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EXISTENTIALLY CLOSED CONTACT ALGEBRAS

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"When you're really shipwrecked, you do really find what you want. When you're really on a desert island, you never find it a desert."

(G. K. Chesterton, *Manalive*)

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Chapter 1

Introduction

Region-based theories of space study topological and geometric structures via logic [26, 32, 36]. In these theories, the primitive notion is that of a region rather than of a point. This is why this approach is also called point-free. In this thesis we connect region-based theories of space with model completions and existentially closed algebras—key concepts of model theory [13]. In particular, the recent work on logics for compact Hausdorff spaces [1, 5] had non-standard Π_2-rules as one of the main ingredients. In this thesis we link admissibility of such rules with the model completion of the first-order theory of contact algebras, and provide an axiomatization of this model completion. We will now give a short historic overview of contact algebras: for more details, we refer to Vakarelov [32], see also [5] for the latest results. We also refer to [33] for a modern account of spatial logics.

The notion of contact algebra was first introduced by researchers interested in defining point-free geometry. Indeed, the classical Euclidean geometry can be considered as *point-based*: the notion of point, regarded as the simplest spatial entity without dimension and internal structure, is considered as primitive, while all other geometrical figures are defined as sets of points. However, the notion of point was considered to be too abstract to have an adequate analogue in reality. Hence the idea to develop a new theory of space emerged, where the primitive objects were more closely related to the real world, and points were defined in terms of these new primitive notions. For instance, one can consider solid bodies as primitive, and the basic relations between solids could be "one solid is part of another solid", "two solids overlap", or any other relation of this kind.

The first relevant attempt in this direction was made by De Laguna [16] in 1922 and Whitehead [37] in 1929. According to De Laguna, the notion of solid is primitive and indefinable, but it should intuitively be thought as the *space* occupied by a physical solid (so solids can pass into and through one another). On the other hand, Whitehead called the solids *regions*, and this is the reason why his theory is known as the *region-based theory of space*. According to Whitehead, intuitively, two regions are in contact if they have a common point. However, this can not be considered a definition, because points are not primitive notions: they have to be defined by means of regions and the contact relation. Hence the formal definition of the relation "region a is in contact with region b" (denoted with aCb) is given by an appropriate set of axioms.

Other attempts to provide a simpler pointless version of Euclidean geometry were made by Tarski [31], who presented his system called "Geometry of solids" in 1927, and by Grzegorczyk [24] in 1960. In particular, Grzegorczyk created a point-free theory of space independently from De Laguna and Whitehead: his results were presented in [7]. He assumed as primitive notions a set R whose elements are called *spatial bodies*, the *inclusion relation* and the relation of *being separated*. However, if we substitute the relation of being separated by its negation, which we call *connection relation*, it becomes clear that Grzegorczyk's system is somehow similar to the one of Whitehead. Grzegorczyk also proved two important theorems that show that there is an equivalence between the point-based and point-free theories of space, and make clear the importance of regular (open or closed) sets in topological spaces as models of regions. However, Grzegorczyk's system is not a first-order system, because it includes the second-order definition of point: several authors independently pointed out that a first-order axiomatization of point-free theory of space can be obtained by accepting one additional axiom, called the *normality axiom*.

The first author to point this out was de Vries [17] in 1962. In this context, de Vries defined the *compingent algebras*: these are contact algebras, i. e. Boolean algebras B endowed with a binary relation $\prec$ satisfying a suitable set of first-order axioms, that also satisfy two additional $\forall\exists$-axioms, which we denote by (I6) and (I7) in this thesis. He also proved that each compingent algebra is isomorphic to a subalgebra of the algebra of regular open subsets of a compact Hausdorff space, with the binary relation on regular open sets defined in this way: $a \prec b$ if and only if $Cl(a) \subseteq b$, where Cl is the closure operator. De Vries also defined a notion of point by defining the *compingent filters*, and established a duality between complete compingent algebras (which we call *de Vries algebras* in his thesis) and compact Hausdorff spaces: this duality and the $\forall\exists$-axioms will play a crucial role in this thesis.

De Vries duality led to new logical calculi for compact Hausdorff spaces in [1] for two-sorted modal language and in [5] for a uni-modal language with a strict implication (for other modern theories of spatial logic we refer to [33]). Key to these approaches is a development of logical calculi corresponding to contact algebras. In [5] such a calculus is called the *strict symmetric implication calculus* and is denoted by $\mathsf{S^2IC}$. As we will also discuss later, the extra Π_2-axoms (I6) and (I7) of compingent algebras then correspond to non-standard Π_2-rules, which turn out to be admissible in $\mathsf{S^2IC}$. This generates a natural question of investigating admissibility of Π_2-rules in $\mathsf{S^2IC}$.

We recall that the use of non-standard rules has a long tradition in modal logic starting from the pioneering work of Gabbay [21], who introduced a non-standard rule for irreflexivity. Non-standard rules have been employed in temporal logic in the context of branching time logic [9] and for axiomatization problems [22] concerning the logic of the real line in the language with the Since and Until modalities. General completeness results for modal languages that are sufficiently expressive to define the so-called difference modality have been obtained in [35].

In this thesis, we connect admissibility of non-standard Π_2-rules with the model completion of the first-order theory of contact algebras: the notion of a model completion of a theory was introduced by Robinson in the 50's. We recall that, given a universal theory T in a language $\mathcal{L}$, a *model completion* for T is a theory $T^* \supseteq T$ in the same language $\mathcal{L}$ that admits quantifier elimination and proves the same quantifier-free formulas as T does. If it exists, the model completion of a universal theory T is unique (Proposition 2.1.23), and its models are exactly the models of T which are existentially closed for T (Proposition 2.1.25), i. e. the models $\mathcal{M}'$ of T such that, for every embedding $\mathcal{M} \subseteq \mathcal{M}'$, every existential $\mathcal{L}_{\mathcal{M}}$-sentence which holds in $\mathcal{M}'$ also holds in $\mathcal{M}$. We also have that, if a variety is locally finite and has the amalgamation property, then the corresponding first-order theory T has a model completion (Theorem 2.1.33): therefore, in Chapter 4, we first prove, by using this result, that the model completion of the theory of contact algebras exists.

We then find an *infinite* axiomatization of such a model completion: in order to do that, we take the same approach as in [11], which characterizes the model completion of Brouwerian semilattices. We start from the following characterization of existentially closed contact algebras (Corollary 4.3.1.1): given a contact algebra $(B, \prec)$, it is existentially closed if and only if, for any finite subalgebra $(B_0, \prec) \subseteq (B, \prec)$ and for any finite minimal extension $(C, \prec) \supseteq (B_0, \prec)$, there exists an embedding $(C, \prec) \hookrightarrow (B, \prec)$ fixing $(B_0, \prec)$ pointwise. So we distinguish two kinds of finite minimal extensions of contact algebras (Definition 4.3.8): in order to do that, we introduce and use a duality between the category of contact algebras, with morphisms of contact algebras, and the category whose objects are pairs (X, R), where X is a Stone space and R is a reflexive and symmetric closed relation on X, and whose morphisms are continuous stable morphisms, i. e. continuous maps $f\colon (X_1, R_1) \to (X_2, R_2)$ which satisfy the condition $[xR_1y \;\Rightarrow\; f(x)R_2f(y)]$. This duality is presented in [6, 18] and recalled in Chapter 4. After that, we assign to both kinds of minimal extensions a *signature* (Definition 4.3.12) such that the following result holds (Theorem 4.3.14): if $(B_0, \prec_{B_0})$ is a finite contact algebra, to give a finite minimal extension either of the first or of the second kind of $(B_0, \prec_{B_0})$ (up to isomorphism) is equivalent to give either a signature of the first kind $(b, \tilde{c_1}, \tilde{c_2}, 1)$ in $(B_0, \prec_{B_0})$ or a signature of the second kind $(b, \tilde{c_1}, \tilde{c_2}, 0)$ in $(B_0, \prec_{B_0})$ respectively. Finally, Corollary 4.3.15.1 provides an infinite axiomatization of the model completion of the theory of contact algebras, which is obtained by using the characterization given by Corollary 4.3.1.1.

Chapter 3 discusses the model completion of the theory of S5-algebras: this is an important variety of modal algebras closely related to contact algebras. Also in this case, we first prove the existence of such a model completion by proving that the variety of S5-algebras is locally finite and has the amalgamation property, and later we present a *finite* axiomatization of it, again by taking the same approach as in [11]. We first present a duality between the category of S5-algebras $(B, \Diamond)$ and the category of modal spaces (X, R) where R is an equivalence relation, and then we use this duality to classify two kinds of finite minimal extensions of S5-algebras. After that, we prove that, given an S5-algebra $(B, \Diamond)$, it is existentially closed if and only if, for any finite sub-S5-algebra $(B_0, \Diamond) \subseteq (B, \Diamond)$ and for any finite minimal extension $(C, \Diamond) \supseteq (B_0, \Diamond)$, there exists an embedding $(C, \Diamond) \hookrightarrow (B, \Diamond)$ fixing $(B_0, \Diamond)$ pointwise (Corollary 3.3.1.1). We then use this characterization in order to provide a finite axiomatization of the considered model completion (Theorem 3.3.15).

In Chapter 5, we present the symmetric strict implication calculus $\mathsf{S^2IC}$, that we have already mentioned above. We closely follow [5] while defining it and introducing the main results. Since $\mathsf{S^2IC}$ is strongly sound and complete with respect to both the class of contact algebras **Con** (Theorem 5.2.3) and the class of compingent algebras **Com** (Theorem 5.2.4), neither the axiom (I6) nor (I7) is expressible in our logic. Therefore we show that we can express (I6) and (I7) in our propositional language by means of two admissible Π_2-rules, in the following way: we first prove that (I6) and (I7) are respectively equivalent to two $\forall\exists$-statements ($\Pi6$) and ($\Pi7$) (Lemma 5.2.8). Then, we define the notion of admissible Π_2-rule (Definitions 5.2.9 and 5.2.16), and we assign a $\forall\exists$-statement $\Pi(\rho)$ to every Π_2-rule (ρ): we observe that every $\forall\exists$-statement $\forall x \exists y \Phi(x, y)$ is equivalent to the $\forall\exists$-statement associated to a certain Π_2-rule (ρ_Φ). Afterwards, by means of an Admissibility Criterion (Theorem 5.3.1), we prove that the Π_2-rules $(\rho 6)$ and $(\rho 7)$, which are respectively associated to ($\Pi6$) and ($\Pi7$), are both admissible in $\mathsf{S^2IC}$ (Corollary 5.2.18.1).

Now, since the model completion of any universal theory can be axiomatized by means of $\forall\exists$-axioms (Remark 2.1.26) provided it exists, it looks natural to ask whether there is any relation between the model completion of the theory of contact algebras and the admissible Π_2-rules in $\mathsf{S^2IC}$: the answer to this question is affirmative. Such a relation is given by the following result (Theorem 5.3.1), which is an original contribution of this thesis: a Π_2-rule is admissible in $\mathsf{S^2IC}$ if and only if $\text{SC}\textsc{on}^* \models \Pi(\rho)$, where $\text{SC}\textsc{on}^*$ is the model completion of the theory of contact algebras. It follows that every existentially closed contact algebra is a compingent algebra, and that checking whether a Π_2-rule is admissible or not amounts to checking whether $\text{SC}\textsc{on}^* \models \Pi(\rho)$ holds or not. This can be done because the quantifier elimination in $\text{SC}\textsc{on}^*$ is effective: this is the last result of this thesis.

The thesis also contain an appendix, in which we modify the duality about contact algebras that we present in Chapter 4: we adapt it to the case in which we want to equivalently define the contact algebras in terms of a binary operation $\rightsquigarrow$ instead of a binary relation $\prec$, as explained in the Remark 4.1.15.

We finish by summarizing the main original contributions of this thesis:

- A proof of the existence of the model completion of the theory of S5-algebras (Section 3.2), and a *finite* axiomatization of it (Theorem 3.3.15);

- A proof of the existence of the model completion of the theory of contact algebras (Section 4.2), and an *infinite* axiomatization of it (Corollary 4.3.15.1);

- A result (Theorem 5.3.1) which specifies the relation between the model completion of the theory of contact algebras and the admissible rules of $\mathsf{S}^2\mathsf{IC}$, and a proof of the fact that quantifier elimination in $\mathrm{SC}\textsc{on}^*$ is effective (last part of Section 5.3);

- A modified duality for contact algebras (Appendix).

Chapter 2

Preliminaries

2.1 Basic definitions and results

The aim of this chapter is to provide the necessary background about the theory of model completions. We will mainly refer to [13] and to [23]. We will provide a proof just for the results that are crucial in the context of the topic of this thesis, and for the ones which, although known, are not easily accessible in the literature. We first recall the following definitions and results:

Definition 2.1.1. A first-order theory T is *consistent* if it has a model. It is *complete* if it is consistent and, for any formula φ, we have that $T \models \varphi$ or $T \models \neg\varphi$.

Definition 2.1.2. If $\mathcal{M}$ is a $\mathcal{L}$-structure, then $Th(\mathcal{M})$ is the collection of all $\mathcal{L}$-sentences true in $\mathcal{M}$. If $\mathcal{N}$ is another $\mathcal{L}$-structure, then we write $\mathcal{M} \equiv \mathcal{N}$ and call $\mathcal{M}$ and $\mathcal{N}$ *elementarily equivalent* whenever $Th(\mathcal{M}) = Th(\mathcal{N})$.

Definition 2.1.3. Let $\mathcal{M}$ and $\mathcal{N}$ be two $\mathcal{L}$-structures. A *homomorphism* $h : \mathcal{M} \to \mathcal{N}$ is a function $h : \mathcal{M} \to \mathcal{N}$ such that:

1. $h(c^{\mathcal{M}}) = c^{\mathcal{N}}$ for all constants c in $\mathcal{L}$
2. $h(f^{\mathcal{M}}(m_1, ..., m_n)) = f^{\mathcal{N}}(h(m_1), ..., h(m_n))$ for all function symbols f in $\mathcal{L}$ and elements $m_1, ..., m_n \in \mathcal{M}$
3. $(m_1, ..., m_n) \in R^{\mathcal{M}}$ implies $(h(m_1), ..., h(m_n)) \in R^{\mathcal{N}}$ for all $m_1, ..., m_n \in \mathcal{M}$.

A homomorphism $h : \mathcal{M} \to \mathcal{N}$ is an *embedding* if it is injective and $(h(m_1), ..., h(m_n)) \in R^{\mathcal{N}}$ implies $(m_1, ..., m_n) \in R^{\mathcal{M}}$ for all $m_1, ..., m_n \in \mathcal{M}$.
An embedding $h : \mathcal{M} \to \mathcal{N}$ is *elementary* if $\mathcal{M} \models \varphi(m_1, ..., m_n) \Leftrightarrow \mathcal{N} \models \varphi(m_1, ..., m_n)$ for all $m_1, ..., m_n \in \mathcal{M}$ and *all* formulas $\varphi(x_1, ..., x_n)$.

An *isomorphism* is a homomorphism h which is bijective and whose inverse h^{-1} is a homomorphism as well.

Lemma 2.1.4. *Given a homomorphism $h : \mathcal{M} \to \mathcal{N}$, the following are equivalent:*

1. *h is an embedding*

2. $\mathcal{M} \models \varphi(m_1, ..., m_n) \Leftrightarrow \mathcal{N} \models \varphi(h(m_1), ..., h(m_n))$ *for all* $m_1, ..., m_n \in \mathcal{M}$ *and all atomic formulas* $\varphi(x_1, ..., x_n)$

3. $\mathcal{M} \models \varphi(m_1, ..., m_n) \Leftrightarrow \mathcal{N} \models \varphi(h(m_1), ..., h(m_n))$ *for all* $m_1, ..., m_n \in \mathcal{M}$ *and all quantifier-free formulas* $\varphi(x_1, ..., x_n)$

Lemma 2.1.5. *Any isomorphism* $h : \mathcal{M} \to \mathcal{N}$ *is also an elementary embedding. If* $\mathcal{M}$ *and* $\mathcal{N}$ *are two* $\mathcal{L}$*-structures and* $h : \mathcal{M} \to \mathcal{N}$ *is an elementary embedding, then* $\mathcal{M}$ *and* $\mathcal{N}$ *are elementarily equivalent.*

Definition 2.1.6. If $\mathcal{M}$ and $\mathcal{N}$ are two $\mathcal{L}$-structures and the inclusion $\mathcal{M} \subseteq \mathcal{N}$ is an embedding, then $\mathcal{M}$ is a *substructure* of $\mathcal{N}$ and $\mathcal{N}$ is an *extension* of $\mathcal{M}$. If the inclusion $\mathcal{M} \subseteq \mathcal{N}$ is an elementary embedding, then $\mathcal{M}$ is an *elementary substructure* of $\mathcal{N}$ and $\mathcal{N}$ is an *elementary extension* of $\mathcal{M}$.

We now state other useful definitions and results:

Definition 2.1.7. We say that a first-order theory T in the language $\mathcal{L}$ is *preserved under unions on chains* if the union of any chain of models of T is a model of T too.

Theorem 2.1.8 (Chang - Łoś - Suzko Theorem)**.** *A first-order theory* T *in the language* $\mathcal{L}$ *is preserved under unions on chains if and only if* T *has a set of universal-existential axioms (i. e.,* T *can be axiomatised by means of universal-existential axioms).*

Definition 2.1.9. If $\mathcal{M}$ is a $\mathcal{L}$-structure, then the collection of all quantifier-free $\mathcal{L}_{\mathcal{M}}$-sentences true in $\mathcal{M}$ is called the *diagram* of $\mathcal{M}$, where $\mathcal{L}_{\mathcal{M}}$ denotes the language which is obtained by adding to $\mathcal{L}$ a set of fresh constants $\{c_m \mid m \in \mathcal{M}\}$, where c_m is usually interpreted as $m \in \mathcal{M}$. The diagram of $\mathcal{M}$ is usually denoted with $Diag(\mathcal{M})$ or with $\Delta_{\mathcal{M}}$.

Theorem 2.1.10. *Given two* $\mathcal{L}$*-structures* $\mathcal{M}$*,* $\mathcal{N}$*,* $\mathcal{M}$ *is a model of* $\Delta_{\mathcal{N}}$ *if and only if there is an embedding* $f : \mathcal{N} \to \mathcal{M}$*.*

Theorem 2.1.11 (Elementary Chain Theorem)**.** *Suppose that* $\mathcal{L}$ *is a first-order language and* $A_0, A_1, ...$ *is a sequence (of any length) of* $\mathcal{L}$*-structures such that any structure in the sequence is an elementary substructure of all the later structures in the sequence. Then there is a unique smallest* $\mathcal{L}$*-structure* B *which contains all the structures in the sequence as substructures; this structure* B *is an elementary extension of all the structures in the sequence.*

The following results are crucial in model theory:

Theorem 2.1.12 (Compactness Theorem)**.** *Given a theory* T *in the language* $\mathcal{L}$*, if every finite subset of* T *has a model, then* T *has a model.*

Theorem 2.1.13 (Downwards Löwenheim - Skolem Theorem)**.** *Suppose that* $\mathcal{M}$ *is an* $\mathcal{L}$*-structure and* $X \subseteq \mathcal{M}$*. Then there is an elementary substructure* $\mathcal{N}$ *of* $\mathcal{M}$ *with* $X \subseteq \mathcal{N}$ *and* $card(N) \leq card(X) + card(\mathcal{L}) + \aleph_0$*.*

Theorem 2.1.14 (Upwards Löwenheim - Skolem Theorem)**.** *Suppose that $\mathcal{M}$ is an infinite $\mathcal{L}$-structure and k is a cardinal number with $k \geq card(\mathcal{M}), card(\mathcal{L})$. Then there is an elementary embedding $i : \mathcal{M} \to \mathcal{N}$ with $card(\mathcal{N}) = k$.*

Now we define the main notions that we are going to consider in this thesis. From this point forward, until the end of this chapter, most of the definitions and results can be found in [23, Chapter 2, Section 2].

Definition 2.1.15. A first-order theory T in the language $\mathcal{L}$ admits *quantifier elimination* if, for every formula $\varphi(x_1, ..., x_n)$ in the language $\mathcal{L}$, there exists a quantifier-free formula $\psi(x_1, ..., x_n)$ such that $T \models \varphi(x_1, ..., x_n) \leftrightarrow \psi(x_1, ..., x_n)$.

Definition 2.1.16. A first-order theory T in the language $\mathcal{L}$ is *submodel-complete* if, given two models $\mathcal{M}_1$, $\mathcal{M}_2$ of T, and given a common substructure A[1] of them, we have that $\mathcal{M}_1$ is elementary equivalent to $\mathcal{M}_2$ in the language $\mathcal{L}_A$.

Remark 2.1.17. By Theorem 2.1.10, a theory T is submodel-complete if and only if $T \cup \Delta_A$ is complete in $\mathcal{L}_A$, whenever A is a submodel of a model of T.

Moreover, every embedding among models of a submodel-complete theory is elementary: this follows from the definition of submodel-complete theory, by considering the particular case in which the common substructure specified in the definition coincides with one of the two models.

