MATHEMATICAL STRUCTURES IN LOGIC 2017
HOMEWORK 7

- Deadline: May 22 — at the **beginning** of class.
- In exceptional cases homework can be submitted electronically (in a single pdf-file!) to Frederik Lauridsen (`f.m.lauridsen@uva.nl`)
- Grading is from 0 to 100 points.
- Success!

(1) (30pt) (Finite model property) Give full details of the algebraic proof[1] of the finite model property of HAs (**IPC**).

(2) (40pt) Let $\mathbf{LC} = \mathbf{IPC} + (\varphi \to \psi) \vee (\psi \to \varphi)$.

(a) Show that subdirectly irreducible **LC**-algebras (that is, HAs validating **LC**) are chains with a second greatest element.

(b) Show that **LC** has the finite model property. That is, prove that if $\mathbf{LC} \nvdash \varphi$, then there is a finite **LC**-algebra A such that $A \not\models \varphi$. (Hint: use algebraic completeness and Birkhoff's theorem.)

(c) Characterize the lattice of subvarieties of the variety LC of all **LC**-algebras. (Hint: prove that every finite chain is a subalgebra of any infinite subdirectly irreducible **LC**-algebra.)

(3) (30pt) Let φ be a formula of propositional intuitionistic logic and φ^* its Gödel translation.

(a) Show that if there is a finite Heyting algebra A such that $A \not\models \varphi \approx 1$, then there also exists a finite $\mathsf{S4}$-algebra $(B, \Box)$ such that $(B, \Box) \not\models \varphi^* \approx 1$.

(b) Show that if there is a finite $\mathsf{S4}$-algebra $(B, \Box)$ such that $(B, \Box) \not\models \varphi^* \approx 1$, then there also exists a finite Heyting algebra A such that $A \not\models \varphi \approx 1$.

[1] A sketch of this proof using $\vee$-free reducts can be found at `https://staff.fnwi.uva.nl/n.bezhanishvili//MSL/MSL2017/HA-FMP-Sketch.pdf`. You are free to choose whether to use $\vee$-free reducts or $\to$-free reducts.