolean algebra $\mathbf{2}$ is the only finite subdirectly irreducible Boolean algebra.