We recall that the following results hold:

Proposition 2.1.18. *A first-order theory T in the language $\mathcal{L}$ admits quantifier elimination if and only if it is submodel-complete.*

Proof. Suppose that T admits quantifier elimination. Let $\mathcal{M}_1$, $\mathcal{M}_2$ be models of T, and let A be a common substructure of them. We need to show that $\mathcal{M}_1$ and $\mathcal{M}_2$ are elementarily equivalent in the language $\mathcal{L}_A$. So let $\varphi(a_1, ..., a_n)$ be a $\mathcal{L}_A$-sentence such that $\mathcal{M}_1 \models \varphi(a_1, ..., a_n)$, where $a_1, ..., a_n \in A$. Since T admits quantifier elimination, there exists a quantifier-free formula $\psi(x_1, ..., x_n)$ such that $T \models \forall x_1, ..., x_n(\varphi(x_1, ..., x_n) \leftrightarrow \psi(x_1, ..., x_n))$. Since by Lemma 2.1.4 quantifier-free formulas are preserved under substructures and extensions, $\mathcal{M}_1 \models \psi(a_1, ..., a_n)$ if and only if $A \models \psi(a_1, ..., a_n)$. Since A is a substructure of both $\mathcal{M}_1$ and $\mathcal{M}_2$, we then have that $\mathcal{M}_1 \models \varphi(a_1, ..., a_n) \Leftrightarrow \mathcal{M}_1 \models \psi(a_1, ..., a_n) \Leftrightarrow A \models \psi(a_1, ..., a_n) \Leftrightarrow \mathcal{M}_2 \models \psi(a_1, ..., a_n) \Leftrightarrow \mathcal{M}_2 \models \varphi(a_1, ..., a_n)$. Therefore $\mathcal{M}_2 \models \varphi(a_1, ..., a_n)$, as required.

Suppose now that T is submodel-complete and let $\varphi(x_1, ..., x_n)$ be an arbitrary formula. Consider the new constants $a_1, ..., a_n$ and define the set of sentences $T' := T \cup \{\varphi(a_1, ..., a_n)\} \cup \{\neg\psi(a_1, ..., a_n) \mid \psi \text{ is quantifier-free and } T \models (\psi(a_1, ..., a_n) \to \varphi(a_1, ..., a_n))\}$. We show

[1] A can be empty, if the language $\mathcal{L}$ does not contain any constant symbol.

that T' can't be consistent. Suppose by contradiction that it is consistent, and consider a model $\mathcal{M}$ of it. Let A be the substructure of $\mathcal{M}$ generated by $a_1, ..., a_n$. It holds that $T \cup \Delta_A \models \varphi(a_1, ..., a_n)$. Indeed, suppose by contradiction that $T \cup \Delta_A \not\models \varphi(a_1, ..., a_n)$. Then there exists a model $\mathcal{N}$ of T containing A as a substructure (i. e., a model of $T \cup \Delta_A$) such that $\mathcal{N} \not\models \varphi(a_1, ..., a_n)$. But T is submodel-complete and $\mathcal{M} \models \varphi(a_1, ..., a_n)$ by definition of T', so this situation can't occur. Hence $T \cup \Delta_A \models \varphi(a_1, ..., a_n)$. So $T \models (\psi(a_1, ..., a_n) \rightarrow \varphi(a_1, ..., a_n))$ for some quantifier-free sentence $\psi(a_1, ..., a_n)$ such that $A \models \psi(a_1, ..., a_n)$. However, according to the definition of T', $\mathcal{M} \models \neg\psi(a_1, ..., a_n)$ and $A \models \neg\psi(a_1, ..., a_n)$, being $\psi(a_1, ..., a_n)$ quantifier-free, and this gives a contradiction. So T' is inconsistent. This means that $T \models \varphi(a_1, ..., a_n) \rightarrow \psi_1(a_1, ..., a_n) \vee ... \vee \psi_k(a_1, ..., a_n)$, where the $\psi_j(a_1, ..., a_n)$ are quantifier-free formulas such that $T \models \psi_j(a_1, ..., a_n) \rightarrow \varphi(a_1, ..., a_n)$. Hence $T \models \varphi(a_1, ..., a_n) \leftrightarrow \psi_1(a_1, ..., a_n) \vee ... \vee \psi_k(a_1, ..., a_n)$, i. e., T admits quantifier elimination.

□

From the proof of the Proposition 2.1.18, it follows that we can equivalently modify the definition of submodel-completeness in this way:

Definition 2.1.19. A first-order theory T in the language $\mathcal{L}$ is *submodel-complete* if, given two models $\mathcal{M}_1$, $\mathcal{M}_2$ of T, and given a common substructure A of them *which is finitely generated*, we have that $\mathcal{M}_1$ is elementary equivalent to $\mathcal{M}_2$ in the language $\mathcal{L}_A$.

Moreover, if the theory doesn't have finite models, we can again equivalently modify the previous definition in this way:

Lemma 2.1.20. *If a first-order theory T in the language $\mathcal{L}$ doesn't have finite models, then it is submodel-complete if and only if, given two models $\mathcal{M}_1$, $\mathcal{M}_2$ of T such that $card(\mathcal{M}_1) = card(\mathcal{M}_2) = k$, where k is a fixed cardinal such that $k \geq card(\mathcal{L})$, and given a common substructure A of them which is finitely generated, we have that $\mathcal{M}_1$ is elementary equivalent to $\mathcal{M}_2$ in the language $\mathcal{L}_A$.*

Proof. The statement follows from the Downwards and Upwards Löwenheim - Skolem Theorems, and from the fact that, if there exists an elementary embedding between two $\mathcal{L}$-structures, then they elementarily equivalent.

□

Proposition 2.1.21. *Let T_1, T_2 be first-order theories in the same language $\mathcal{L}$. The following conditions are equivalent:*

1. *each model of T_1 embeds into a model of T_2*

2. *for every quantifier-free formula $\psi(x_1, ..., x_n)$, the condition $T_2 \models \psi(x_1, ..., x_n)$ implies that $T_1 \models \psi(x_1, ..., x_n)$.*

Proof. (1. $\Rightarrow$ 2.) Suppose that $\psi(x_1, ..., x_n)$ is a quantifier-free formula such that $T_2 \models \psi(x_1, ..., x_n)$, and suppose by contradiction that $T_1 \not\models \psi(x_1, ..., x_n)$. This means that there exists a model $\mathcal{M}$ of T_1 and $a_1, ..., a_n \in \mathcal{M}$ such that $\mathcal{M} \not\models \psi(a_1, ..., a_n)$. By the hypothesis, we know that there exists a model $\mathcal{M}'$ of T_2 such that $\mathcal{M} \subseteq \mathcal{M}'$. Since ψ is quantifier-free, $\mathcal{M}' \not\models \psi(a_1, ..., a_n)$, and so $T_2 \not\models \psi(a_1, ..., a_n)$, which is a contradiction.
(2. $\Rightarrow$ 1.) Let $\mathcal{M}$ be a model of T_1, and consider $T' := T_2 \cup \Delta_M$. We need to prove that T' is consistent. Suppose by contradiction that this is not the case. Then there exists a quantifier-free formula ψ such that $T_2 \models \psi(a_1, ..., a_n)$ and $\mathcal{M} \not\models \psi(a_1, ..., a_n)$. By hypothesis, the fact that $T_2 \models \psi(a_1, ..., a_n)$ implies that $T_1 \models \psi(a_1, ..., a_n)$, but this is a contradiction because $\mathcal{M}$ is a model of T_1 and $\mathcal{M} \not\models \psi(a_1, ..., a_n)$. Hence T' is consistent, as required.

□

Now we can finally define:

Definition 2.1.22. Let T be a universal theory in a language $\mathcal{L}$. A theory $T^* \supseteq T$ in the same language $\mathcal{L}$ is a *model completion* of T if T^* admits quantifier elimination and T^* proves the same quantifier-free formulas as T does.

If it exists, the model completion of a theory is unique:

Proposition 2.1.23. *Let T_1^*, T_2^* be two model completions of the same first-order theory T. Then, for every formula φ, we have that $T_1^* \models \varphi$ if and only if $T_2^* \models \varphi$.*

Proof. It is sufficient to show that any model $\mathcal{M}_0$ of T_2^* is also a model of T_1^*. Since $T \subseteq T_2^*$ and $\mathcal{M}_0$ is a model of T_2^*, $\mathcal{M}_0$ is also a model of T. Since T_1^* is a model completion of T, T_1^* proves the same quantifier-free formulas as T does. This means that $T_1^* \models \psi \Rightarrow T \models \psi$ for any quantifier-free formula ψ. Then, by Proposition 2.1.21, $\mathcal{M}_0$ can be embedded into a model $\mathcal{M}_1$ of T_1^*. In the same way, $\mathcal{M}_1$ can be embedded into a model $\mathcal{M}_2$ of T_2^*, and so on. Hence we have obtained a chain in which the odd indices chain is entirely formed by models of T_1^*, while the even indices chain is entirely formed by models of T_2^*. Both these sub-chains satisfy the hypotheses of the Elementary Chain Theorem, according to the Remark 2.1.17, being both T_1^* and T_2^* submodel complete. So let $\mathcal{M}$ be the union of such a chain: by the Elementary Chain Theorem, $\mathcal{M}$ is elementarily equivalent to both $\mathcal{M}_0$ and $\mathcal{M}_1$. It follows that $\mathcal{M}_0$ is also a model of T_1^*, as required.

□

Definition 2.1.24. A $\mathcal{L}$-structure $\mathcal{M}$ is said to be *existentially closed* for a theory T if, for every embedding $\mathcal{M} \subseteq \mathcal{M}'$ where $\mathcal{M}'$ is a model of T, every existential $\mathcal{L}_{\mathcal{M}}$-sentence which holds in $\mathcal{M}'$ holds in $\mathcal{M}$ too.

Proposition 2.1.25. *If a first-order theory T has a model completion T^*, then the class of models of T^* is just the class of models of T which are existentially closed for T.*

Proof. T^* is submodel-complete, being the model completion of T. Hence, by Remark 2.1.17, any embedding among models of T^* is elementary. So, by the Elementary Chain Theorem, T^* is preserved under union on chains. By the Chang-Łoś-Suzko Theorem, this implies that T^* can be axiomatised by means of $\forall\exists$-axioms. So let $\mathcal{M}$ be an existentially closed model of T and let $\forall x_1, ..., x_n \exists y_1, ..., y_m \psi(x_1, ..., x_n, y_1, ..., y_m)$ be an axiom for T^*, where ψ is quantifier-free. We need to show that $\mathcal{M} \models \forall x_1, ..., x_n \exists y_1, ..., y_m \psi(x_1, ..., x_n, y_1, ..., y_m)$, i. e., that $\mathcal{M} \models \exists y_1, ..., y_m \psi(a_1, ..., a_n, y_1, ..., y_m)$ for every $a_1, ..., a_n \in \mathcal{M}$. By definition of model completion and by Proposition 2.1.21, we know that we can embed $\mathcal{M}$ into a model $\mathcal{N}$ of T^*. Since $\mathcal{N}$ is a model of T^*, we know that $\mathcal{N} \models \exists y_1, ..., y_m \psi(a_1, ..., a_n, y_1, ..., y_m)$ for every $a_1, ..., a_n \in \mathcal{M}$. Hence $\mathcal{M} \models \exists y_1, ..., y_m \psi(a_1, ..., a_n, y_1, ..., y_m)$ for every $a_1, ..., a_n \in \mathcal{M}$, being $\mathcal{M}$ existentially closed for $T \subseteq T^*$.

Conversely, suppose that $\mathcal{M}$ is a model of T^*: we have to prove that, given a model $\mathcal{N}$ of T and an extension of $\mathcal{M}$, if $\mathcal{N} \models \varphi$ then $\mathcal{M} \models \varphi$, where φ is an existential $\mathcal{L}_{\mathcal{M}}$-sentence. Again by definition of model completion and by Proposition 2.1.21, we know that we can embed $\mathcal{N}$ into a model $\mathcal{M}'$ of T^*. Since φ is existential, we have that $\mathcal{M}' \models \varphi$. Moreover, by the Remark 2.1.17, the embedding $\mathcal{M} \subseteq \mathcal{M}'$ is elementary, being both $\mathcal{M}$ and $\mathcal{M}'$ models of T^*, which is submodel-complete. Therefore, by Lemma 2.1.5, $\mathcal{M} \models \varphi$, as required.

□

Remark 2.1.26. In the proof of the Proposition 2.1.25, we proved that, if T^* is the model completion of a first-order theory T, then it can be axiomatised by means of $\forall\exists$-axioms, being T^* submodel-complete.

In order to provide the last relevant result about the model completion of a theory, we first recall the following definitions:

Definition 2.1.27. A class of algebras K is called a *variety* if $S(K) \subseteq K$, $P(K) \subseteq K$ and $H(K) \subseteq K$, where $S(K)$, $P(K)$ and $H(K)$ denote the classes of subalgebras of algebras in K, products of algebras in K and homomorphic images of algebras in K respectively.

Definition 2.1.28. We say that an equation $\varphi(x_1, ..., x_n) \approx \psi(y_1, ..., y_m)$ *holds* or *is valid* on an algebra A, and write $A \models \varphi \approx \psi$, if for every $a_1, ..., a_n, b_1, ..., b_m \in A$ we have $\varphi(a_1, ..., a_n) \approx \psi(b_1, ..., b_m)$.

Theorem 2.1.29 (Birkhoff's Theorem)**.** *A class of algebras V is a variety if and only if it is equationally definable. That is, there is a set of equations Σ such that, for each algebra A, we have that $A \in V$ if and only if $A \models \varphi \approx \psi$ for every $\varphi \approx \psi$ is Σ.*

So, by Birkhoff's Theorem, we can associate to any variety V the theory Σ of the equations that define V.

Definition 2.1.30. A variety V is *locally finite* if every finitely generated V-algebra is finite.

Example 2.1.31. The variety **BA** of Boolean algebras is locally finite.

Definition 2.1.32. Given a class of algebras $\mathcal{K}$, an *amalgam* is a tuple (A, f, B, g, C), where $A, B, C \in \mathcal{K}$ and $f : A \to B$, $g : A \to C$ are embeddings. We say that the class $\mathcal{K}$ of algebras has the *amalgamation property* if, for every amalgam (A, f, B, g, C) with $A, B, C \in \mathcal{K}$ and $A \neq \emptyset$, there exist an algebra $D \in \mathcal{K}$ and two embeddings $f^{'} : B \to D$, $g^{'} : C \to D$ such that $f^{'} \circ f = g^{'} \circ g$:

$$\begin{tikzcd} A \arrow[r, hook, "f"] \arrow[d, hook, "g"'] & B \arrow[d, hook, "f'"] \\ C \arrow[r, hook, "g'"'] & D \end{tikzcd}$$

We now introduce the following crucial result, that can be found in [28, Theorem 1]:

Theorem 2.1.33. *If a variety is locally finite and has the amalgamation property, then the corresponding first-order theory T has a model completion.*

Proof. We prove that the class of existentially closed models of T is axiomatised by $T^* := T \cup \{\forall \vec{x}(\Delta_A(\vec{x}) \to \varphi(\vec{x})) \mid \varphi$ is an existential formula, and there are two models $B \supseteq A = \langle \vec{a} \rangle$ with $B \models T \cup \{\varphi(\vec{a})\}\}$.

Let $C \models T^*$, and let $\varphi(\vec{C})$ be an existential formula with constants from C (we can assume that there is at least one constant). If $C \subseteq D$ and $D \models T \cup \{\varphi(\vec{c})\}$, from the fact that $C \models \Delta_C(\vec{c}) \wedge \forall \vec{x}(\Delta_C(\vec{x}) \to \varphi(\vec{x}))$ it follows that $C \models \varphi(\vec{c})$, i. e., C is existentially closed.

Conversely, let C be existentially closed, $\varphi(\vec{x})$ an existential formula, $B \supseteq A = \langle \vec{a} \rangle$ and $B \models T \cup \{\varphi(\vec{a})$. For any $\vec{c} \in C$, if $C \models \Delta_A(\vec{c})$, then $\langle \vec{c} \rangle$ is isomorphic to $\langle \vec{a} \rangle = A$ because they have the same diagram. Since the variety we are dealing with has the amalgamation property by hypothesis, there exists D making the following diagram commute:

$$\begin{tikzcd} \langle \vec{c} \rangle \cong A \arrow[r, hook] \arrow[d, hook] & B \arrow[d, hook] \\ C \arrow[r, hook] & D \end{tikzcd}$$

Hence $D \models \varphi(\vec{c})$, and since C is existentially closed, $C \models \varphi(\vec{c})$. So we have just proved that $C \models \forall \vec{x}(\Delta_A(\vec{x}) \to \varphi(\vec{x}))$, i. e., $C \models T^*$.

Therefore we have found a first-order axiomatisation of the class of existentially closed models of T: by the Proposition 2.1.25, we then have that T^* is the model completion of T. $\square$

2.2 The model completion of the theory of Boolean algebras

In this section, we study a relevant example: we prove that the model completion of the theory of Boolean algebras (denoted with T) is the theory of atomless Boolean algebras (denoted with T^*). So we first recall the following:

Definition 2.2.1. A Boolean algebra B is said to be *atomless* if $\forall b \in B\ \exists a \in B\ (b \neq 0 \rightarrow 0 < a < b)$.

Observe that the axiom $\forall b \in B\ \exists a \in B\ (b \neq 0 \rightarrow 0 < a < b)$ is a $\forall\exists$-axiom, according to the Remark 2.1.26.

Recall that the following result holds:

Proposition 2.2.2. *The categories* **FinBA** *of finite Boolean algebras and* **FinSet** *of finite sets are dually equivalent.*

Proof. We just sketch the idea of the proof. The duality between these categories is defined in the following way. For every finite set X, $\mathcal{P}(X)$ is a finite Boolean algebra. Conversely, given a finite Boolean algebra B, $B \cong \mathcal{P}(X)$, where $X = At(B) = \{a \in B \mid a \text{ is an atom}\}$ is a finite set: the isomorphism is given by $\eta(a) = \{x \in At(B) \mid x \leq a\}$. So we can associate $At(B)$ to B.

Given a morphism $h : A \rightarrow B$ between finite Boolean algebras, $At(h) : At(B) \rightarrow At(A)$ such that $At(h)(b) = \bigwedge\{a \in A \mid b \leq h(a)\}$ is a well-defined function between finite sets. Conversely, given a function $f : X \rightarrow Y$ between finite sets, $\mathcal{P}(f) : \mathcal{P}(Y) \rightarrow \mathcal{P}(X)$ such that $\mathcal{P}(f)(A) = f^{-1}(A)$ is a well-defined homomorphism of finite Boolean algebras.

□

Corollary 2.2.2.1. *Every finite Boolean algebra is isomorphic to a Boolean algebra of the kind* $(\mathcal{P}(X), \cup, \cap, \setminus, \emptyset, X)$*, for a certain finite set* X*.*

We now start our investigation about the model completion of Boolean algebras with the following:

Proposition 2.2.3. *Every non-degenerate*[2] *Boolean algebra can be embedded into an atomless Boolean algebra.*

Proof. Let B be a non-degenerate Boolean algebra, and consider the diagram of B Δ_B. Let $\{c_b \mid b \in B \setminus \{0\}\}$ be a set of new constants[3], and define $T' := T \cup \Delta_B \cup \{0 < c_b < \bar{b} \mid b \in B\}$.

[2]With *degenerate Boolean algebra* we mean the one-element Boolean algebra, which is trivially atomless (so there is nothing to prove in this case).

[3]We **don't** require that the interpretation of c_b is $b \in B$: this won't be the case. We denote with $\bar{b}$ the constant symbol that will be interpreted as $b \in B$.

We now show that T' is consistent.

By Compactness Theorem, it is sufficient to prove that any theory having the same shape of T' is consistent, where now Δ_B is the diagram of a **finite** non-degenerate Boolean algebra. In fact, the formulas which are contained in any finite subset of Δ_B involve finitely many constant symbols $\bar{b}$ of the language $\mathcal{L}_B = \mathcal{L} \cup \{\bar{b} \mid b \in B\}$. Therefore we can consider the subalgebra of B generated by the elements corresponding to those constant symbols: this subalgebra of B is finitely generated and so it is finite, being **BA** locally finite. Moreover, it is a model of the finite subset of Δ_B that we are considering.

So now our aim is to prove that $T' = T \cup \Delta_B \cup \{0 < c_b < \bar{b} \mid b \in B\}$ is consistent, where B is a finite Boolean algebra. By the Corollary 2.2.2.1, we have that $B \cong \mathcal{P}(X)$ for a finite set X. $\mathcal{P}(X)$ can be embedded into $\mathcal{P}(X \times \{0,1\})$, in the following way. Consider the projection $\pi_1 : X \times \{0,1\} \to X$ such that $\pi_1(x,i) = x$ for $i \in \{0,1\}$ and $x \in X$. Then $h : \mathcal{P}(X) \hookrightarrow \mathcal{P}(X \times \{0,1\})$ such that $h(A) = \pi_1^{-1}(A)$ for every $A \subseteq X$ is an embedding. Therefore $\mathcal{P}(X \times \{0,1\}) \models \varphi$ for every $\varphi \in \Delta_B$, by Lemma 2.1.4.

Now we define how to interpret the constants of the kind c_b in $\mathcal{P}(X \times \{0,1\})$: the interpretation of c_b is $\{(x,0) \mid x \in b\} \in \mathcal{P}(X \times \{0,1\})$ (recall that $b \in B \cong \mathcal{P}(X)$). It remains to prove that, with this interpretation, $\mathcal{P}(X \times \{0,1\}) \models (0 < c_b < \bar{b})$ for every $b \in B \setminus \{0\}$. Since by definition every constant c_b is associated to $b \in B \setminus \{0\}$, it holds that $\{(x,0) \mid x \in b\} \neq \emptyset$. This means that the condition $0 < c_b$ is satisfied in $\mathcal{P}(X \times \{0,1\})$. We also observe that the constant $\bar{b}$ is interpreted as $b \in B \cong \mathcal{P}(X)$, and as $h(b) = \pi_1^{-1}(b) = b \times \{0,1\}$ in $\mathcal{P}(X \times \{0,1\})$. Hence $b \times \{0,1\} = \{(x,0) \mid x \in b\} \subsetneq b \times \{0,1\}$, so also $c_b < \bar{b}$ is valid in $\mathcal{P}(X \times \{0,1\})$. This means that $\mathcal{P}(X \times \{0,1\})$ is a model of T', so it is also atomless, as required.

