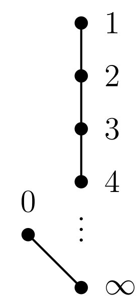
MATHEMATICAL STRUCTURES IN LOGIC 2017
HOMEWORK 6

- Deadline: May 8 — at the **beginning** of class.
- In exceptional cases homework can be submitted electronically (in a single pdf-file!) to Frederik Lauridsen (`f.m.lauridsen@uva.nl`)
- Grading is from 0 to 100 points.
- Success!

(1) (20pt) Prove that for Heyting algebras A and B we have $A \times B \simeq \mathsf{ClpUp}(X_A \sqcup X_B)$, where $X_A \sqcup X_B$ is the disjoint union of X_A and X_B. (For the definition of a disjoint union of posets see page 17 of "Lattices and Order" and for a disjoint union of topological spaces, see page 267.)

(2) (30pt) Consider the ordered topological space $(\mathfrak{X}, \leq)$ drawn below where the space $\mathfrak{X}$ is the Alexandroff compactification $\alpha\mathbb{N}$ of $\mathbb{N}$. In other words, the clopen sets are finite subsets of $\mathbb{N}$ and cofinite subsets of $\mathbb{N}$ together with the point ∞.



(a) Show that this space is a Priestley space.

(b) Show that it is not an Esakia space.

(c) Draw the distributive lattice dual to this space.

(3) (30pt) Let X be a poset. X is called *rooted* if there is $r \in X$ such that $X = {\uparrow}r$. Then r is called the *root* of X.

(a) Let X be a non-trivial Esakia space. Show that X is rooted if and only if in its dual Heyting algebra A for each $a, b \in A$ we have $a \vee b = 1$ implies $a = 1$ or $b = 1$.

(b) Call a rooted Esakia space X with a root r *strongly rooted* if $\{r\}$ is open. Show that an Esakia space X is strongly rooted if and only if the Heyting algebra dual to X has the second largest element. (Recall that in Hausdorff spaces every point is closed.)

(c) Give an example of a rooted Esakia space, which is not strongly rooted.

(4) (20pt) Give an example of a Stone space X and a partial order $\leq$ on X such that $\uparrow x$ is a closed set for each $x \in X$, but $(X, \leq)$ is not a Priestley space. (Hint: it might be useful to work with the two-point compactification of $\mathbb{N}$. That is, consider the space $\mathbb{N} \cup \{\infty_1, \infty_2\}$, whose topology is generated by the set

$$\mathcal{S} = \{\text{finite subsets of } \mathbb{N}, \text{cofinite subsets of } \mathbb{N} \text{ with } \{\infty_1, \infty_2\}, E \cup \{\infty_1\}, O \cup \{\infty_2\}\},$$

where E is the set of even numbers and O is the set of odd numbers. In other words we are taking the least topology containing $\mathcal{S}$).