□

Corollary 2.2.3.1. *T and T^* prove the same quantifier-free formulas.*

Proof. It follows directly from Lemma 2.1.4 and Proposition 2.2.3. □

Now we have to prove that the theory of atomless Boolean algebras is submodel-complete. In order to prove this result, we first recall that the following hold:

Lemma 2.2.4. *For any nonzero element y of an atomless Boolean algebra, and for any $k \geq 2$, there exist nonzero elements $y_1, ..., y_k$ such that $y_1 \vee ... \vee y_k = y$ and $y_i \wedge y_j = 0$ for $i \neq j$. Such $y_1, ..., y_k$ are called* k-partition *of y.*

Lemma 2.2.5. *Given an atomless Boolean algebra A, and given two finite Boolean algebra A_1, A_2 with two embeddings $m : A_1 \hookrightarrow A$ and $n : A_1 \hookrightarrow A_2$, there exists an embedding $m' : A_2 \hookrightarrow A$ such that $m' \circ n = m$.*

$$\begin{tikzcd} A_1 \arrow[r, hook, "n"] \arrow[d, hook, "m"'] & A_2 \arrow[dl, hook, dashed, "m'"] \\ A & \end{tikzcd}$$

Proof. By the finite duality provided by the Proposition 2.2.2, $n : A_1 \hookrightarrow A_2$ is dual to a surjective map $h : X_2 \twoheadrightarrow X_1$ between finite sets. Let $k \in \mathbb{N}$ be such that $card(h^{-1}(x_1)) < k$, and let $A' \subseteq A$ be big enough so that every element of the kind $m(a)$ has a k-partition in A'. It is then sufficient to prove the statement with A' in place of A.

A' is a finite Boolean algebra. In fact, since A_1 is finite, there are finitely many elements of A of the kind $m(a)$. For each element of this kind, we consider **one** k-partition $m(a) = a'_1 \vee ... \vee a'_k$. Hence we can define A' as the subalgebra of A generated by the elements $a'_1, ..., a'_k$ for each $m(a)$, $a \in A_1$: A' is finitely generated and then finite (being **BA** a locally finite variety).

Therefore, again by finite duality, $m : A_1 \hookrightarrow A'$ is dual to a surjective function $l : X \twoheadrightarrow X_1$ between finite sets. Because of the definition of A', each element of X_1 has more than k preimages along l. In fact, let $\bar{a} \in X_1 = At(A_1)$, and let $\bar{a}'_1, ..., \bar{a}'_k$ as above (so $m(\bar{a}) = \bar{a}'_1 \vee ... \vee \bar{a}'_k$). We show that $\bar{a} = \bigwedge\{a \in A_1 \mid \bar{a}'_i \leq m(a)\} = At(m)(\bar{a}'_i)$ for every $i \in \{...,k\}$, if $\bar{a}'_1, ..., \bar{a}'_k \in At(A')$ (if this is not the case, since A' is finite, we know that $\bar{a}'_i = \bigvee\{x \in At(A') \mid x \leq \bar{a}'_i\}$, so we can replace $\bar{a}'_i$ with those x and argue in a similar way). Since $m(\bar{a}) = \bar{a}'_1 \vee ... \vee \bar{a}'_k$, we have that $m(\bar{a}) \geq \bar{a}'_i \; \forall i \in \{...,k\}$, and so $\bar{a} \geq \bigwedge\{a \in A_1 \mid \bar{a}'_i \leq m(a)\}$. Suppose by contradiction that $\tilde{a} := \bigwedge\{a \in A_1 \mid \bar{a}'_i \leq m(a)\} < \bar{a}$. Then $\tilde{a} \geq \bar{a}'_i \; \forall i \in \{1,...,k\}$ (otherwise, if $\tilde{a} < \bar{a}'_j$ for some $j \in \{...,k\}$, then $\tilde{a} = 0$ because $\bar{a}'_j$ is an atom. But then $0 = m(\tilde{a}) \geq \bar{a}'_i \Rightarrow \bar{a}'_i = 0$, so there is a contradiction.). Therefore $m(\tilde{a}) \geq \bar{a}'_1 \vee ... \vee \bar{a}'_k = m(\bar{a})$. But, since m is an embedding, $\tilde{a} < \bar{a} \Rightarrow m(\tilde{a}) < m(\bar{a})$, so we have a contradiction.

So now we have:

$$\begin{tikzcd} X_1 & X_2 \arrow[l, two heads, "h"'] \\ X \arrow[u, two heads, "l"] \arrow[ur, dashed, two heads, "l'"'] & \end{tikzcd}$$

We know that, $\forall x \in X_1$, $card(h^{-1}(x)) < k$ and $card(l^{-1}(x)) \geq k$: this is sufficient to build a surjective map $l' : X \to X_2$ such that $h \circ l' = l$. The homomorphism of finite Boolean algebras which is dual to l' is the required m'.

□

Proposition 2.2.6. *Given a finite Boolean algebra B_0 and given two countable atomless Boolean algebras C, D such that $B_0 \subseteq C$ and $B_0 \subseteq D$, there is an isomorphism $f : C \to D$ fixing B_0.*

Proof. Let $c_1, c_2, ...$ and $d_1, d_2, ...$ be enumerations of the elements of $C \setminus B_0$ and $D \setminus B_0$ respectively. Let C_i the Boolean algebra generated by B_0 and by $c_1, ..., c_i$, and let D_i be the Boolean algebra generated by B_0 and by $d_1, ..., d_i$. Thanks to the Lemma 2.2.5, it is easy to build the required isomorphism by a back-and-forth construction. In fact, we can start from $f_0 = id_{B_0} : B_0 \to B_0$. We can then define $f_1 : C_1 \to D'_1$ (where D'_1 is a subalgebra of

D including B_0) by sending c_1 into $d'_1 \in D'_1$ in such a way that f_1 is a local isomorphism[4]. Then we define $f_2 : D'_2 \to C'_2$ (where $D'_2 \supseteq D'_1$ is the subalgebra of D generated by B_0, d_1 and d'_1, while C'_2 is a subalgebra of C containing C_1) by sending $d_1 \in D'_1 \subseteq D'_2$ into c'_2 in such a way that f_2 is a local isomorphism. So, in general, at the stage i (for i odd) we define an isomorphism $f_i : C'_i \to D'_i$ fixing B_0 (where C'_i is a subalgebra of C including C_i and D'_i is a subalgebra of D including D_i). At the stage i (for i even), we just reverse the role of C and D. We define the maps f_i in such a way that the sequence $\{f_i\}_i$ is increasing. "In the limit", as $i \in \mathbb{N}$ increases, we will get the required isomorphism thanks to the Lemma 2.2.5.

□

Corollary 2.2.6.1. *The theory of atomless Boolean algebras is submodel-complete.*

Proof. The statement follows directy from the Proposition 2.2.6 and the Lemma 2.1.20, keeping in mind that the theory of atomless Boolean algebras doesn't have finite models (if we include the axiom $0 \neq 1$ to this theory).

□

Hence, by the Proposition 2.1.18, the theory of atomless Boolean algebras has quantifier elimination. So the fact that the theory of atomless Boolean algebras is the model completion of the theory of Boolean algebras follows from what we proved in this section.

[4] Given two $\mathcal{L}$-structures M and N, a function $f : A \to N$, where $A = \{a_1, ..., a_n\} \subseteq M$, is a *local isomorphism* if $M \models \varphi(a_1, ..., a_n) \Leftrightarrow N \models \varphi(a_1, ..., a_n)$ for any quantifier-free $\mathcal{L}$-formula φ.

Chapter 3

The model completion of the theory of S5-algebras

In this chapter, we present the first new results of this thesis: we prove that the model completion of the theory of S5-algebras exists, and we provide its finite axiomatization.

3.1 Basic definitions and duality

In this section, we introduce S5-algebras. We also define a duality between them and modal spaces (X, R) where R is an equivalence relation: this will be useful while looking for the model completion of the theory of S5-algebras. We use [8, 12, 27, 34] as our main references for the dual equivalence between modal algebras and modal spaces.

Definition 3.1.1. A *modal algebra* is a pair $(B, \Diamond)$, where B is a Boolean algebra and $\Diamond : B \to B$ satisfies:

i. $\Diamond 0 = 0$

ii. $\Diamond(a \vee b) = \Diamond a \vee \Diamond b$

A modal algebra is an *S4-algebra* if it also satisfies

iii. $\Diamond\Diamond a \leq \Diamond a$

iv. $a \leq \Diamond a$

Dually, we can define a modal algebra by means of a map $\Box : B \to B$, defined as $\Box a = \neg\Diamond\neg a$ for every $a \in B$.

Definition 3.1.2. Given two modal algebras $(B, \Diamond)$ and $(B', \Diamond')$, a map $h\colon B \to B'$ is a *modal homomorphism* if it is a Boolean homomorphism and $h(\Diamond a) = \Diamond' h(a)$ for every $a \in B$.

Definition 3.1.3. A *modal space* is a pair (X, R) where X is a Stone space and $R \subseteq X^2$ such that:

i. $R[x] = \{y \in X \mid xRy\}$ is closed for every $x \in X$,

ii. if $U \in \mathsf{Clop}(X)$, then $\Diamond_R(U) \in \mathsf{Clop}(X)$,

where $\Diamond_R(U) = \{x \in X \mid \exists y \in U \text{ such that } xRy\}$.

Definition 3.1.4. Given two modal spaces (X, R) and (X', R'), a map $f\colon X \to X'$ is a *continuous p-morphism* if it is continuous and, $\forall x, w \in X$, $\forall y \in X'$, the two following conditions hold:

i. xRw implies $f(x)R'f(w)$

ii. $f(x)R'y$ implies there is $z \in X$ with xRz and $f(z) = y$

Lemma 3.1.5. *If (X, R) is a modal space, then $(\mathsf{Clop}(X), \Diamond_R)$ is a modal algebra.*

Given a modal algebra $(B, \Diamond)$, we construct the dual modal space (X, R) in this way[1]: X is the Stone dual of B, and R is defined by xRy iff $\Box a \in x \Rightarrow a \in y$ iff $b \in y \Rightarrow \Diamond b \in x$. If we consider a modal homomorphism $h\colon B \to B'$, then $h_* = h^{-1}(-)\colon X_{B'} \to X_B$ is the continuous p-morphism dual to h. Moreover, if $f\colon X \to X'$ is a continuous p-morphism, then $f^* = f^{-1}(-)\colon \mathsf{Clop}(X') \to \mathsf{Clop}(X)$ is the modal homomorphism dual to f.

Definition 3.1.6. An S4-algebra $(B, \Diamond)$ is called an *S5-algebra* if it satisfies, for all $a \in B$, the inequality $a \leq \Box\Diamond a$.

Proposition 3.1.7. *$(B, \Diamond)$ is an S5-algebra if and only if, in its dual modal space (X, R), the relation R is an equivalence relation.*

Proof. ($\Rightarrow$) Let $x \in X$ (so x is an ultrafilter of $(B, \Diamond)$), and let $a \in x$. Since $(B, \Diamond)$ is an S5-algebra (and then, in particular, an S4-algebra), it holds that $a \leq \Diamond a$. Since x is an upset (being an ultrafilter), it also holds that $\Diamond a \in x$. This implies that xRx, and so R is *reflexive*. Now let $x, y, z \in X$ be such that xRy and yRz, and let $a \in z$. Since yRz, it holds that $\Diamond a \in y$. Therefore $\Diamond\Diamond a \in x$, because xRy. Since $(B, \Diamond)$ is an S5-algebra (and then, in particular, an S4-algebra), it holds that $\Diamond\Diamond a \leq \Diamond a$. Hence $\Diamond a \in x$, being x an ultrafilter. This implies that R is *transitive*. Let now $x, y \in X$ be such that xRy, and let $a \in x$. Since $(B, \Diamond)$ is an S5-algebra, $a \leq \Box\Diamond a$, and then $\Box(\Diamond a) = \Box\Diamond a \in x$, being x an ultrafilter. Since xRy, this implies that $\Diamond a \in y$, and so R is *symmetric*.
($\Leftarrow$) Assume that R is an equivalence relation on X. We prove that then $a \leq \Box\Diamond a$ $\forall a \in B$, by using the duality: since $(B, \Diamond) \cong (\mathsf{Clop}(X), \Diamond_R)$, it is sufficient to show that $U \subseteq \Box_R\Diamond_R U$ $\forall U \in \mathsf{Clop}(X)$, where we recall that $\Box_R U = \{x \in X \mid \forall y \in X (xRy \Rightarrow y \in U)\}$ and $\Diamond_R U = \{x \in X \mid \exists y \in U \text{ s. t. } xRy\}$. So, given $U \in \mathsf{Clop}(X)$, it holds that $\Box_R\Diamond_R U = \{x \in X \mid \forall y \in X (xRy \Rightarrow \exists z \in U \text{ s. t. } yRz)\}$. Now let $x \in U$. Since R is symmetric, $\forall y \in X$ s.

[1] The category of modal algebras and modal algebra homomorphisms **MA** is dually equivalent to the category of modal spaces and continuous p-morphisms **MS**: for a proof of this fact, see [4, Theorem 2.9].

t. xRy, it holds that yRx, and clearly $x \in U$. Therefore $x \in \Box_R \Diamond_R U$ and then $U \subseteq \Box_R \Diamond_R U$. This implies that $(B, \Diamond)$ is an S5-algebra.

$\Box$

3.2 Proof of the existence of the model completion

Now we are going to show that the model completion of the theory of S5-algebras exists, by proving that the variety of the S5-algebras is locally finite and has the amalgamation property, according to the Theorem 2.1.33.

We first recall the following useful criterion, which is mentioned in [2], and proved in [2] and in [29]:

Lemma 3.2.1. *A variety V is locally finite if and only if, for every $n \in \mathbb{N}$, there exists $M(n) \in \mathbb{N}$ such that, for every n-generated subdirectly irreducible $A \in V$, we have $|A| \leq M(n)$.*

Now we have two preliminary lemmas:

Lemma 3.2.2. *If (X, R) is a modal space such that $R = X \times X$, then, $\forall U \in \mathsf{Clop}(X)$, it holds that*

$$\Diamond_R(U) = \begin{cases} X & \text{if } U \neq \emptyset \\ \emptyset & \text{otherwise} \end{cases}$$

Lemma 3.2.3. *Given an S5-algebra $(B, \Diamond)$ and its dual modal space (X, R), $(B, \Diamond)$ is subdirectly irreducible if and only if X is a cluster, i. e., if and only if xRy for each $x, y \in X$.*

Proof. ($\Rightarrow$) Assume that $(B, \Diamond)$ is subdirectly irreducible. We know that $(B, \Diamond) \cong (\mathsf{Clop}(X), \Diamond_R) \cong \prod_{i \in I}(\mathsf{Clop}(X_i), \Diamond_{R_i})$ where $X = \biguplus_{i \in I} X_i$ and $X_i = R[x_i]$ for some $x_i \in X$. Since $(B, \Diamond)$ is subdirectly irreducible, we have that $(B, \Diamond) \cong (\mathsf{Clop}(X_i), \Diamond_{R_i})$ for some $i \in I$. So also the dual spaces are homeomorphic: $X \cong R[x_i]$. Since R is an equivalence relation, this implies that $R = X \times X$, i. e., X is a cluster.
($\Leftarrow$) Let (X, R) be a cluster, and consider a non-diagonal congruence $\sim \subseteq B \times B$. Then there are $a, b, \in B$ such that $a \neq b$ (without loss of generality, we can suppose that $a \not\leq b$) and $a \sim b$. So $a \wedge \neg b \neq 0$. Since $a \sim b$ and $\neg b \sim \neg b$, we have that $0 \neq a \wedge \neg b \sim b \wedge \neg b = 0$. By Lemma 3.2.2, it holds that $1 = \Diamond(a \wedge \neg b) \sim \Diamond 0 = 0$, i. e., $0 \in [1]_\sim$. Since $[1]_\sim$ is a filter, $[1]_\sim = B$. So, $\forall \bar{a}, \bar{b} \in B$, $\bar{a} \sim 1$ and $\bar{b} \sim 1$: then $\bar{a} \sim \bar{b}$, i. e., $\sim = B \times B$. This implies that there are only two congruences: $\Delta = \{(a, a) | a \in B\}$ and $B \times B$. So we have a least non diagonal congruence (which is $B \times B$), and this means that $(B, \Diamond)$ is subdirectly irreducible.

$\Box$

The two lemmas above allow us to prove the following, where we denote with V_{S5} the variety of S5-algebras:

Proposition 3.2.4. *The variety V_{S5} is locally finite.*

Proof. We prove the proposition by applying Lemma 3.2.1. Let $(B, \Diamond)$ be an n-generated subdirectly irreducible S5-algebra. By Lemma 3.2.3, we know that (X_B, R_B) is a cluster. Therefore, by Lemma 3.2.2, we have that, for every $b \in B$,

$$\Diamond b = \begin{cases} 1 & \text{if } b \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

So, if we consider the set of the generators of $(B, \Diamond)$, $S = \{s_1, ..., s_n\}$, and we consider the Boolean algebra generated by them, we have that it is also an S5-subalgebra of B (which coincides with B), being closed under $\Diamond$. Since the variety of the Boolean algebra is locally finite (being finitely generated), we have that B is finite. So it is sufficient to show that there is a bound on the cardinality of B that is a function of n. This bound is given by $M(n) = 2^{2^n}$ (this is a result which holds for n-generated Boolean algebras, in general). So we can conclude that the variety V_{S5} of S5-algebras is locally finite.

$\square$

If we also show that V_{S5} has the amalgamation property, then we can conclude that the theory of this variety has a model completion. In order to prove this, we use the following theorem [25, Theorem 3, page 352]:

Theorem 3.2.5. *Let $\mathcal{K}$ be an equational class of algebras satisfying the Congruence Extension Property, and let every subalgebra of each subdirectly irreducible algebra in $\mathcal{K}$ be subdirectly irreducible. Then $\mathcal{K}$ satisfies the Amalgamation Property if and only if, whenever A, B, C are subdirectly irreducible algebras in $\mathcal{K}$ with A a common subalgebra of B and C, the amalgam $\langle A; B, C\rangle$ can be amalgamated in $\mathcal{K}$.*

where we have that:

Definition 3.2.6. A class $\mathcal{K}$ of algebras is said to satisfy the Congruence Extension Property if, given any algebra B and subalgebra A, both in $\mathcal{K}$, and any congruence Θ on A, there is a congruence $\bar{\Theta}$ on B such that the restriction of $\bar{\Theta}$ to A, $\bar{\Theta}_A$, satisfies $\bar{\Theta}_A = \Theta$.

Thanks to an observation that we can find at the end of page 274 of [14], we know that the variety **MA** of modal algebras has the Congruence Extension Property. Therefore, in particular V_{S5} has the Congruence Extension Property, being a subvariety of **MA**. Moreover, we know that V_{S5} is equational. Hence, if we prove that every subalgebra of each subdirectly irreducible algebra in V_{S5} is subdirectly irreducible, then we can prove that V_{S5} has the amalgamation property by proving the equivalent condition given by the theorem above.

Lemma 3.2.7. *Every subalgebra of each subdirectly irreducible algebra in V_{S5} is subdirectly irreducible.*

Proof. We consider a subdirectly irreducible S5-algebra $(B, \Diamond)$, and a subalgebra $(A, \Diamond)$ of $(B, \Diamond)$. We are going to show that $(A, \Diamond)$ is subdirectly irreducible too. By Lemma 3.2.3, it is sufficient to prove that the modal space (X_A, R_A) dual to $(A, \Diamond)$ is such that X_A is a cluster: since $(A, \Diamond)$ is a subalgebra of $(B, \Diamond)$, we know that the inclusion $A \hookrightarrow B$ is an embedding. This means that the dual map $X_A \twoheadleftarrow X_B$ is surjective: for any ultrafilter x of A, there exists an ultrafilter z of B such that $x = A \cap z$. We also know that, by definition, for every $x, y \in X_A$, xR_Ay iff $\forall a \in A$ $\Box a \in x \Rightarrow a \in y$. So, we need to show that, given $x, y \in X_A$, and given $a \in A$, $\Box a \in x \Rightarrow a \in y$. We know that there exist two ultrafilters F_1 and F_2 of B such that $x = F_1 \cap A$ and $y = F_2 \cap A$. So, if $\Box a \in x = F_1 \cap A$, then $a \in F_2$ because B is subdirectly irreducible, and clearly $a \in A$. Hence $a \in F_2 \cap A = y$, as required. Then $\forall x, y \in X_A$, xR_Ay, i. e., X_A is a cluster, and so $(A, \Diamond)$ is a subdirectly irreducible S5-algebra too.

□

Therefore, in order to prove that V_{S5} has the amalgamation property, it is sufficient to show that, whenever A, B, C are subdirectly irreducible algebras in V_{S5} with A a common subalgebra of B and C, the amalgam $\langle A; B, C\rangle$ can be amalgamated in V_{S5}. In order to prove this, we use the duality again: we will add the hypothesis of A, B, C being subdirectly irreducible only when necessary.

Given three modal spaces (X_A, R_A), (X_B, R_B), (X_C, R_C) dual to three S5-algebras $(A, \Diamond_A)$, $(B, \Diamond_B)$, $(C, \Diamond_C)$, and given two surjective continuous p-morphisms $f : X_B \twoheadrightarrow X_A$, $g : X_C \twoheadrightarrow X_A$, we have to find a modal space (X_D, R_D) and two surjective maps $f' : X_D \twoheadrightarrow X_B$, $g' : X_D \twoheadrightarrow X_C$ such that $f \circ f' = g \circ g'$.

We know that $X_B \times X_C$ (with the product topology) is a Stone space, being X_B and X_C both Stone spaces. Therefore, we can consider the subspace $X_D :=\{(x, y) \in X_B \times X_C \mid f(x) = g(y)\} \subseteq X_B \times X_C$: we now prove that it is closed in $X_B \times X_C$, so that we can deduce from this that X_D is a Stone space too[2]. In order to show that, we need the following lemma:

Lemma 3.2.8. *Let (X, τ) be a topological space. (X, τ) is Hausdorff if and only if the diagonal $D := \{(x, x) \mid x \in X\} \subseteq X \times X$ is closed, with respect to the product topology.*

Proof. ($\Leftarrow$) Suppose that D is closed, and let $x, y \in X$ such that $x \neq y$. Then, $(x, y) \notin D$. This implies that there exists an open set $U \times V$, which belongs to the basis of the product topology $\mathcal{B} = \{U \times V \mid U, V \in \tau\}$, such that $(x, y) \in U \times V \subseteq (X \times X) \setminus D$. Now, suppose by contradiction that $U \cap V \neq \emptyset$. Then, there exists $z \in U \cap V$. This means that $(z, z) \in (U \times V) \cap D = \emptyset$, which is absurd. Therefore, since $U \cap V = \emptyset$, $x \in U$ and $y \in V$, we can conclude that X is Hausdorff.

[2]It is a general fact (and an easy exercise to show) that every closed subset of a Stone space with the induced topology is a Stone space.

($\Rightarrow$) Now, suppose that X is Hausdorff. We show that D is closed by proving that $(X\times X)\setminus D$ is open. Let $(x,y)\in(X\times X)\setminus D$. Then, $x\neq y$. Therefore, since X is Hausdorff, there exist $U,V\in\tau$ such that $U\cap V=\emptyset$, $x\in U$ and $y\in V$. So, $(x,y)\in U\times V$. Since $(\forall z\in X)$ ($z\in U\cap V$ if and only if $(z,z)\in(U\times V)\cap D$), and since $U\cap V=\emptyset$, we have that $U\times V\subseteq(X\times X)\setminus D$. Hence $(X\times X)\setminus D$ is open, as required.

□

Lemma 3.2.9. *Given three topological spaces X_A, X_B, X_C, where X_A is Hausdorff, and given two continuous maps $f\colon X_B\to X_A$ and $g\colon X_C\to X_A$, $X_D:=\{(x,y)\in X_B\times X_C\mid f(x)=g(y)\}\subseteq X_B\times X_C$ is closed with respect to the product topology.*

Proof. Since X_A is Hausdorff, by the previous lemma we have that the diagonal $D_A:=\{(x,x)\mid x\in X_A\}\subseteq X_A\times X_A$ is closed. Moreover, since f and g are both continuous, also the map $(f\times g)\colon X_B\times X_C\to X_A\times X_A$ defined by $(f\times g)(x,y):=(f(x),g(y))$ is continuous. So, $X_D=(f\times g)^{-1}(D_A)\subseteq X_B\times X_C$ is closed with respect to the product topology, as required.

□

Therefore, since every Stone space is Hausdorff, we have that X_D is a Stone space, being a closed subset of a Stone space.

We can now define the following relation on $X_D\subseteq X_B\times X_C$, by using the two equivalence relations R_B and R_C:

$$(x,y)R_D(x',y') \text{ iff } (xR_Bx' \ \&\ yR_Cy') \tag{3.1}$$

In this way, we have that:

Lemma 3.2.10. *Let X_D be ad in the statement of Lemma 3.2.9, and R_D as in (3.1). Then (X_D,R_D) is a modal space and R_D is an equivalence relation.*

Proof. The fact that R_D is an equivalence relation follows easily from the fact that R_B and R_C are both equivalence relations. Then we have to prove that $R_D[(x,y)]=\{(x',y')\in X_D\mid(x,y)R_D(x',y')\}$ is closed for every $(x,y)\in X_D$. This follows from the fact that $R_D[(x,y)]=\{(x',y')\in X_D\mid(x,y)R_D(x',y')\}=\{(x',y')\in X_D\mid x'\in R_B[x]\ \&\ y'\in R_C[y]\}=(R_B[x]\times R_C[y])\cap X_D$, being $R_B[x]\subseteq X_B$ and $R_C[y]\subseteq X_C$ both closed.

Now let $U\in\mathsf{Clop}(X_D)$. We need to show that $\Diamond_{R_D}(U)\in\mathsf{Clop}(X_D)$. By definition, $\Diamond_{R_D}(U)=\{(x,y)\in X_D\mid\exists(x',y')\in U \text{ s. t. } (x,y)R_D(x',y')\}=\{(x,y)\in X_D\mid\exists(x',y')\in U \text{ s. t. } xR_Bx'\ \&\ yR_Cy'\}$. Therefore, since both X_B and X_C are clusters (being $(B,\Diamond)$ and $(C,\Diamond)$ subdirectly irreducible), we have that $\Diamond_{R_D}(U)=\begin{cases}X_D & \text{if } U\neq\emptyset\\ \emptyset & \text{otherwise}\end{cases}$. In any case, $\Diamond_{R_D}(U)\in\mathsf{Clop}(X_D)$, as required.

□

Now we consider the maps $f' := \pi_1 \colon X_D \to X_B$ such that $f'((x,y)) = \pi_1((x,y)) = x$, and $g' := \pi_2 \colon X_D \to X_C$ such that $g'((x,y)) = \pi_2((x,y)) = y$. We have that:

Lemma 3.2.11. *The projections π_1 and π_2 are continuous p-morphisms, i. e., they are morphisms of the category* **MS**. *Moreover, they are surjective.*

Proof. π_1 is continuous. In fact, let $U \subseteq X_B$ be open. Then, $\pi_1^{-1}(U) =\{(x,y) \in X_B \times X_C \mid f(x) = g(y) \text{ and } x \in U\}= X_D \cap (U \times X_C)$. By definition of product topology, $(U \times X_C)$ is open in $X_B \times X_C$, and so, by definition of subspace topology, $\pi_1^{-1}(U)$ is open in X_D. In a similar way, we can prove that also π_2 is continuous.

π_1 is a p-morphism: if xR_Bx' and $(x,y) \in X_D$ is such that $\pi_1(x,y) = x$, then we have to show that there exists $(x',y') \in X_D$ such that $(x,y)R_D(x',y')$. This is equivalent to find $y' \in X_C$ such that yR_Cy'. But since R_C is an equivalence relation, we can just consider $y' = y$. Similarly for π_2.

Moreover, π_1 is surjective. In fact, let $x \in X_B$, and consider $f(x) \in X_A$. Since g is surjective, there exixts $y \in X_C$ such that $g(y) = f(x)$. Then, $(x,y) \in X_D$, and $\pi_1((x,y)) = x$, as required. Similarly, we can prove that also π_2 is surjective.

$\square$

Since π_1 and π_2 are surjective, the morphisms of **MA** which are the dual of those functions are injective (according to the duality recalled above). Moreover, because of the definition of X_D, we have that $\forall (x,y) \in X_D$ $f(f'((x,y))) = f(\pi_1((x,y))) = f(x) = g(y) = g(\pi_2((x,y))) = g(g'((x,y)))$, and so $f \circ f' = g \circ g'$. Hence, we can conclude that V_{S5} has the amalgamation property. Therefore, we can conclude that the variety of S5-algebras has a model completion.

3.3 Axiomatisation of the model completion of the theory of S5-algebras

We are now interested in finding a nice axiomatisation of the model completion of the theory of S5-algebras. Hence, according to the Proposition 2.1.25, we are interested in studying the existentially closed S5-algebras. The following result about existentially closed S5-algebras holds (cf. [11, Proposition 2.16]):

Theorem 3.3.1. *Let $(B, \Diamond)$ be an S5-algebra. $(B, \Diamond)$ is existentially closed iff for any finite sub-S5-algebra $(B_0, \Diamond) \subseteq (B, \Diamond)$ and for any finite extension $(C, \Diamond) \supseteq (B_0, \Diamond)$ there exists an embedding $(C, \Diamond) \hookrightarrow (B, \Diamond)$ fixing $(B_0, \Diamond)$ pointwise.*

$$\begin{tikzcd}
(B_0, \Diamond) \arrow[r, hook] \arrow[d, hook] & (B, \Diamond) \\
(C, \Diamond) \arrow[ur] &
\end{tikzcd}$$

Proof. ($\Leftarrow$) Let $(D,\Diamond)$ be an extension of $(B,\Diamond)$ and $\exists x_1,...,x_m\phi(x_1,...,x_m,a_1,...,a_n)$ an existential $\mathcal{L}_{(\mathcal{B},\Diamond)}$-sentence, where $\phi(x_1,...,x_m,a_1,...,a_n)$ is quantifier-free and $a_1,...,a_n \in B$. Suppose that $(D,\Diamond) \vDash \exists x_1,...,x_m\phi(x_1,...,x_m,a_1,...,a_n)$. Let $d_1,...,d_m$ be elements of D such that $(D,\Diamond) \vDash \phi(d_1,...,d_m,a_1,...,a_n)$. Consider the sub-S5-algebra $(B_0,\Diamond)$ of $(B,\Diamond)$ generated by $a_1,...,a_n$ and the sub-S5-algebra $(C,\Diamond) \subseteq (D,\Diamond)$ generated by $d_1,...,d_m$, $a_1,...,a_n$. They are both finite because they are finitely generated and the variety of S5-algebras is locally finite. By hypothesis there exists an embedding $(C,\Diamond) \hookrightarrow (B,\Diamond)$ fixing $(B_0,\Diamond)$ pointwise. Let $d'_1,...,d'_m$ be the images of $d_1,...,d_m$ by this embedding. Thus $(B,\Diamond) \vDash \phi(d'_1,...,d'_m,a_1,...,a_n)$ because ϕ is quantifier-free. Therefore $(B,\Diamond) \vDash \exists x_1,...,x_m\phi(x_1,...,x_m,a_1,...,a_n)$: it follows that $(B,\Diamond)$ is existentially closed.
($\Rightarrow$) Now suppose that $(B,\Diamond)$ is existentially closed, and let $(C,\Diamond) \supseteq (B_0,\Diamond)$ be a finite extension of a finite sub-S5-algebra $(B_0,\Diamond) \subseteq (B,\Diamond)$. Since the variety of S5-algebras has the amalgamation property, there exists an S5-algebra $(D,\Diamond)$ amalgamating $(C,\Diamond)$ and $(B,\Diamond)$ over $(B_0,\Diamond)$:

$$\begin{tikzcd}
(B_0,\Diamond) \arrow[r, hook, "f"] \arrow[d, hook, "g"'] & (B,\Diamond) \arrow[d, hook, "f'"] \\
(C,\Diamond) \arrow[r, hook, "g'"'] & (D,\Diamond)
\end{tikzcd}$$

Let Σ be the set of quantifier-free $\mathcal{L}_{(C,\Diamond)}$-sentences of the form $c \star c' = c''$ or $\bullet c = c'$ true in $(C,\Diamond)$, where $c,c',c'' \in C$, $\star$ is either $\wedge$ or $\vee$ and $\bullet$ is either $\neg$ or $\Diamond$. Now let $c_1,...,c_r,a_1,...,a_n$ be an enumeration of elements in C where the a_i's are elements in B. We obtain the quantifier-free $\mathcal{L}_{(C,\Diamond)}$-sentence $\sigma(c_1,...,c_r,a_1,...,a_n)$ by taking the conjunction of all the sentences in Σ and all the sentences of the form $\neg(c = c')$ for every $c,c' \in C$ such that $c \neq c'$. Clearly, $\exists x_1,...,x_r\sigma(x_1,...,x_r,a_1,...,a_n)$ is an existential $\mathcal{L}_{(\mathcal{B},\Diamond)}$-sentence true in $(D,\Diamond)$. Since $(B,\Diamond)$ is existentially closed, $(B,\Diamond) \vDash \exists x_1,...,x_r\sigma(x_1,...,x_r,a_1,...,a_n)$. Let $c'_1,...,c'_r \in B$ be such that $(B,\Diamond) \vDash \sigma(c'_1,...,c'_r,a_1,...,a_n)$. The map $(C,\Diamond) \hookrightarrow (B,\Diamond)$ fixing $(B_0,\Diamond)$ pointwise and mapping c_i to c'_i is the required embedding, because it is injective and it is a homomorphism (by definition of the sentence σ).

□

Since every finite extension of a finite S5-algebra $(B_0,\Diamond)$ is composition of minimal extensions, we are interested in characterizing the minimal extensions of a finite S5-algebra $(B_0,\Diamond)$. In fact, we have the following:

Corollary 3.3.1.1. *Let $(B,\Diamond)$ be an S5-algebra. $(B,\Diamond)$ is existentially closed iff for any finite sub-S5-algebra $(B_0,\Diamond) \subseteq (B,\Diamond)$ and for any finite* minimal *extension $(C,\Diamond) \supseteq (B_0,\Diamond)$ there exists an embedding $(C,\Diamond) \hookrightarrow (B,\Diamond)$ fixing $(B_0,\Diamond)$ pointwise.*

Proof. ($\Rightarrow$) Direct application of the previous theorem: any finite minimal extension $(C,\Diamond) \supseteq (B_0,\Diamond)$ is a finite extension.

($\Leftarrow$) Given a finite extension $(B_0, \Diamond) \overset{\iota}{\hookrightarrow} (C, \Diamond)$, there exists a chain of minimal extensions $(B_0, \Diamond) \overset{\iota_1}{\hookrightarrow} (B_1, \Diamond) \overset{\iota_2}{\hookrightarrow} ... \overset{\iota_n}{\hookrightarrow} (B_n, \Diamond) = (C, \Diamond)$ such that $\iota = \iota_n \circ \iota_{n-1} \circ ... \circ \iota_1$: this chain is finite because ι is a finite extension. By hypothesis, for each of these inclusions there exits an embedding $(B_k, \Diamond) \overset{g_k}{\hookrightarrow} (C, \Diamond)$ fixing $(B_{k-1}, \Diamond)$ pointwise. Hence we can consider $(B_n, \Diamond) = (C, \Diamond) \overset{g_n}{\hookrightarrow} (B, \Diamond)$: this is an embedding which fixes $(B_{n-1}, \Diamond)$ pointwise. This implies that g_n fixes also $(B_0, \Diamond)$ pointwise, being $(B_0, \Diamond) \cong (\iota_{n-1} \circ \iota_{n-2} \circ ... \circ \iota_1)((B_0, \Diamond)) \subseteq (B_{n-1}, \Diamond)$. Hence, thanks to the previous theorem, $(B, \Diamond)$ is existentially closed.

$\square$

Therefore, in order to study the minimal extensions of the finite S5-algebras, we use the duality which is recalled above. We observe that:

Remark 3.3.2. If B is a finite Boolean algebra, then the Stone space X_B dual to B has the discrete topology. In fact, given any ultrafilter F of a finite Boolean algebra, it has a minimum element a[3], and so $F = \uparrow a$, i. e., it is principal. Such an a has to be an atom[4] of B, and so every ultrafilter of a finite Boolean algebra B is principal and it contains a unique[5] atom of B. We also know that, according to Stone duality, every atom of a Boolean algebra B corresponds to an isolated point in the dual Stone space X_B. In fact, we have that $a \in B$ is an atom if and only if $\phi(a) = \{\uparrow a\}$, where $\phi(a) = \{x \in X_B \mid a \in x\}$ is one of the elements of the basis of clopens of X_B, and $\uparrow a$ is the unique ultrafilter of B containing a. Therefore, if B is finite, we have that $\forall x = \uparrow a \in X_B$, $\{x\} = \phi(a)$, i. e., every singleton of X_B is open. This means that the topology on X_B is discrete[6].

So, suppose that $(C, \Diamond)$ is a finite S5-algebra, and let (X_C, R_C) be its dual modal space. Since X_C is the Stone space dual to C, the topology on X_C is discrete, and so the two conditions on R_C given by the definition of modal space[7] are trivially satisfied. Hence we

[3]Let F be an ultrafilter of a finite Boolean algebra B. Since B is finite, F is a finite poset too (with respect to the restriction of the order of B to F). So, it has at least one minimal element, because every finite partially ordered set has a minimal element. Suppose by contradiction that F has two minimal elements which are incomparable, b and c. Then $b \neq b \wedge c \neq c$, and $b \wedge c \in F$, by definition of filter. But, since b and c are incomparable, $b \wedge c < b$ and $b \wedge c < b$, and this contradicts the fact that b and c are minimal in F.

[4]Suppose by contradiction that $\exists b \in B$ such that $0 < b < a$. Then $\uparrow a \subsetneq \uparrow b \subsetneq B$, so $\uparrow a$ can't be a maximal filter, and then it can't be an ultrafilter of B.

[5]If F contains two different atoms a and b, the it has to contain also $a \wedge b = 0$, and so it can't be a proper filter, but this contradicts the definition of ultrafilter.

[6]This means, by definition, that one of the following equivalent conditions holds:

- every singleton of X_B is open
- every subset of X_B is open
- every subset of X_B is closed

[7]The two conditions I am referring to are the follwing:

- $R_C[x]$ is closed $\forall x \in X_C$
- $\Diamond_{R_C}(U) \in \mathsf{Clop}(X_C)$ $\forall U \in \mathsf{Clop}(X_C)$

can regard X_C just as a set, equipped with an equivalence relation R_C.
Therefore, if we dualize the diagram that we find in the statement of Theorem 3.3.1, we obtain the following diagram:

$$\begin{tikzcd}
(X_{B_0}, R_{B_0}) & (X_B, R_B) \arrow[l, two heads] \arrow[dl, two heads] \\
(X_C, R_C) \arrow[u, two heads] &
\end{tikzcd}$$

Hence, according to the duality between modal algebras and modal spaces and to Corollary 3.3.1.1, we are interested in studying the p-morphisms which are dual to the minimal extensions of finite S5-algebras. There are two kinds of such p-morphisms.

Definition 3.3.3. A p-morphism $f : (X_C, R_C) \twoheadrightarrow (X_{B_0}, R_{B_0})$ between finite modal spaces, such that R is an equivalence relation, is *of the first kind* if $X_C = X_{B_0} \cup \{x'\}$, $x' R_C x$ for some $x \in X_C \backslash \{x'\}$, $f(x') = f(x)$ and $f \restriction_{X_{B_0}} = id_{X_{B_0}}$.

Lemma 3.3.4. *The embedding between finite S5-algebras $(B_0, \Diamond) \hookrightarrow (C, \Diamond)$, which is dual to a p-morphism of the first kind, is a minimal extension.*

Proof. The embedding which is dual to a p-morphism of the first kind f is given by $f^* = f^{-1}(-) : (B_0, \Diamond) \cong (\mathcal{P}(X_{B_0}), \Diamond_{R_{B_0}}) \hookrightarrow (\mathcal{P}(X_C), \Diamond_{R_C}) \cong (C, \Diamond)$. This is a minimal extension. In fact, let $n = |X_{B_0}|$. Since $X_C = X_{B_0} \cup \{x'\}$, it holds that $|X_C| = n + 1$. Therefore, keeping in mind that $\mathcal{P}(X_{B_0}) \cong B_0$ [8] and $\mathcal{P}(X_C) \cong C$, we can conclude that $|B_0| = 2^n$, while $|C| = 2^{(n+1)}$ [9]. Hence the homomorphism of modal algebras which is dual to f is a minimal extension, because (for cardinality reasons) there can't be any proper intermediate extension.

□

Definition 3.3.5. A finite minimal extension of finite S5-algebras which is dual to a p-morphism of the first kind is called *minimal extension of the first kind.*

The second kind of p-morphism that we have to consider occurs when, in order to obtain (X_C, R_C), we decide to add to (X_{B_0}, R_{B_0}) ad element x' which is not in relation with any other element of X_{B_0}, and we map it to an element of X_{B_0}, $f(x')$ (also in this situation, we define $f \restriction_{X_{B_0}} = id_{X_{B_0}}$). By definition of p-morphism, f has to satisfy the following condition: $(\forall x \in X_C)(f(x) R_{B_0} y \Rightarrow \exists z \in X_C$ s. t. $x R_C z$ and $f(z) = y)$. Therefore, since x'

[8] This isomorphism comes from the duality recalled above, because $\mathsf{Clop}(X) = \mathcal{P}(X)$ in the finite case, being the topology on X discrete if X is the Stone space which is dual to a finite Boolean algebra.

[9] We can observe that $f_*(\mathcal{P}(X_{B_0}))$ contains all the elements of $\mathcal{P}(X_C)$, except for the subsets of X_C containing *only one* of the elements x, x'. We know that the number of those subsets is $2^{n+1} - 2^n = 2^n$. In fact, $X_C \backslash \{x, x'\}$ has $2^{(n+1)-2} = 2^{n-1}$ subsets: for each of those subsets U, we have to consider $U \cup \{x\}$ and $U \cup \{x'\}$. So the number of the considered subsets is given by $2 \times 2^{n-1} = 2^n$.

is not in relation with any other element of X_{B_0}, we also have to add to X_C one element for each element of $[f(x')]_{R_{B_0}}$: all those additional elements together form a new equivalence class of X_C, which is another copy of the equivalence class $[f(x')]_{R_{B_0}}$. The map f sends each of those additional elements into an element of $[f(x')]_{R_{B_0}}$. So we define:

Definition 3.3.6. A p-morphism $f : (X_C, R_C) \twoheadrightarrow (X_{B_0}, R_{B_0})$ between finite modal spaces, such that R is an equivalence relation, is *of the second kind* if $X_C = X_{B_0} \uplus T$, where T is a copy of an equivalence class $R_{B_0}[x]$ of X_{B_0} which form another equivalence class of X_C, distinct from the others, $f \restriction_{X_{B_0}} = id_{X_{B_0}}$, and $f \restriction_T : T \longrightarrow R_{B_0}[x]$ is a bijection which sends each element of T into its copy in $R_{B_0}[x]$.

Lemma 3.3.7. *The embedding between finite S5-algebras $(B_0, \Diamond) \hookrightarrow (C, \Diamond)$, which is dual to a p-morphism of the second kind, is a minimal extension.*

Proof. Suppose by contradiction that there exists a proper intermediate extension $(B_0, \Diamond) \subsetneq (\bar{B}, \Diamond) \subsetneq (C, \Diamond)$, where $|B_0| = 2^n$, $|\bar{B}| = 2^k$ and $|C| = 2^m$, with $m > k > n$. Dually, we have $(X_C, R_C) \overset{f_2}{\twoheadrightarrow} (X_{\bar{B}}, R_{\bar{B}}) \overset{f_1}{\twoheadrightarrow} (X_{B_0}, R_{B_0})$, where $f = f_1 \circ f_2$ is the continuous p-morphism which is dual to the inclusion $(B_0, \Diamond) \subsetneq (C, \Diamond)$. From the assumption about the cardinalities of the considered S5-algebras, we know that $|X_C| - |X_{B_0}| = |T| = |[f(x')]_{R_{B_0}}| = m - n$ (where $x' \in X_C$ is such that $[x']_{R_C} = T$), and that $|X_{\bar{B}}| - |X_{B_0}| = |[f_2(x')]_{R_{\bar{B}}}| = k - n$.[10] Then f_1 can't be a p-morphism, because $k - n = |[f_2(x')]_{R_{\bar{B}}}| < |[(f_1 \circ f_2)(x')]_{R_{B_0}}| = |[f(x')]_{R_{B_0}}| = m - n$, so $[f_2(x')]_{R_{\bar{B}}}$ doesn't contain enough elements to satisfy the condition given by the definition of p-morphism. This implies that such a proper intermediate extension $(\bar{B}, \Diamond)$ can't exist, and so the considered extension is minimal.

□

Definition 3.3.8. A finite minimal extension of finite S5-algebras which is dual to a p-morphism of the second kind is called *minimal extension of the second kind.*

Remark 3.3.9. It is easy to see that there can't exist other kinds of minimal finite extensions. In fact, suppose that $B_0 \subsetneq C$ is a proper finite extension of B_0, and let $X_C \overset{f}{\twoheadrightarrow} X_{B_0}$ be the dual continuous p-morphism. Since $C \cong \mathsf{Clop}(X_C) = \mathcal{P}(X_C)$, $B_0 \cong \mathsf{Clop}(X_{B_0}) = \mathcal{P}(X_{B_0})$ and $C \setminus B_0 \neq \emptyset$, it holds that $|X_{B_0}| < |X_C|$. Then there is at least one element $x' \in X_C$ such that $\exists x \in X_C \setminus \{x'\}$ with $f(x') = f(x)$. There are two possibilities: either $x' R_C x$ or $[x']_{R_C} \neq [x]_{R_C}$. In the first case, $f = f_1 \circ f_2$, where $f_1 : X_C \to X_A$, $X_A = X_C \setminus \{x'\}$, $f_1 \restriction X_A = id_{X_A}$. So f_1 is dual to a minimal extension of the first kind. In the second case, by definition of p-morphism, both $[x']_{R_C}$ and $[x]_{R_C}$ contain

[10] These equalities hold because we know that $C \cong \mathcal{P}(X_C)$ (because of duality). The elements of X_C are the ultrafilters of C, which correspond to the ultrafilters of $\mathcal{P}(X_C)$. Those ultrafilters coincide with the principal filters of $\mathcal{P}(X_C)$ generated by the atoms (i. e., the singletons contained $\mathcal{P}(X_C)$). Since there is a bijection between the elements $x \in X_C$ and the singletons $\{x\} \in \mathcal{P}(X_C)$, we can conclude that, if $|C| = 2^m$, then C has m atoms, hence $|X_C| = m$. The same holds for the other considered Boolean algebras.

a copy of $[f(x)]_{R_{B_0}} = [f(x')]_{R_{B_0}}$. Hence we can regard f as the composition $f = f_1 \circ f_2$, where $f_2 : X_A \to X_{B_0}$ is a continuous p-morphism which is dual to a minimal extension of the second kind, being $X_A = X_{B_0} \uplus T$, where T is a copy of $[f(x')]_{R_{B_0}}$.

Now we prove two useful results:

Proposition 3.3.10. *Let $(B_0, \Diamond)$ be a finite S5-algebra, and let $(B_0, \Diamond) \overset{\iota}{\hookrightarrow} (C, \Diamond)$ be a finite minimal extension of the first kind. Then there exists a unique atom b of B_0 such that there exist two different atoms $b_1, b_2 \in C$ with $\iota(b) = b_1 \vee b_2$ and $\Diamond\iota(b) = \Diamond b_1 = \Diamond b_2$.*

Proof. Since ι is a finite minimal extension of the first kind, we know that the p-morphism which is dual to ι, $f : (X_C, R_C) \twoheadrightarrow (X_{B_0}, R_{B_0})$, is such that $X_C = X_{B_0} \cup \{x'\}$, $x' R_C x$ for some $x \in X_C \setminus \{x'\}$, $f(x') = f(x)$ and $f \restriction_{X_{B_0}} = id_{X_{B_0}}$. We also know that $(B_0, \Diamond) \cong (\mathcal{P}(X_{B_0}), \Diamond)$, and that $(C, \Diamond) \cong (\mathcal{P}(X_C), \Diamond)$. Therefore the singletons $\{f(x) = f(x')\}$, $\{x\}$ and $\{x'\}$ correspond to the atoms $b \in B_0$, $b_1, b_2 \in C$ respectively. Because of the definition of the duality, we know that $\iota = f^{-1}(-)$. Hence it holds that $\iota(b) \cong f^{-1}(\{f(x) = f(x')\}) = \{x, x'\} = \{x\} \cup \{x'\} \cong b_1 \vee b_2$. Moreover, it holds that $\Diamond b \cong \Diamond\{f(x) = f(x')\} = R_{B_0}[f(x)]$, and similarly for b_1 and b_2. So we have that $\Diamond\iota(b) = \iota(\Diamond b) \cong f^{-1}(\Diamond\{f(x)\}) = f^{-1}(R_{B_0}[f(x)]) = R_C[x] = \Diamond\{x\} \cong \Diamond b_1$, and similarly $\Diamond\iota(b) = \iota(\Diamond b) \cong f^{-1}(\Diamond\{f(x')\}) = f^{-1}(R_{B_0}[f(x')]) = R_C[x'] = \Diamond\{x'\} \cong \Diamond b_2$.

In order to prove the uniqueness of such an atom $b \in B_0$, we suppose by contradiction that there are two different atoms $b, \bar{b} \in B_0$ such that $\exists b_1, b_2, \bar{b_1}, \bar{b_2}$ different atoms of C with $\iota(b) = b_1 \vee b_2$, $\Diamond\iota(b) = \Diamond b_1 = \Diamond b_2$, $\iota(\bar{b}) = \bar{b_1} \vee \bar{b_2}$ and $\Diamond\iota(\bar{b}) = \Diamond\bar{b_1} = \Diamond\bar{b_2}$. It is easy to see that then $\iota = \iota_2 \circ \iota_1$ isn't a minimal extension, being the composition of two minimal extensions of the first kind ι_1 and ι_2. ι_1 is dual to the p-morphism $f_1 : (X_A, R_A) \twoheadrightarrow (X_{B_0}, R_{B_0})$ such that $X_A = X_{B_0} \cup \{x'\}$, $x' R_A x$ for some $x \in X_A \setminus \{x'\}$, $f_1(x') = f_1(x)$ and $f_1 \restriction_{X_{B_0}} = id_{X_{B_0}}$, while ι_2 is dual to the p-morphism $f_2 : (X_C, R_C) \twoheadrightarrow (X_A, R_A)$ such that $X_C = X_A \cup \{\bar{x'}\}$, $\bar{x'} R_C \bar{x}$ for some $\bar{x} \in X_C \setminus \{\bar{x'}\}$, $f_2(\bar{x'}) = f_2(\bar{x})$ and $f_2 \restriction_{X_A} = id_{X_A}$, where the singletons $\{f_1(x) = f_1(x')\}$, $\{x\}$, $\{x'\}$, $\{f_2(\bar{x}) = f_2(\bar{x'})\}$, $\{\bar{x}\}$ and $\{\bar{x'}\}$ correspond to $b, b_1, b_2, \bar{b}, \bar{b_1}, \bar{b_2}$ respectively. This is sufficient to prove the uniqueness of such an atom $b \in B_0$.

$\square$

In order to prove the second relevant result, we need the following lemma:

Lemma 3.3.11. *Given an S5-algebra $(B, \Diamond)$, we have that $\mathcal{B} = \{\Diamond b \mid b \in B\}$ is a Boolean algebra.*

Proof. By definition of S5-algebra, $0 = \Diamond 0$ and $\Diamond a \vee \Diamond b = \Diamond(a \vee b)$ $\forall a, b \in B$, so $0 \in \mathcal{B}$ and $\Diamond a \vee \Diamond b \in \mathcal{B}$. It also holds that $1 = \Diamond 1 \in \mathcal{B}$, being $a \leq \Diamond a$ $\forall a \in B$. Thanks to the fact that $(B, \Diamond) \cong (\mathsf{Clop}(X_B), \Diamond_R)$, it holds that $\forall a, b \in B$ $\exists U, V \in \mathsf{Clop}(X_B))$ such that $\Diamond a \wedge \Diamond b \cong \Diamond_R U \cap \Diamond_R V = (\bigcup_{x \in U} R[x]) \cap (\bigcup_{x \in V} R[x]) \overset{(*)}{=} \bigcup_{z \in (\Diamond_R U \cap \Diamond_R V)} R[z] = \Diamond_R(\Diamond_R U \cap \Diamond_R V) \cong \Diamond(\Diamond a \wedge \Diamond b)$, where the equality denoted by $(*)$ holds because $z \in \Diamond_R U \cap \Diamond_R V \Leftrightarrow$

$z \in R[x] \cap R[y]$ for some $x \in U$, $y \in V$ and, in this situation, $R[z] = R[x] = R[y]$, being R an equivalence relation. Therefore $\forall a, b \in B\ \Diamond a \wedge \Diamond b \in \mathcal{B}$. Moreover, $\forall a \in B\ \exists U \in \mathsf{Clop}(X_B)$ such that $\neg \Diamond a \cong X_B \setminus \Diamond_R U = X_B \setminus \bigcup_{x \in U} R[x] \overset{(*)}{=} \bigcup_{z \in X_B \setminus \Diamond U} R[z] = \Diamond(X_B \setminus \Diamond U) \cong \Diamond(\neg \Diamond a) \in \mathcal{B}$, where the equality denoted by $(*)$ holds because $z \in X_B \setminus \bigcup_{x \in U} R[x] \Leftrightarrow \nexists y \in U$ s. t. $zRy \Leftrightarrow R[z] \subseteq X_B \setminus \Diamond U$.

□

Proposition 3.3.12. *Let $(B_0, \Diamond)$ be a finite S5-algebra, and let $(B_0, \Diamond) \overset{\iota}{\hookrightarrow} (C, \Diamond)$ be a finite minimal extension of the second kind. Then there exists a unique atom $\Diamond b$ of the Boolean algebra $\mathcal{B} = \{\Diamond b \mid b \in B_0\}$ such that there exist $b_1, b_2 \in C$ with $\Diamond \iota(b) = \Diamond b_1 \vee \Diamond b_2$, where $\Diamond b_1$ and $\Diamond b_2$ are two different atoms of the Boolean algebra $\mathcal{C} = \{\Diamond c \mid c \in C\}$.*

Proof. Since ι is a finite minimal extension of the second kind, we know that the p-morphism which is dual to ι, $f : (X_C, R_C) \twoheadrightarrow (X_{B_0}, R_{B_0})$, is such that $X_C = X_{B_0} \cup T$, where T is a copy of an equivalence class $R_{B_0}[x]$ of X_{B_0} which form another equivalence class of X_C, distinct from the others, $f \restriction_{X_{B_0}} = id_{X_{B_0}}$, and $f \restriction_T : T \longrightarrow R_{B_0}[x]$ is a bijection which sends each element of T into its copy in $R_{B_0}[x]$. We also know that $(B_0, \Diamond) \cong (\mathcal{P}(X_{B_0}), \Diamond)$, and that $(C, \Diamond) \cong (\mathcal{P}(X_C), \Diamond)$. Therefore the atoms of $\mathcal{B} = \{\Diamond b \mid b \in B_0\}$ are dually given by the elements of the type $\Diamond U = R_{B_0}[y]$ for some $y \in X_{B_0}$, where $U \in \mathcal{P}(X_{B_0})$ is such that $U \subseteq R_{B_0}[y]$. This condition is satisfied, in particular, by all the singletons. Hence we can consider the atom $b \in B_0$ which dually corresponds to the singleton $\{x\}$ such that T is a copy of $R_{B_0}[x]$: it holds that $\Diamond \iota(b) = \iota(\Diamond b) \cong f^{-1}(\Diamond\{x\}) = f^{-1}(R_{B_0}[x]) = T \cup R_C[x] = \Diamond\{x'\} \cup \Diamond\{x\} \cong \Diamond b_1 \vee \Diamond b_2$, where $x' \in T$ (so $T = R_C[x']$) and b_1, b_2 are the elements of C dually corresponding to the singletons $\{x'\}$ and $\{x\}$. Since $\diamond b_1 \cong \Diamond\{x'\} = R_C[x'] = T$ and $\diamond b_2 \cong \Diamond\{x\} = R_C[x]$, $\Diamond b_1$ and $\Diamond b_2$ are atoms of the Boolean algebra $\mathcal{C}$.

The idea that we can use in order to prove the uniqueness of the atom $\Diamond b$ is similar to the one we used in the Proposition 3.3.10: if by contradiction there are two different atoms $\Diamond b$ and $\Diamond \bar{b}$ which satisfy the condition given by the statement of the proposition, then ι can't be a minimal extension, being the composition of two minimal extensions of the second kind. This shows the uniqueness of $\Diamond b$.

□

The Propositions 3.3.10 and 3.3.12 allow us to prove the following theorem:

Theorem 3.3.13. *An S5-algebra $(B, \Diamond)$ is existentially closed if and only if, for any finite sub-S5-algebra $(B_0, \Diamond) \overset{\bar{\iota}}{\hookrightarrow} (B, \Diamond)$, the two following conditions hold:*

1. *for every finite minimal extension of the first kind $(B_0, \Diamond) \overset{\iota}{\hookrightarrow} (C, \Diamond)$ there exist $b_1, b_2 \in \boldsymbol{B} \setminus \{0\}$ with $\bar{\iota}(b) = b_1 \vee b_2$, $b_1 \wedge b_2 = 0$ and $\Diamond \bar{\iota}(b) = \Diamond b_1 = \Diamond b_2$, where $b \in B_0$ is the unique atom that satisfies the condition given by the Proposition 3.3.10*

2. *for every finite minimal extension of the second kind $(B_0, \Diamond) \overset{\iota}{\hookrightarrow} (C, \Diamond)$ there exist $b_1, b_2 \in \boldsymbol{B}$ such that $\Diamond b_1 \neq 0 \neq \Diamond b_2$, $\Diamond b_1 \wedge \Diamond b_2 = 0$ and $\Diamond \bar{\iota}(b) = \Diamond b_1 \vee \Diamond b_2$, where*

$\Diamond b$ is the unique atom of $\mathcal{B}$ =$\{\Diamond b \mid b \in B_0\}$ that satisfies the condition given by the Proposition 3.3.12

Proof. ($\Rightarrow$) Given a finite sub-S5-algebra $(B_0, \Diamond) \overset{\bar{\iota}}{\hookrightarrow} (B, \Diamond)$, and given a finite minimal extension $(B_0, \Diamond) \overset{\iota}{\hookrightarrow} (C, \Diamond)$, it can be either of the first kind or of the second kind. In the first case, thanks to the Proposition 3.3.10, there exists a unique atom b of B_0 such that there exist two different atoms $\bar{b}_1, \bar{b}_2 \in C$ with $\iota(b) = \bar{b}_1 \vee \bar{b}_2$ and $\Diamond\iota(b) = \Diamond\bar{b}_1 = \Diamond\bar{b}_2$. Since we are supposing that $(B, \Diamond)$ is existentially closed, thanks to the Corollary 3.3.1.1, we know that there exists an embedding $(C, \Diamond) \overset{\tilde{\iota}}{\hookrightarrow} (B, \Diamond)$ fixing $(B_0, \Diamond)$ pointwise, i. e., such that $\tilde{\iota} \circ \iota = \bar{\iota}$. Therefore we can consider $b_1 := \tilde{\iota}(\bar{b}_1) \in B$ and $b_2 := \tilde{\iota}(\bar{b}_2) \in B$: they are such that $\bar{\iota}(b) = \tilde{\iota}(\iota(b)) = \tilde{\iota}(\bar{b}_1 \vee \bar{b}_2) = \tilde{\iota}(\bar{b}_1) \vee \tilde{\iota}(\bar{b}_2) = b_1 \vee b_2$, $b_1 \wedge b_2 = \tilde{\iota}(\bar{b}_1) \wedge \tilde{\iota}(\bar{b}_2) = \tilde{\iota}(\bar{b}_1 \wedge \bar{b}_2) = \tilde{\iota}(0) = 0$, $\Diamond\bar{\iota}(b) = \bar{\iota}(\Diamond b) = \tilde{\iota}(\iota(\Diamond b)) = \tilde{\iota}(\Diamond\bar{b}_1) = \Diamond\tilde{\iota}(\bar{b}_1) = \Diamond b_1$ and similarly $\Diamond\bar{\iota}(b) = \bar{\iota}(\Diamond b) = \tilde{\iota}(\iota(\Diamond b)) = \tilde{\iota}(\Diamond\bar{b}_2) = \Diamond\tilde{\iota}(\bar{b}_2) = \Diamond b_2$. So, the condition *1.* is satisfied.

In a similar way, we can prove that also the condition *2.* is satisfied: if $(B_0, \Diamond) \overset{\iota}{\hookrightarrow} (C, \Diamond)$ is a finite minimal extension of the second kind, then, thanks to the Proposition 3.3.12, there exists a unique atom $\Diamond b$ of the Boolean algebra $\mathcal{B}$ =$\{\Diamond b \mid b \in B_0\}$ such that there exist $\bar{b}_1, \bar{b}_2 \in C$ with $\Diamond\iota(b) = \Diamond\bar{b}_1 \vee \Diamond\bar{b}_2$, where $\Diamond\bar{b}_1$ and $\Diamond\bar{b}_2$ are two different atoms of the Boolean algebra $\mathcal{C}$ =$\{\Diamond c \mid c \in C\}$. Again, thanks to the Corollary 3.3.1.1, we know that there exists an embedding $(C, \Diamond) \overset{\tilde{\iota}}{\hookrightarrow} (B, \Diamond)$ fixing $(B_0, \Diamond)$ pointwise, i. e., such that $\tilde{\iota} \circ \iota = \bar{\iota}$. Hence we can consider $b_1 := \tilde{\iota}(\bar{b}_1) \in B$ and $b_2 := \tilde{\iota}(\bar{b}_2) \in B$: it holds that $\Diamond b_1 = \Diamond\tilde{\iota}(\bar{b}_1) = \tilde{\iota}(\Diamond\bar{b}_1) \neq 0$ because $\tilde{\iota}$ is injective and $\Diamond\bar{b}_1$ is an atom of $\mathcal{C}$. Similarly we have that $\Diamond b_2 \neq 0$. Moreover, $\Diamond b_1 \wedge \Diamond b_2 = \tilde{\iota}(\Diamond\bar{b}_1 \wedge \Diamond\bar{b}_2) = \tilde{\iota}(0) = 0$ and $\Diamond\bar{\iota}(b) = \Diamond\tilde{\iota}(\iota(b)) = \tilde{\iota}(\Diamond\iota(b)) = \tilde{\iota}(\Diamond\bar{b}_1 \vee \Diamond\bar{b}_2) = \Diamond\tilde{\iota}(\bar{b}_1) \vee \Diamond\tilde{\iota}(\bar{b}_1) = \Diamond b_1 \vee \Diamond b_2$, as required.
($\Leftarrow$) We show that $(B, \Diamond)$ is existentially closed by proving that the condition given by Corollary 3.3.1.1 is satisfied. Let $(B_0, \Diamond) \overset{\iota}{\hookrightarrow} (C, \Diamond)$ be a finite minimal extension of the first kind. We need to provide an embedding $(C, \Diamond) \overset{\tilde{\iota}}{\hookrightarrow} (B, \Diamond)$ such that $\tilde{\iota} \circ \iota = \bar{\iota}$. In order to do that, we use the duality: ι dually corresponds to a continuous p-morphism $X_{B_0} \overset{f}{\twoheadleftarrow} X_C$ such that $X_C = X_{B_0} \cup \{x'\}$, $x' R_C x$ for some $x \in X_C \backslash \{x'\}$, $f(x') = f(x)$ and $f \restriction_{X_{B_0}} = id_{X_{B_0}}$. We also have that the embedding $(B_0, \Diamond) \overset{\bar{\iota}}{\hookrightarrow} (B, \Diamond)$ dually corresponds to a continuous p-morphism $X_{B_0} \overset{\bar{f}}{\twoheadleftarrow} X_B$. So, to provide the required $\tilde{\iota}$ is equivalent to provide a continuous p-morphism $X_B \overset{\tilde{f}}{\twoheadrightarrow} X_C$ such that $f \circ \tilde{f} = \bar{f}$:

$$\begin{tikzcd}
(X_{B_0}, R_{B_0}) & (X_B, R_B) \arrow[l, two heads, "\bar{f}"'] \arrow[dl, dashed, two heads, "\tilde{f}"] \\
(X_C, R_C) \arrow[u, two heads, "f"] &
\end{tikzcd}$$

So we now define such a p-morphism $\tilde{f}$. We know that $(B_0, \Diamond) \cong (\mathcal{P}(X_{B_0}), \Diamond_{R_{B_0}})$, $(C, \Diamond) \cong (\mathcal{P}(X_C), \Diamond_{R_C})$ and $(B, \Diamond) \cong (\mathsf{Clop}(X_B), \Diamond_{R_B})$. Therefore, by hypothesis (condition *1.*), we know that there exist two non-empty clopen subsets $U, V \subseteq X_B$ such that $\bar{f}^{-1}(\{f(x) =$

$f(x')\}) = U \cup V$, $U \cap V = \emptyset$ and $\bar{f}^{-1}(\Diamond\{f(x) = f(x')\}) = \Diamond U = \Diamond V$. Hence we can define $\tilde{f}$ as follows. Consider $y \in X_B$. Then either $\bar{f}(y) = f(x) = f(x')$, or $\bar{f}(y) \in X_{B_0}\backslash\{f(x) = f(x')\}$. In the first case, we know that either $y \in U$ or $y \in V$, being $\bar{f}^{-1}(\{f(x) = f(x')\}) = U \cup V$. So we define $\tilde{f}(y) = x$ if $y \in U$, and $\tilde{f}(y) = x'$ if $y \in V$. This is well defined, being $U \cap V = \emptyset$. In the second case, we define $\tilde{f}(y) = f^{-1}(\bar{f}(y))$: this is well defined because $f \restriction_{X_{B_0}} = id_{X_{B_0}}$, so $f^{-1}(\bar{f}(y))$ is a singleton, being $\bar{f}(y) \in X_{B_0}\backslash\{f(x) = f(x')\}$. By definition of $\tilde{f}$, it is clear that $\tilde{f}$ is surjective and that $f \circ \tilde{f} = \bar{f}$.

Now we prove that $\tilde{f}$ is a continuous p-morphism. Since X_C has the discrete topology, in order to show that $\tilde{f}$ is continuous it is sufficient to show that $\tilde{f}^{-1}(\{z\}) = \tilde{f}^{-1}(z) \subseteq X_B$ is open $\forall z \in X_C$. So suppose that $z \in X_C\backslash\{x, x'\}$. Then $\tilde{f}^{-1}(z) = \{y \in X_B \mid f^{-1}(\bar{f}(y)) = \tilde{f}(y) = z\} = \{y \in X_B \mid \bar{f}(y) = f(z) = z\} = \bar{f}^{-1}(z)$, which is open because X_{B_0} has the discrete topology and $\bar{f}$ is continuous. Now suppose that $z = x \in X_C$. Then $\tilde{f}^{-1}(z) = \tilde{f}^{-1}(x) = U \subseteq X_B$[11], which is open because it is clopen. Similarly, if $z = x' \in X_C$, it holds that $\tilde{f}^{-1}(z) = \tilde{f}^{-1}(x') = V \subseteq X_B$, which is also open. Hence $\tilde{f}$ is continuous.

It remains to show that it is a p-morphism. So suppose that $z, z' \in X_B$ are such that zR_Bz'. Then $\bar{f}(z)R_{B_0}\bar{f}(z')$, because $\bar{f}$ is a p-morphism. By definition of f, which is a continuous p-morphism of the first kind, this implies that $f^{-1}(\bar{f}(z))$ and $f^{-1}(\bar{f}(z'))$ are contained in the same equivalence class of X_C. Since $f \circ \tilde{f} = \bar{f}$, this implies that $\tilde{f}(z)R_C\tilde{f}(z')$, as required. Moreover, suppose that $\tilde{f}(z)R_Cw$ for $z \in X_B$ and $w \in X_C$. We need to show that there exists $\bar{x} \in X_B$ such that $zR_B\bar{x}$ and $\tilde{f}(\bar{x}) = w$. Since $\tilde{f}(z)R_Cw$, it holds that $\bar{f}(z) = f(\tilde{f}(z))R_{B_0}f(w)$, because f is a p-morphism. Since also $\bar{f}$ is a p-morphism, it also holds that $\exists y \in X_B$ s. t. zR_By and $\bar{f}(y) = f(w)$. Since $\bar{f}(y) = f(\tilde{f}(y))$, then, by definition of f, it follows that one of the following situations can occur: $\tilde{f}(y) = w$ (and, in this case, we finish the proof by taking $\bar{x} = y$), or $\tilde{f}(y) = x$ and $w = x'$, or $\tilde{f}(y) = x'$ and $w = x$. If $\tilde{f}(y) = x$, then $y \in U = \tilde{f}^{-1}(x)$. So, by using the condition *1.* of the hypothesis, we obtain that $y \in [y]_{R_B} \subseteq \bigcup_{x \in U}[x]_{R_B} = \Diamond_{R_B}U = \Diamond_{R_B}V = \bigcup_{x \in V}[x]_{R_B}$. Therefore $y \in [l]_{R_B}$ for some $l \in V$. Then we have obtained that $\exists l \in V \subseteq X_B$ such that yR_Bl (and so zR_Bl, being R_B an equivalence relation) and $\tilde{f}(l) = x'$, being $\tilde{f}(V) = x'$. So we can consider $\bar{x} = l$. We can argue in a similar way if $\tilde{f}(y) = x'$ and $w = x$. So $\tilde{f}$ is a p-morphism.

We can argue in a similar way if $(B_0, \Diamond) \overset{\iota}{\hookrightarrow} (C, \Diamond)$ is a finite minimal extension of the second kind, i. e., if f is a continuous p-morphism of the second kind. In fact, we know that in this situation $X_C = X_{B_0} \uplus T$, where T is a copy of an equivalence class $R_{B_0}[x]$ of X_{B_0} which form another equivalence class of X_C, distinct from the others, $f \restriction_{X_{B_0}} = id_{X_{B_0}}$, and $f \restriction_T$: $T \longrightarrow R_{B_0}[x]$ is a bijection which sends each element of T into its copy in $R_{B_0}[x]$. We also know that, by hypothesis (condition *2.*), there exist two clopen subsets $U, V \subseteq X_B$ such that $\Diamond_{R_B}U \neq \emptyset \neq \Diamond_{R_B}V$, $\Diamond_{R_B}U \cap \Diamond_{R_B}V = \emptyset$ and $\bar{f}^{-1}(R_{B_0}[x]) = \Diamond_{R_B}U \cup \Diamond_{R_B}V$. Therefore we

[11] We defined $\tilde{f}$ in such a way that $\tilde{f}(U) = x$, so $U \subseteq \tilde{f}^{-1}(x)$. If $w \in X_B \setminus U$, then either $w \in V$ (and so $\tilde{f}(x) = x'$, according to the definition of $\tilde{f}$), or $w \in X_B \setminus U \cup V$ (and then $\bar{f}(w) \in X_{B_0}\backslash\{f(x) = f(x')\}$, so $\tilde{f}(w) = f^{-1}(\bar{f}(w)) \neq x$). So $\tilde{f}^{-1}(x) \subseteq U$.

can define the continuous p-morphism $X_B \xrightarrow{\tilde{f}} X_C$ such that $f \circ \tilde{f} = \bar{f}$ in the following way: let $y \in X_B$. If $\bar{f}(y) \in X_{B_0} \setminus R_{B_0}[x]$, then we define $\tilde{f}(y) := f^{-1}(\bar{f}(y))$, which is well defined because $f^{-1}(\bar{f}(y))$ is a singleton. If $\bar{f}(y) \in R_{B_0}[x]$, then $y \in \bar{f}^{-1}(R_{B_0}[x]) = \Diamond_{R_B} U \cup \Diamond_{R_B} V$. So we define $\tilde{f}(y) := f^{-1}(\bar{f}(y)) \cap R_C[x]$ if $y \in \Diamond_{R_B} U$, and $\tilde{f}(y) := f^{-1}(\bar{f}(y)) \cap T$ if $y \in \Diamond_{R_B} V$, which is well defined because $\Diamond_{R_B} U \cap \Diamond_{R_B} V = \emptyset$. It is clear that $\tilde{f}$ is surjective and that $f \circ \tilde{f} = \bar{f}$.

Now we prove that $\tilde{f}$ is continuous: suppose first that $z \in X_C \setminus (R_C[x] \cup T)$. Then $\tilde{f}^{-1}(z) = \{y \in X_B \mid f^{-1}(\bar{f}(y)) = \tilde{f}(y) = z\} = \{y \in X_B \mid \bar{f}(y) = f(z) = z\} = \bar{f}^{-1}(z)$, which is open because X_{B_0} has the discrete topology and $\bar{f}$ is continuous. If $z \in R_C[x]$, then $\tilde{f}^{-1}(z) = \{y \in X_B \mid \tilde{f}(y) = z\} = \{y \in X_B \mid f^{-1}(\bar{f}(y)) \cap R_C[x] = \{z\}\} = \{y \in X_B \mid y \in \Diamond_{R_B} U \ \& \ \bar{f}(y) = z \in R_{B_0}[x]\} = \bar{f}^{-1}(z) \cap \Diamond_{R_B} U$ [12], which is open because X_{B_0} has the discrete topology, $\bar{f}$ is continuous and U (and so $\Diamond_{R_B} U$) is clopen. If $z \in T$, then the situation is similar to the case when $z \in R_C[x]$, so $\tilde{f}$ is continuous.

It remains to show that $\tilde{f}$ is a p-morphism. Then consider $z, z' \in X_B$ such that $z R_B z'$. Since $\bar{f}$ is a p-morphism, it follows that $\bar{f}(z) R_{B_0} \bar{f}(z')$. So we consider the two following cases: $\bar{f}(z), \bar{f}(z') \in X_{B_0} \setminus R_{B_0}[x]$ or $\bar{f}(z), \bar{f}(z') \in R_{B_0}[x]$. In the first case, since f is a p-morphism of the second kind, $\tilde{f}(z) = f^{-1}(\bar{f}(z)) R_C f^{-1}(\bar{f}(z')) = \tilde{f}(z')$, according to the definition of $\tilde{f}$, as required. In the second case, since $z R_B z'$ and $\Diamond_{R_B} U \cap \Diamond_{R_B} V = \emptyset$, according to the definition of $\tilde{f}$ we have that either $z, z' \in \Diamond_{R_B} U$ (and so $\tilde{f}(z), \tilde{f}(z') \in R_C[x]$, as required) or $z, z' \in \Diamond_{R_B} V$ (and so $\tilde{f}(z), \tilde{f}(z') \in T$, as required). In any case, we have that $\tilde{f}(z) R_C \tilde{f}(z')$. Finally, suppose that $\tilde{f}(z) R_C w$ for $z \in X_B$ and $w \in X_C$. We need to show that there exists $\bar{x} \in X_B$ such that $z R_B \bar{x}$ and $\tilde{f}(\bar{x}) = w$. Since $\tilde{f}(z) R_C w$ and f is a p-morphism, it holds that $\bar{f}(z) = f(\tilde{f}(z)) R_{B_0} f(w)$. Since $\bar{f}$ is a p-morphism too, it follows that $\exists y \in X_B$ such that $z R_B y$ and $f(\tilde{f}(y)) = \bar{f}(y) = f(w)$. By definition of f, the condition $f(\tilde{f}(y)) = f(w)$ implies that one of the following situations occur: $\tilde{f}(y) = w$ (and so, in this case, it is sufficient to consider $\bar{x} = y$ in order to obtain what we need), or $\tilde{f}(y) \in R_C[x]$ and $w \in T$, or $\tilde{f}(y) \in T$ and $w \in R_C[x]$. However, the last two cases can't occur, because $y R_B z \Rightarrow \tilde{f}(y) R_C \tilde{f}(z) R_C w \Rightarrow \tilde{f}(y) R_C w$, being R_C an equivalence relation. So we can conclude that $\tilde{f}$ is a p-morphism.

□

Now we prove the last useful lemma before the main result:

Lemma 3.3.14. *Let B be an atomless Boolean algebra. Then $\forall b \in B \setminus \{0\}$ $\exists b_1, b_2 \in B$ such that $b_1 \neq b_2$, $b_1 \neq 0 \neq b_2$, $b_1 \neq b \neq b_2$, $b_1 \wedge b_2 = 0$ and $b = b_1 \vee b_2$.*

[12] The equality $\{y \in X_B \mid f^{-1}(\bar{f}(y)) \cap R_C[x] = \{z\}\} = \{y \in X_B \mid y \in \Diamond_{R_B} U \ \& \ \bar{f}(y) = z \in R_{B_0}[x]\}$ holds because $\tilde{f}(y) = z \Rightarrow \bar{f}(y) = f(\tilde{f}(y)) = f(z) = z \in R_{B_0}[x]$ and $\tilde{f}(y) = z \in R_C[x] \Rightarrow y \in \tilde{f}^{-1}(R_C[x]) = \Diamond_{R_B} U$, so $\{y \in X_B \mid f^{-1}(\bar{f}(y)) \cap R_C[x] = \{z\}\} \subseteq \{y \in X_B \mid y \in \Diamond_{R_B} U \ \& \ \bar{f}(y) = z \in R_{B_0}[x]\}$, and $\bar{f}(y) = z \in R_{B_0}[x] \ \& \ y \in \Diamond_{R_B} U \Rightarrow \tilde{f}(y) = f^{-1}(\bar{f}(y)) \cap R_C[x]$, and since $\bar{f}(y) = z$, this implies that $\tilde{f}(y) = z$, so $\{y \in X_B \mid f^{-1}(\bar{f}(y)) \cap R_C[x] = \{z\}\} \supseteq \{y \in X_B \mid y \in \Diamond_{R_B} U \ \& \ \bar{f}(y) = z \in R_{B_0}[x]\}$.

Proof. Since B is atomless, given $b \in B \backslash \{0\}$, there exists $b_1 \in B$ such that $0 < b_1 < b$. By Stone duality, $B \cong \mathsf{Clop}(X_B)$. Therefore b dually corresponds to a clopen $U \neq \emptyset$ of the Stone space X_B dual to B, while b_1 corresponds to a clopen $\emptyset \neq V \subsetneq U$. Then we can consider the clopen $W := U \cap (X_B \setminus V)$: it is such that $W \neq \emptyset$, $U \neq W \neq V$, $V \cap W = \emptyset$ and $U = V \cup W$. So W dually corresponds to $b_2 \in B$ such that $b \neq b_2 \neq 0$, $b_1 \neq b_2$, $b_1 \wedge b_2 = 0$ and $b = b_1 \vee b_2$, as required.

□

Now we can prove the key result, from which we can obtain a finite axiomatization of the model completion of the theory of the S5-algebras:

Theorem 3.3.15. *An S5-algebra $(B, \Diamond)$ is existentially closed if and only if it satisfies the following axioms:*

1. $\forall b \in B \setminus \{0\}\ [(b = \Diamond b) \to \exists b_1, b_2 \in B \backslash \{0\}\ [(b_1, b_2 \leq b),\ (b_1 \wedge b_2 = 0),\ (b_1 \vee b_2 = b),\ (\Diamond b_1 = b = \Diamond b_2)]]$
2. $\forall b \in B \setminus \{0\}\ [(b = \Diamond b) \to [\forall b_1, b_2 \in B \backslash \{0\}\ [(b_1 \wedge b_2 = 0),\ (b_1 \vee b_2 = b),\ (\Diamond b_1 = b = \Diamond b_2)] \to \exists c_1 \neq 0,\ c_2 \neq 0\ (c_1 \wedge c_2 = 0,\ c_1 \vee c_2 = b_1,\ \Diamond c_1 = b = \Diamond c_2)]]$.
3. *$\mathcal{B} = \{\Diamond b \mid b \in B\}$ is an atomless Boolean algebra.*

Proof. ($\Rightarrow$) Suppose that the S5-algebra $(B, \Diamond)$ is existentially closed. We prove that then the axiom *1.* is satisfied. So consider $b \in B \backslash \{0\}$ such that $b = \Diamond b$, and consider the sub-S5-algebra $(B_0, \Diamond) \overset{\bar{\iota}}{\hookrightarrow} (B, \Diamond)$ generated by b. It is finite, because it is finitely generated and the variety of the S5-algebras is locally finite. Moreover, we have that b is an atom of B_0. In fact, $\forall a \in B_0$ there exists a term $t(x)$ in the language of the S5-algebras such that $a = t(b)$. Since $b = \Diamond b$, we can replace every subterm of $t(x)$ of the form $\Diamond z$ with z, where $z = x$ or $z = \neg x$[13]: we obtain a Boolean term $t'(x)$ such that $a = t'(b)$. So a belongs to the Boolean algebra generated by b, which contains only $b, \neg b, 1, 0$. Therefore b is an atom of B_0. Hence we can consider the finite minimal extension of the first kind $(B_0, \Diamond) \overset{\iota}{\hookrightarrow} (C, \Diamond)$ which is dual to the p-morphism $X_{B_0} \overset{f}{\twoheadleftarrow} X_C$, where $X_C = X_{B_0} \cup \{x'\}$, $f(x') = f(x) = x_b \in X_{B_0}$ for some $x \in X_C \backslash \{x'\}$ and x_b is such that $b \cong \{x_b\}$ (according to the fact that $(B_0, \Diamond) \cong (\mathcal{P}(X_{B_0}), \Diamond_R)$). So b is the unique atom of B_0 that satisfies the condition given by the Proposition 3.3.10. Therefore, thanks to the Theorem 3.3.13, there exist $b_1, b_2 \in B \backslash \{0\}$ such that $b = \bar{\iota}(b) = b_1 \vee b_2$, $b_1 \wedge b_2 = 0$ and $b = \Diamond b = \Diamond \bar{\iota}(b) = \Diamond b_1 = \Diamond b_2$. Hence the axiom *1.* is satisfied.
Now we prove that the axiom *2.* is satisfied. So consider $b_1, b_2, b \in B \backslash \{0\}$ such that $b = \Diamond b, b_1 \vee b_2 = b, b_1 \wedge b_2 = 0$ and $\Diamond b_1 = b = \Diamond b_2$. Consider then the sub-S5-algebra

[13] $b = \Diamond b \Rightarrow \neg b = \Diamond \neg b$. In fact, the condition $b = \Diamond b$ means that b dually corresponds to a clopen U_b of X_B which is union of equivalence classes of R_B: this implies that $\neg b$ dually corresponds to $U_{\neg b} = X_B \setminus U_b$, which is still a clopen that is union of equivalence classes.

$(B_0, \Diamond) \overset{\bar{\iota}}{\hookrightarrow} (B, \Diamond)$ generated by b_1 and b_2. It is finite, because it is finitely generated and the variety of the S5-algebras is locally finite. Moreover, b_1 is an atom of B_0. In fact, if $a \in B_0$, there exists a term $t(x_1, x_2)$ in the language of the S5-algebras such that $a = t(b_1, b_2)$. Since $\Diamond b_1 = \Diamond b_2 = \Diamond b = b = b_1 \vee b_2$, we can replace every subterm of $t(x_1, x_2)$ of the form $\Diamond x_1$ or $\Diamond x_2$ with $x_1 \vee x_2$: we obtain a Boolean term $t'(x_1, x_2)$ such that $a = t'(b_1, b_2)$. This means that a belongs to the Boolean algebra generated by b_1, and b_2, and so b_1 is an atom of it[14]. Hence we can consider the finite minimal extension of the first kind $(B_0, \Diamond) \overset{\iota}{\hookrightarrow} (C, \Diamond)$ which is dual to the p-morphism $X_{B_0} \overset{f}{\twoheadleftarrow} X_C$, where $X_C = X_{B_0} \cup \{x'\}$, $f(x') = f(x) = x_{b_1} \in X_{B_0}$ for some $x \in X_C \setminus \{x'\}$ and x_{b_1} is such that $b_1 \cong \{x_{b_1}\}$ (according to the fact that $(B_0, \Diamond) \cong (\mathcal{P}(X_{B_0}), \Diamond_R)$). So b_1 is the unique atom of B_0 that satisfies the condition given by the Proposition 3.3.10. Therefore, thanks to the Theorem 3.3.13, there exist $c_1, c_2 \in B \setminus \{0\}$ such that $b_1 = \bar{\iota}(b_1) = c_1 \vee c_2$, $c_1 \wedge c_2 = 0$ and $b = \Diamond b_1 = \Diamond \bar{\iota}(b_1) = \Diamond c_1 = \Diamond c_2$. Hence the axiom *2.* is satisfied.

Now we prove that the axiom *3.* is satisfied. Let $\Diamond a \in \mathcal{B}$, and consider the sub-S5-algebra $(B_0, \Diamond) \overset{\bar{\iota}}{\hookrightarrow} (B, \Diamond)$ generated by $\Diamond a$: since the variety of S5-algebras is locally finite, $(B_0, \Diamond)$ is finite. Now consider the Boolean algebra $\bar{\mathcal{B}} = \{\Diamond b \mid b \in B_0\}$. If $\Diamond a$ isn't an atom of $\bar{\mathcal{B}}$, then there exists $c \in B_0$ such that $0 < \Diamond c < \Diamond a$. Therefore, since $\bar{\iota}$ is injective, it follows that $\Diamond \bar{\iota}(c)$ is such that $0 < \Diamond \bar{\iota}(c) < \Diamond \bar{\iota}(a) = \Diamond a$, i. e., $\Diamond a$ isn't an atom of $\mathcal{B} = \{\Diamond b \mid b \in B\}$. On the other hand, if $\Diamond a$ is an atom of $\bar{\mathcal{B}}$, then we can consider the finite minimal extension of the second kind which is dual to the p-morphism $X_{B_0} \overset{f}{\twoheadleftarrow} X_C$, where $X_C = X_{B_0} \uplus T$ (T is a copy of the equivalence class $R_{B_0}[x_a]$ such that $R_{B_0}[x_a] \cong \Diamond a$, according to the fact that $(B_0, \Diamond) \cong (\mathcal{P}(X_{B_0}), \Diamond_R)$), $f \restriction_{X_{B_0}} = id_{X_{B_0}}$, and $f \restriction_T : T \longrightarrow R_{B_0}[x_a]$ is a bijection which sends each element of T into its copy in $R_{B_0}[x_a]$. So $\Diamond a$ is the unique atom of $\bar{\mathcal{B}}$ that satisfies the condition given by the Proposition 3.3.12. Hence, thanks to the Theorem 3.3.13, there exist $b_1, b_2 \in B$ such that $\Diamond b_1 \neq 0 \neq \Diamond b_2$, $\Diamond b_1 \wedge \Diamond b_2 = 0$ and $\Diamond \bar{\iota}(a) = \Diamond b_1 \vee \Diamond b_2$. From the fact that $\Diamond b_1 \wedge \Diamond b_2 = 0$, it follows that $\Diamond b_1 \neq \Diamond \bar{\iota}(a) \neq \Diamond b_2$. Hence $\Diamond a$ isn't an atom of $\mathcal{B}$: since $\Diamond a$ is arbitrary, this implies that $\mathcal{B}$ is atomless.

($\Leftarrow$) We show that $(B, \Diamond)$ is existentially closed by proving that the two conditions provided by Theorem 3.3.13 are satisfied. So let $(B_0, \Diamond) \overset{\bar{\iota}}{\hookrightarrow} (B, \Diamond)$ be a finite sub-S5-algebra, and let $(B_0, \Diamond) \overset{\iota}{\hookrightarrow} (C, \Diamond)$ be a finite minimal extension of the first kind, where $\bar{b} \in B_0$ is the unique atom that satisfies the condition given by the Proposition 3.3.10. We need to show that $\exists b_1, b_2 \in B \setminus \{0\}$ with $\bar{\iota}(\bar{b}) = b_1 \vee b_2$, $b_1 \wedge b_2 = 0$ and $\Diamond \bar{\iota}(\bar{b}) = \Diamond b_1 = \Diamond b_2$. In order to do that, we can consider $b := \bar{\iota}(\bar{b})$, and we distinguish two cases:

- $b = \Diamond b$
- $b < \Diamond b$

[14]In fact, we have that $b_1 \wedge b_2 = 0$, $b_1 \vee b_2 = b$, $\neg b_2 = \neg b \vee b_1 = \neg(b_1 \vee b_2) \vee b_1 = (\neg b_1 \wedge \neg b_2) \vee b_1 = (\neg b_1 \vee b_1) \wedge (\neg b_2 \vee b_1) = 1 \wedge (\neg b_2 \vee b_1) = \neg b_2 \vee b_1$, $\neg b_1 \wedge \neg b_2 = \neg(b_1 \vee b_2) = \neg b$, $\neg b_1 \vee \neg b_2 = \neg(b_1 \wedge b_2) = \neg 0 = 1$, $b_2 \vee \neg b_1 = \neg b_1$, $b_1 \wedge \neg b_2 = b_1$, $b_2 \wedge \neg b_1 = b_2$ and so on, so it doesn't exist a way to obtain an element of $B_0 \setminus \{0\}$ which is strictly smaller than b_1.

In the first case, thanks to the axiom *1.*, we have that $\exists b_1, b_2 \in B\backslash\{0\}$ $[(b_1, b_2 \leq b),\ (b_1 \wedge b_2 = 0),\ (b_1 \vee b_2 = b),\ (\Diamond b_1 = b = \Diamond b_2)]$. Hence the condition *1.* of the Theorem 3.3.13 is satisfied. In the second case, we define $d := \neg b \wedge \Diamond b \in B\backslash\{0\}$ and we distinguish two subcases:

- d is such that $\Diamond d = \Diamond b$
- d is such that $\Diamond d < \Diamond b$

In the first case, we consider $\tilde{b} := \Diamond b$, $b_1 := b$ and $b_2 := d$: they are such that $\tilde{b} = \Diamond \tilde{b}$[15], $b_1 \wedge b_2 = b \wedge d = b \wedge (\neg b \wedge \Diamond b) = (b \wedge \neg b) \wedge \Diamond b = 0 \wedge \Diamond b = 0$, $b_1 \vee b_2 = b \vee (\neg b \wedge \Diamond b) = (b \vee \neg b) \wedge (b \vee \Diamond b) = 1 \wedge \Diamond b = \Diamond b = \tilde{b}$, $\Diamond b_1 = \Diamond b = \tilde{b} = \Diamond \tilde{b}$ and $\Diamond b_2 = \Diamond d = \Diamond b = \tilde{b} = \Diamond \tilde{b}$. So, in this case, the condition *1.* of the Theorem 3.3.13 follows from the axiom *2.*: $\exists c_1, c_2$ such that $c_1 \neq 0 \neq c_2$, $c_1 \wedge c_2 = 0$, $c_1 \vee c_2 = b_1 = b$ and $\Diamond c_1 = \Diamond c_2 = \tilde{b} = \Diamond b$, as required.

In the second case, we consider $a := \Diamond b \wedge \neg \Diamond d \in B \setminus \{0\}$. It holds that $a = \Diamond a$. In fact, using the duality (i. e., the fact that $(B, \Diamond) \cong (\mathsf{Clop}(X_B), \Diamond_{R_B})$), we can deduce that, if $b \cong U_b \in \mathsf{Clop}(X_B)$ and $d \cong U_d \in \mathsf{Clop}(X_B)$, then $a = \Diamond b \wedge \neg \Diamond d \cong (\bigcup_{x \in U_b} R_B[x]) \cap (X_B \backslash \bigcup_{y \in U_d} R_B[y]) = (\bigcup_{x \in U_b} R_B[x]) \cap (\bigcup_{z \in X_B \backslash \Diamond U_d} R_B[z])$ which is still union of equivalence classes of R_B. Then can the apply the axiom *1.* to a: we obtain $a_1, a_2 \in B\backslash\{0\}$ such that $a_1 \wedge a_2 = 0$, $a_1 \vee a_2 = a$ and $\Diamond a_1 = a = \Diamond a_2$.

We can also consider $h := \Diamond d$, $h_1 := \neg d \wedge \Diamond d$ and $h_2 := d$. Clearly, $h = \Diamond h$, and $h_2 \neq 0$ being $h_2 = d = \neg b \wedge \Diamond b$ and $b < \Diamond b$. It also holds that $h_1 \neq 0$, because $d < \Diamond d$[16]. Moreover, $h_1 \wedge h_2 = (\neg d \wedge \Diamond d) \wedge d = (\neg d \wedge d) \wedge \Diamond d = 0 \wedge \Diamond d = 0$, and $h_1 \vee h_2 = (\neg d \wedge \Diamond d) \vee d = (\neg d \vee d) \wedge (\Diamond d \vee d) = 1 \wedge \Diamond d = \Diamond d = h$. We also have that $\Diamond h_2 = \Diamond d = h$ and $\Diamond h_1 = h$. In fact, $h_1 = \neg d \wedge \Diamond d \leq \Diamond d \Rightarrow \Diamond h_1 \leq \Diamond\Diamond d = \Diamond d$. Suppose by contradiction that $\Diamond h_1 < \Diamond d$. Then, since $b = a \vee h_1$[17] and since we already observed that $a = \Diamond a$, we have that $\Diamond b = \Diamond(a \vee h_1) = \Diamond a \vee \Diamond h_1 = a \vee \Diamond h_1 = (\Diamond b \wedge \neg \Diamond d) \vee \Diamond h_1 = (\Diamond b \vee \Diamond h_1) \wedge (\neg \Diamond d \vee \Diamond h_1) = \Diamond b \wedge (\neg \Diamond d \vee \Diamond h_1) < \Diamond b$ (being $\Diamond h_1 < \Diamond d < \Diamond b$), but this is a contradiction. Hence $\Diamond h_1 = h$, so we can apply the axiom *2.*: we obtain $\tilde{h}, \bar{h} \in B\backslash\{0\}$ such that $\tilde{h} \wedge \bar{h} = 0$, $\tilde{h} \vee \bar{h} = h_1$ and $\Diamond \tilde{h} = h = \Diamond \bar{h}$.

Now we define $\tilde{c} := a_1 \vee \tilde{h}$ and $\bar{c} := a_2 \vee \bar{h}$: they are such that $\tilde{c} \wedge \bar{c} = (a_1 \vee \tilde{h}) \wedge (a_2 \vee \bar{h}) = [(a_1 \vee \tilde{h}) \wedge a_2] \vee [(a_1 \vee \tilde{h}) \wedge \bar{h}] = [(a_1 \wedge a_2) \vee (\tilde{h} \wedge a_2)] \vee [(a_1 \wedge \bar{h}) \vee (\tilde{h} \wedge \bar{h})] = [0 \vee (\tilde{h} \wedge a_2)] \vee [(a_1 \wedge \bar{h}) \vee 0] = (\tilde{h} \wedge a_2) \vee (a_1 \wedge \bar{h}) \overset{(*)}{=} 0 \vee 0 = 0$ (where the equality denoted with $(*)$ holds because $\tilde{h}, \bar{h} \leq h = \Diamond d$ and $a_1, a_2 \leq a = \Diamond b \wedge \neg \Diamond d \leq \neg \Diamond d$, so $(\tilde{h} \wedge a_2) \leq \Diamond d \wedge \neg \Diamond d = 0$,

[15] $\tilde{b} \leq \Diamond \tilde{b}$ and $\Diamond \tilde{b} = \Diamond\Diamond b \leq \Diamond b = \tilde{b}$, by definition of S5-algebra.

[16] In fact, suppose by contradiction that $d = \Diamond d$. Then $d = \Diamond d \Rightarrow a = \Diamond b \wedge \neg \Diamond d = \Diamond b \wedge \neg d = \Diamond b \wedge \neg(\neg b \wedge \Diamond b) = \Diamond b \wedge (b \vee \neg \Diamond b) = (\Diamond b \wedge b) \vee (\Diamond b \wedge \neg \Diamond b) = b \vee 0 = b$. Since we already observed that $a = \Diamond a$, we have that then $\Diamond b = \Diamond a = a = \Diamond b \wedge \neg \Diamond d$, which is strictly smaller than $\Diamond b$ (otherwise, $\Diamond b \wedge \neg \Diamond d = \Diamond b \Rightarrow \Diamond b \leq \neg \Diamond d$, and since we are in the case that $d < \Diamond b$, we have that $\Diamond d \leq \Diamond\Diamond b = \Diamond b \leq \neg \Diamond d$, so $0 = \Diamond d \wedge \neg \Diamond d = \Diamond d$, and since $d \leq \Diamond d$, we have that $d = 0$, which is a contradiction), but $\Diamond b < \Diamond b$ is a contradiction.

[17] In fact, $a \vee h_1 = (\Diamond b \wedge \neg \Diamond d) \vee (\neg d \wedge \Diamond d) = [(\Diamond b \wedge \neg \Diamond d) \vee \neg d] \wedge [(\Diamond b \wedge \neg \Diamond d) \vee \Diamond d] = [(\Diamond b \vee \neg d) \wedge (\neg \Diamond d \vee \neg d)] \wedge [(\Diamond b \vee \Diamond d) \wedge (\neg \Diamond d \vee \Diamond d)] = [(\Diamond b \vee \neg d) \wedge \neg d] \wedge [\Diamond b \wedge 1] = [(\Diamond b \wedge \neg d) \vee (\neg d \wedge \neg d)] \wedge \Diamond b = [(\Diamond b \wedge \neg d) \vee \neg d] \wedge \Diamond b = [(\Diamond b \wedge \neg(\neg b \wedge \Diamond b)) \vee \neg(\neg b \wedge \Diamond b)] \wedge \Diamond b = [(\Diamond b \wedge (b \vee \neg \Diamond b)) \vee (b \vee \neg \Diamond b)] \wedge \Diamond b = [(\Diamond b \wedge b) \vee (\Diamond b \wedge \neg \Diamond b) \vee (b \vee \neg \Diamond b)] \wedge \Diamond b = [b \vee 0 \vee b \vee \neg \Diamond b] \wedge \Diamond b = (b \vee \neg \Diamond b) \wedge \Diamond b = (b \wedge \Diamond b) \vee (\neg \Diamond b \wedge \Diamond b) = b \vee 0 = b$.

and similarly for $(a_1 \wedge \bar{h})$), $\tilde{c} \vee \bar{c} = (a_1 \vee \tilde{h}) \vee (a_2 \vee \bar{h}) = (a_1 \vee a_2) \vee (\tilde{h} \vee \bar{h}) = a \vee h_1 = b$, $\Diamond\tilde{c} = \Diamond(a_1 \vee \tilde{h}) = \Diamond a_1 \vee \Diamond\tilde{h} = a \vee h = (\Diamond b \wedge \neg\Diamond d) \vee \Diamond d = (\Diamond b \vee \Diamond d) \wedge (\neg\Diamond d \vee \Diamond d) = \Diamond b \wedge 1 = \Diamond b$ and similarly $\Diamond\bar{c} = \Diamond b$. Hence the condition *1.* of the Theorem 3.3.13 is satisfied also in this case.

Now we prove that also the condition *2.* of the Theorem 3.3.13 is satisfied: if we show this, we can conclude that $(B, \Diamond)$ is an existentially closed S5-algebra. So let $(B_0, \Diamond) \overset{\bar{\iota}}{\hookrightarrow} (B, \Diamond)$ be a finite sub-S5-algebra, and let $(B_0, \Diamond) \overset{\iota}{\hookrightarrow} (C, \Diamond)$ be a finite minimal extension of the second kind, where $\Diamond b \in \bar{\mathcal{B}} = \{\Diamond b \mid b \in B_0\}$ is the unique atom that satisfies the condition given by the Proposition 3.3.12. By the hypotheses (axiom *3.*), we know that $\mathcal{B} = \{\Diamond c \mid c \in B\}$ is an atomless Boolean algebra. So, thanks to the Lemma 3.3.14, we know that $\exists \Diamond b_1, \Diamond b_2 \in \mathcal{B}$ such that $\Diamond b_1 \neq \Diamond b_2$, $\Diamond b_1 \neq 0 \neq \Diamond b_2$, $\Diamond b_1 \neq \Diamond\bar{\iota}(b) \neq \Diamond b_2$, $\Diamond b_1 \wedge \Diamond b_2 = 0$ and $\Diamond\bar{\iota}(b) = \Diamond b_1 \vee \Diamond b_2$, as required.

□

Chapter 4

The model completion of the theory of contact algebras

In this chapter, we present the main result of this thesis: we first prove that the model completion of the theory of contact algebras exists, and then we provide its infinite axiomatization.

4.1 Basic definitions and duality

In this section, we introduce contact algebras. We also recall from [6] and [5] a duality between contact algebras and a subcategory of the category **StR**: this will be useful while looking for the model completion of the theory of contact algebras.

With a geometric motivation in mind, the concept of a precontact relation is introduced in [20] (see also [19]): we present it below. However, we prefer to follow [6] and [5] in defining contact algebras by means of subordinations, as specified below, since these are our main references for logics of compact Hausorff spaces. Moreover, in our view the definition of morphism of contact algebras will look more natural from this perspective.

Definition 4.1.1. A *proximity* or a *precontact relation* on a Boolean algebra B is a binary relation δ satisfying:

(P1) $a\delta b \Rightarrow a, b \neq 0$

(P2) $a\delta(b \vee c) \Leftrightarrow a\delta b$ or $a\delta c$

(P3) $(a \vee b)\delta c \Leftrightarrow a\delta c$ or $b\delta c$

Definition 4.1.2. A *subordination* or *strong inclusion* on a Boolean algebra B is a binary relation $\prec$ satisfying:

(S1) $0 \prec 0$ and $1 \prec 1$

(S2) $a \prec b, c$ implies $a \prec b \wedge c$

(S3) $a, b \prec c$ implies $a \vee b \prec c$

(S4) $a \leq b \prec c \leq d$ implies $a \prec d$

Definition 4.1.3. A *subordination algebra* is a pair $(B, \prec)$, where B is a Boolean algebra and $\prec$ is a subordination on B. We denote with **Sub** the category whose objects are the subordination algebras, and whose morphisms are the Boolean homomorphisms h satisfying the condition $[a \prec b \;\Rightarrow\; h(a) \prec h(b)]$.

Remark 4.1.4. Precontact and subordination are dual concepts: if δ is a precontact relation on a Boolean algebra B, then we can define $\prec_\delta$ by $a \prec_\delta b$ if and only if $a \not\delta \neg b$. It is easy to check that $\prec_\delta$ is a subordination on B. Conversely, if $\prec$ is a subordination on B, then we can define define $\delta_\prec$ by $a \delta_\prec b$ if and only if $a \not\prec \neg b$. It is easy to verify that $\delta_\prec$ is a precontact relation on B. Moreover, $a\delta b$ if and only if $a\delta_{\prec_\delta} b$, and $a \prec b$ if and only if $a \prec_{\delta_\prec} b$. Hence there is a bijection between precontact relations and subordinations on B. It is easy to check that **Sub** is isomorphic to the category **PCon** whose objects are precontact algebras and whose morphisms are Boolean homomorphisms h satisfying the condition $[h(a)\delta h(b) \Rightarrow a\delta b]$.

The standard definition of precontact algebras (and contact algebras, that we will define later) uses the relation δ: however, in this thesis we refer to the definition that we presented above, including the notion of subordination $\prec$.

Definition 4.1.5. We denote with **StR** the category whose objects are pairs (X, R), where X is a Stone space and R is a closed[1] relation on X, and whose morphisms are continuous *stable* morphisms, i. e., continuous maps $f\colon (X_1, R_1) \to (X_2, R_2)$ which satisfy the condition $[xR_1y \;\Rightarrow\; f(x)R_2f(y)]$.

The two categories that we have defined above are related in the following way (see [6, Theorem 2.22]):

Theorem 4.1.6. *The categories* **Sub** *and* **StR** *are dually equivalent.*

The duality provided by the previous theorem is given by the two following controvariant functors (see [6, Definition 2.16, Definition 2.19]):

Definition 4.1.7. Define $(-)_* : \mathbf{Sub} \to \mathbf{StR}$ as follows. If $(B, \prec)$ is a subordination algebra, then $(B, \prec)_* := (X, R)$, where X is the Stone space dual to B and $[xRy \Leftrightarrow \uparrow x \subseteq y]$[2] $\forall x, y \in X$. If $h : (B_1, \prec_1) \to (B_2, \prec_2)$ is a morphism in **Sub**, then define $h_* := h^{-1}(-) : (X_2, R_2) \to (X_1, R_1)$ such that $\forall x \in X_2$ $h_*(x) = h^{-1}(x)$.

[1] A binary relation R on a topological space X is said to be *closed* if R is a closed set in the product topology on $X \times X$.

[2] If $(B, \prec)$ is a subordination algebra and $S \subseteq B$, then $\uparrow S$:=$\{b \in B \mid \exists a \in S \text{ with } a \prec b\}$, and $\downarrow S$:=$\{b \in B \mid \exists a \in S \text{ with } b \prec a\}$.

Definition 4.1.8. Define $(-)^* : \mathbf{StR} \to \mathbf{Sub}$ as follows. If (X, R) is an object of $\mathbf{StR}$, then $(X, R)^* := (\mathsf{Clop}(X), \prec)$, where $[U \prec V \Leftrightarrow R[U] \subseteq V]$[3] $\forall U, V \in \mathsf{Clop}(X)$. If $f : (X_1, R_1) \to (X_2, R_2)$ is a morphism in $\mathbf{StR}$, then define $f^* := f^{-1}(-) : (\mathsf{Clop}(X_2), \prec_2) \to (\mathsf{Clop}(X_1), \prec_1)$ such that $\forall U \in \mathsf{Clop}(X_2)$ $f^*(U) = f^{-1}(U)$.

In this section, we are interested in the following algebraic structures:

Definition 4.1.9. We say that a subordination algebra $(B, \prec)$ is *reflexive* if it also satisfies the following axiom:

(S5) $a \prec b$ implies $a \leq b$

We say that a reflexive subordination algebra is a *contact algebra* if it also satisfies the following axiom:

(S6) $a \prec b$ implies $\neg b \prec \neg a$

We say that a contact algebra is a *compingent algebra* if it also satisfies the following axioms:

(S7) $a \prec b$ implies $\exists c \in B$ with $a \prec c \prec b$

(S8) $a \neq 0$ implies $\exists b \neq 0$ with $b \prec a$

We say that a compingent algebra is a *de Vries algebra* if B is a complete Boolean algebra.

We denote with **RSub** the full subcategory of **Sub** having as objects the reflexive subordination algebras, and with **Con** the full subcategory of **RSub** having as objects the contact algebras.

We also denote with **Com** the class of compingent algebras and with **DeV** the class of de Vries algebras.

We have the following characterization [5, Lemma 2.2], which is obtained by means of the duality:

Lemma 4.1.10. *Let B be a Boolean algebra, X the Stone space of B, $\prec$ a subordination on B, and R the corresponding closed relation on X. Then, $(B, \prec)$ satisfies (S5) iff R is reflexive, $(B, \prec)$ satisfies (S6) iff R is symmetric and $(B, \prec)$ satisfies (S7) iff R is transitive.*

It also holds that a contact algebra $(B, \prec)$ satisfies *(S8)* iff R is an irreducible equivalence relation, where R is said to be *irreducible* provided $R[U]$ is a proper subset of X for each proper clopen subset U of X. To characterize dually de Vries algebras, we recall that a

[3] If (X, R) is an object of $\mathbf{StR}$ and $U \subseteq X$, then $R[U] := \{x \in X \mid \exists y \in U \text{ with } yRx\}$.

Boolean algebra B is complete iff its Stone space X is extremally disconnected, i. e., X is such that the closure of each open set is clopen. Therefore compingent algebras dually correspond to pairs (X, R) where X is a Stone space and R is an ireducible equivalence relation, while de Vries algebras dually correspond to pairs (X, R), where X is an extremally disconnected Stone space and R is an ireducible equivalence relation. Such pairs are also called *Gleason spaces*.

Remark 4.1.11. We point out that there is also a duality between the category of de Vries algebras and the category of compact Hausdorff spaces. In fact, given a compact Hausdorff space X, we can consider its Gleason cover, i. e., a pair (Y, π) where Y is an extremally disconnected Stone space and $\pi : Y \to X$ is an irreducible map (i. e., a surjective continuous map such that the π-image of each proper closed subset of Y is a proper subset of X). It can be shown that Gleason covers are unique up to homeomorphism. We can then define R on Y in this way: $xRy \Leftrightarrow \pi(x) = \pi(y)$. Then (Y, R) is a Gleason space. Conversely, if (Y, R) is a Gleason space, then the quotient space $X := Y/R$ is compact Hausdorff. This establishes a one-to-one correspondence between Gleason spaces and compact Hausdorff spaces, which extends to a categorical duality (see [6, Section 6] for details). Since **DeV** dually corresponds to the class of Gleason spaces (see [6, Section 6]), it follows that **DeV** dually corresponds to the class **KHaus** of compact Hausdorff spaces. The correspondence between **DeV** and **KHaus** can also be obtained directly, as was done by de Vries in [17].

We now state the following lemmas, which will be useful in Chapter 5: we only report the proof of the second one, as it will be useful later. The interested reader can find a proof of the third lemma in [5] (Lemma 5.6), together with the first two lemmas that we are going to state.

Lemma 4.1.12. *Let R be a binary relation on a set X. Define $\prec_R$ on $\mathcal{P}(X)$ by $U \prec_R V \Leftrightarrow R[U] \subseteq V$. Then we have that:*

1. *$\prec_R$ is a subordination on $\mathcal{P}(X)$*
2. *R is reflexive iff $(\mathcal{P}(X), \prec_R)$ satisfies $(S5)$*
3. *R is symmetric iff $(\mathcal{P}(X), \prec_R)$ satisfies $(S6)$*
4. *R is transitive iff $(\mathcal{P}(X), \prec_R)$ satisfies $(S7)$*

Lemma 4.1.13. *Every $(B, \prec) \in$***RSub** *can be embedded into $(C, \prec) \in$***RSub** *satisfying $(S7)$. Similarly, every $(B, \prec) \in$***Con** *can be embedded into $(C, \prec) \in$***Con** *satisfying $(S7)$.*

Proof. Suppose first that (X, R) is the dual of $(B, \prec) \in$**RSub**. By Lemma 4.1.10, R is reflexive. Let $Y = \{\{x, y\} \subseteq X \mid xRy\}$ and let $X' = \{(x, \alpha) \in X \times Y \mid x \in \alpha\}$. Define R' on X' by $(x, \alpha)R'(y, \beta) \Leftrightarrow xRy$ and $\alpha = \beta$. We show that R' is reflexive and transitive. That R' is reflexive follows from the reflexivity of R. To see that R' is transitive, let $(x, \alpha)R'(y, \beta)R'(z, \gamma)$. Then $xRyRz$ and $\alpha = \beta = \gamma$. Therefore, either $x = y$, $y = z$, or $z = x$. Since R is reflexive, we see that in each of these cases we have xRz. Thus, $(x, \alpha)R'(z, \gamma)$, and so R' is transitive.

Now define $f : X' \to X$ by $f(x, \alpha) = x$. Clearly f is onto. Therefore, $f^{-1} : \mathsf{Clop}(X) \to \mathcal{P}(X')$ is a Boolean embedding. For $U, V \in \mathsf{Clop}(X)$, we have $U \prec_R V$ iff $f^{-1}(U) \prec_{R'} f^{-1}(V)$. In fact, it follows from the definition of R' that $(x, \alpha)R'(y, \beta)$ implies $f(x, \alpha)Rf(y, \beta)$. So $U \prec_R V$ implies $f^{-1}(U) \prec_{R'} f^{-1}(V)$. For the converse, suppose $U \not\prec_R V$. Then $R[U] \nsubseteq V$. Therefore, there are $x \in U$ and $y \notin V$ such that xRy. Let $\alpha = \{x, y\}$. Then $(x, \alpha)R'(y, \alpha)$, $(x, \alpha) \in f^{-1}(U)$, and $(y, \alpha) \notin f^{-1}(V)$. Thus, $R'[f^{-1}(U)] \nsubseteq f^{-1}(V)$, and hence $f^{-1}(U) \not\prec_{R'} f^{-1}(V)$.

So let $(C, \prec) = (\mathcal{P}(X'), \prec_{R'})$. By Lemma 4.1.12, $(C, \prec)$ satisfies $(S1) - (S5)$ and $(S7)$, and by what we have just shown, f^{-1} is the required embedding of $(B, \prec)$ into $(C, \prec)$.
Now let $(B, \prec) \in$**Con**: then, by Lemma 4.1.10, R is also symmetric. Therefore, so is R', and hence R' is an equivalence relation. Thus, by Lemma 4.1.12, $(C, \prec)$ satisfies $(S1) - (S7)$, concluding the proof.

□

Lemma 4.1.14. *Every $(B, \prec) \in$***RSub** *can be embedded into $(C, \prec) \in$***RSub** *satisfying $(S8)$. In addition, if $(B, \prec)$ satisfies either $(S6)$ or $(S7)$, then so does $(C, \prec)$.*

Moreover, we can observe the following:

Remark 4.1.15. Subordinations on a Boolean algebra B can be described by means of binary operations $\rightsquigarrow: B \times B \to B$ with values in $\{0, 1\}$ satisfying:

(I1) $0 \rightsquigarrow a = a \rightsquigarrow 1 = 1$

(I2) $(a \vee b) \rightsquigarrow c = (a \rightsquigarrow c) \wedge (b \rightsquigarrow c)$

(I3) $a \rightsquigarrow (b \wedge c) = (a \rightsquigarrow b) \wedge (a \rightsquigarrow c)$

In fact, if $\prec$ is a subordination on B, we can define $\rightsquigarrow: B \times B \to B$ by

$$a \rightsquigarrow b = \begin{cases} 1 & \text{if } a \prec b \\ 0 & \text{otherwise} \end{cases}$$

It is easy to see that $\rightsquigarrow$ has values in $\{0, 1\}$ and satisfies (I1)-(I3). Conversely, given $\rightsquigarrow$, we can define $\prec$ by setting $[a \prec b \iff a \rightsquigarrow b = 1]$. It is easy to see that $\prec$ is a subordination on B and that this correspondence is one-to-one.

Moreover, the axioms (S5)-(S8) correspond respectively to the following axioms:

(I4) $a \rightsquigarrow b \leq a \to b$

(I5) $a \rightsquigarrow b = \neg b \rightsquigarrow \neg a$

(I6) $a \rightsquigarrow b = 1$ implies that there is c with $a \rightsquigarrow c = 1$ and $c \rightsquigarrow b = 1$

(I7) $a \neq 0$ implies that there is $b \neq 0$ with $b \rightsquigarrow a = 1$

Definition 4.1.16. We call $(B, \rightsquigarrow)$ a *strict implication algebra* if $(B, \rightsquigarrow)$ belongs to the variety generated by **RSub**. We denote with **SIA** the variety generated by **RSub**[4], i. e., the variety of strict implication algebras.

4.2 Proof of the existence of the model completion

Now we are going to show that the model completion of the theory of contact algebras exists, by proving that the variety of the contact algebras is locally finite and has the amalgamation property, according to the Theorem 2.1.33.

As in the previous chapter, we will use the criterion provided by 3.2.1. Moreover, we have the following result, which can be found in [5, Proposition 3.3]:

Proposition 4.2.1. *The following results hold:*

1. *The variety* **SIA** *is a semisimple variety*[5]*.*
2. *The simple algebras in* **SIA** *are exactly the members of* **RSub**.

Using the binary operation $\rightsquigarrow$ described above and the two previous results, we can prove that also the following important result holds (see [5, Proposition 3.7]):

Theorem 4.2.2. *The variety* **SIA** *is locally finite.*

Proof. Let $(B, \rightsquigarrow) \in$ **RSub** be n-generated, with generators $a_1, ..., a_n \in B$. For each $a \in B$, there is a term $t(x_1, ..., x_n)$ such that $a = t(a_1, ..., a_n)$. Since $(B, \rightsquigarrow) \in$ **RSub**, for each $b, c \in B$, we have $b \rightsquigarrow c \in \{0, 1\}$. Therefore, by replacing each subterm of $t(x_1, ..., x_n)$ of the form $x \rightsquigarrow y$ with either 0 or 1, we obtain a Boolean term $t'(x_1, ..., x_n)$ such that $a = t'(a_1, ..., a_n)$. Thus, B is n-generated as a Boolean algebra, and hence has at most 2^{2^n} elements. Thanks to the Proposition 4.2.1, we know that **RSub** is the class of simple algebras is **SIA**, which is a semisimple variety. Then there is a uniform bound $m(n) = 2^{2^n}$ on the cardinality of all n-generated subdirectly irreducible members of **SIA**. Hence, by Lemma 3.2.1, **SIA** is locally finite.

□

[4]Observe that **RSub** isn't a variety itself, even if the remark above provides an axiomatization by equations. In fact, in this axiomatization there is a condition missing, i. e., the fact that $\rightsquigarrow$ has values in $\{0, 1\}$: this can't be expressed by means of an equation.

[5]An algebra A belonging to a variety is *simple* if $Con(A) = \{\triangle, \nabla\}$, where $Con(A)$ is the congruence lattice of A, $\triangle$ is the diagonal congruence and $\nabla = A \times A$. A variety is *semisimple* if every subdirectly irreducible member of it is simple.

Since **Con** $\subseteq$ **RSub**, it holds that the variety generated by **Con** is contained in **SIA**. We recall that a variety V is *locally finite* if every finitely generated V-algebra is finite. Therefore, the previous theorem implies that the variety generated by **Con** is locally finite too.

Moreover, we can use the duality that we have introduced above in order to prove that the class **Con** has the amalgamation property: given an amalgam $((A, \prec_A), f, (B, \prec_B), g, (C, \prec_C))$, we have to show that there exist a contact algebra $(D, \prec_D)$ and two embeddings $f^{'} : B \to D$, $g^{'} : C \to D$ such that $f^{'} \circ f = g^{'} \circ g$. So we consider the pairs (X_A, R_A), (X_B, R_B) and (X_C, R_C), which dually correspond to the contact algebras $(A, \prec_A)$, $(B, \prec_B)$ and $(C, \prec_C)$ respectively. We also consider the maps $\bar{f} \colon X_B \to X_A$ and $\bar{g} \colon X_C \to X_A$, which are the dually correspondent of $f \colon A \to B$ and $g \colon A \to C$ respectively. Since f and g are injective, $\bar{f}(x) := f^{-1}(x)$ and $\bar{g}(x) := g^{-1}(x)$, we have that $\bar{f}$ and $\bar{g}$ are surjective continuous stable maps. So, we need to define $(X_D, R_D) \in$ **StR** (with R_D reflexive and symmetric) and two surjective continuous stable maps $\bar{f}' : X_D \to X_B$, $\bar{g}' : X_D \to X_C$ such that the morphisms of contact algebras which are dual to $\bar{f}'$ and $\bar{g}'$ are embeddings, and such that $\bar{f} \circ \bar{f}' = \bar{g} \circ \bar{g}'$:

$$\begin{tikzcd} X_A & X_B \arrow[l, two heads, "\bar{f}"'] \\ X_C \arrow[u, two heads, "\bar{g}"] & X_D \arrow[l, two heads, "\bar{g}'"] \arrow[u, two heads, "\bar{f}'"'] \end{tikzcd}$$

Actually, we can observe that it is sufficient to prove that the amalgamation property holds just for the finite contact algebras, i. e., it is sufficient to consider A, B and C finite in the amalgam that we considered above. In fact, we have the following useful result:

Lemma 4.2.3. *Suppose that V is a locally finite variety such that its theory T is written in a language $\mathcal{L}$ that contains a finite number $n \geq 1$ of constants. Suppose that V is such that the finite algebras contained in V have the amalgamation property (i. e., for every amalgam (A, f, B, g, C) with $A, B, C \in V$* finite *algebras and $A \neq \emptyset$, there exist an algebra $D \in V$ and two embeddings $f^{'} : B \to D$, $g^{'} : C \to D$ such that $f^{'} \circ f = g^{'} \circ g$). Then V has the amalgamation property (satisfied by* any *amalgam of algebras and morphisms in V).*

Proof. Let (A, f, B, g, C) be any amalgam (with A, B and C algebras of V of any cardinality). We have to show that there exist an algebra D and two embeddings $f^{'} : B \to D$, $g^{'} : C \to D$ such that $f^{'} \circ f = g^{'} \circ g$.

We recall that in general, given a first-order language $\mathcal{L}$ and two $\mathcal{L}$-structures M and N, it holds that M can be embedded in N if and only if $N \models Diag(M)$, where $Diag(M)$ is the *diagram* of M, i. e., the collection of quantifier-free $\mathcal{L}_M$-sentences true in M.

Therefore, in order to show what we need, it is sufficient to prove that there exists an algebra D in the variety V such that $D \models Diag(B) \cup Diag(C)$ (clearly, since D has to be an algebra of the considered variety V, we should also have that $D \models T$, where T is the theory of the variety V, and hence we require that $D \models T \cup Diag(B) \cup Diag(C)$). Hence what we need to prove is that the theory $T' := T \cup Diag(B) \cup Diag(C)$ has a model D.

By Compactness Theorem, in order to show that such a model D exists, it is sufficient to show that every finite subset of T' has a model. So let $S \subseteq T'$ be a finite subset: then $S = S_0 \cup S_1 \cup S_2$, where $S_0 \subseteq T$, $S_1 \subseteq Diag(B)$ and $S_2 \subseteq Diag(C)$. Consider the constants $\bar{b}_1, ..., \bar{b}_n$ of the language $\mathcal{L}_B$ appearing in S_1, and the constants $\bar{c}_1, ...\bar{c}_m$ of the language $\mathcal{L}_C$ appearing in S_2: since S_1 and S_2 are finite, they both contain a finite number of constants. They correspond to elements $b_1, ..., b_n \in B$ and $c_1, ..., c_m \in C$. So let B_0 be the subalgebra of B generated by $b_1, ..., b_n$, and C_0 the subalgebra of C generated by $c_1, ..., c_m$: they are finite, being V locally finite. Since the language $\mathcal{L}$ of the theory T has at least one constant $\bar{c}$, then we know that there exists a nonempty algebra A_0 of V (it has to contain at least the interpretations of the constants in $\mathcal{L}$) which can be embedded both in B_0 and in C_0. Since the finite subalgebras of V satisfy the amalgamation property by hypothesis, we have that there exists an algebra $D_0 \in V$ such that both B_0 and C_0 can be embedded in D_0. This means that $D_0 \models T \cup Diag(B_0) \cup Diag(C_0)$, as we recalled above. Since we have that $S_0 \subseteq T$, $S_1 \subseteq Diag(B_0)$ and $S_2 \subseteq Diag(C_0)$, it holds that $D_0 \models S$, as required.

□

Hence we can focus just on the amalgams $((A, \prec_A), f, (B, \prec_B), g, (C, \prec_C))$ where A, B and C are **finite** contact algebras. However, at the beginning we can study the problem in general: we will use the finiteness of A, B and C only at the end[6], when we will have to show that the morphisms of contact algebras which are dual to $\bar{f}'$ and $\bar{g}'$ are embeddings.

So now we use the duality as we described above. We know that $X_B \times X_C$ (with the product topology) is a Stone space, being X_B and X_C both Stone spaces. Therefore, we can consider the subspace $X_D :=\{(x, y) \in X_B \times X_C \mid \bar{f}(x) = \bar{g}(y)\} \subseteq X_B \times X_C$. It follows from the two following lemmas[7] that X_D is closed in $X_B \times X_C$, and this is sufficient to deduce that X_D is a Stone space too.

Lemma 4.2.4. *Let (X, τ) be a topological space. (X, τ) is Hausdorff if and only if the diagonal $D := \{(x, x) \mid x \in X\} \subseteq X \times X$ is closed, with respect to the product topology.*

Lemma 4.2.5. *Given three topological spaces X_A, X_B, X_C, where X_A is Hausdorff, and given two continuous maps $\bar{f} \colon X_B \to X_A$ and $\bar{g} \colon X_C \to X_A$, $X_D := \{(x, y) \in X_B \times X_C \mid \bar{f}(x) = \bar{g}(y)\} \subseteq X_B \times X_C$ is closed with respect to the product topology.*

Therefore, since every Stone space is Hausdorff, we have that X_D is a Stone space, being a closed subset of a Stone space.

[6] Observe that a contact algebra $(A, \prec_A)$ is finite if and only if its dual space (X_A, R_A) is finite.

[7] A proof of those two lemmas can be found in the previous section.

We can now define the following relation on $X_D \subseteq X_B \times X_C$, by using the closed reflexive and symmetric relations R_B and R_C:

$$(x, y)R_D(x', y') \text{ iff } (xR_Bx' \ \& \ yR_Cy') \tag{4.1}$$

Lemma 4.2.6. *The relation R_D defined in (4.1) is closed, reflexive and symmetric.*

Proof. Since both R_B and R_C are reflexive, $\forall(x, y) \in X_D \subseteq X_B \times X_C$ we have that xR_Bx & yR_Cy, and then (x, y)R_D(x, y). Hence, R_D is reflexive.

Moreover, $\forall(x, y), (x', y') \in X_D \subseteq X_B \times X_C$, $(x, y)R_D(x', y')$ means that xR_Bx' & yR_Cy'. So, since both R_B and R_C are symmetric, then $x'R_Bx$ & $y'R_Cy$, i. e., $(x', y')R_D(x, y)$. Therefore also R_D is symmetric.

Moreover, because of the definition of R_D, we have that $R_D \subseteq (X_B \times X_C) \times (X_B \times X_C)$ is homeomorphic to $R_B \times R_C \subseteq (X_B \times X_B) \times (X_C \times X_C)$. Then, since both R_B and R_C are closed, we can conclude that also R_D is closed, as required.

□

Therefore we can conclude that the dual of (X_D, R_D) belongs to **Con**, as required. Now we consider the maps $\bar{f}' := \pi_1 \colon X_D \to X_B$ such that $\bar{f}'((x, y)) = \pi_1((x, y)) = x$, and $\bar{g}' := \pi_2 \colon X_D \to X_C$ such that $\bar{g}'((x, y)) = \pi_2((x, y)) = y$.

Lemma 4.2.7. *The projections π_1 and π_2 are continuous stable maps, i. e., they are morphisms of the category **StR**. Moreover, they are surjective.*

Proof. π_1 is continuous. In fact, let $U \subseteq X_B$ be open. Then, $\pi_1^{-1}(U) = \{(x, y) \in X_B \times X_C \mid f(x) = g(y) \text{ and } x \in U\} = X_D \cap (U \times X_C)$. By definition of product topology, $(U \times X_C)$ is open in $X_B \times X_C$, and so, by definition of subspace topology, $\pi_1^{-1}(U)$ is open in X_D. In a similar way, we can prove that also π_2 is continuous.

π_1 is stable. In fact, suppose that $(x, y), (x', y') \in X_D$ are such that $(x, y)R_D(x', y')$. Then, by definition of R_D, we have that $(\pi_1((x, y)) = x)R_B(x' = \pi_1((x', y')))$. Similarly, we can prove that also π_2 is stable.

Moreover, π_1 is surjective. In fact, let $x \in X_B$, and consider $\bar{f}(x) \in X_A$. Since $\bar{g}$ is surjective, there exixts $y \in X_C$ such that $\bar{g}(y) = \bar{f}(x)$. Then, $(x, y) \in X_D$, and $\pi_1((x, y)) = x$, as required. Similarly, we can prove that also π_2 is surjective.

□

Since π_1 and π_2 are surjective, the morphisms of **Sub** which are the dual of those functions are injective (according to the duality recalled above). In order to show that they are also embeddings[8], we observe the following:

[8]Recall: in general, a homomorphism $h : M \to N$ is an embedding if it is injective and $(h(m_1), ..., h(m_n)) \in R^N \Rightarrow (m_1, ..., m_n) \in R^M$ for every n-ary relation symbol of the considered language.

Remark 4.2.8. In order to prove that **Con** has the amalgamation property, we have to guarantee that the injective morphisms of **Sub** which are the dual of π_1 and π_2 are embeddings.

We recall that, in general, an injective morphism of contact algebras $h : (B, \prec_B) \to (B', \prec_{B'})$ is an embedding if it satisfies the condition $[h(a) \prec_{B'} h(b) \;\Rightarrow\; a \prec_B b]$. Therefore, given a continuous stable morphism which is surjective $(X_1, R_1) \xrightarrow{f} (X_2, R_2)$, we have that the injective morphism of contact algebras $f^* = f^{-1}(-) : (\mathsf{Clop}(X_2), \prec_2) \hookrightarrow (\mathsf{Clop}(X_1), \prec_1)$ is an embedding if and only if f satisfies the condition $[\forall U, V \in \mathsf{Clop}(X_2)\ (f^{-1}(U) \prec_1 f^{-1}(V) \;\Rightarrow\; U \prec_2 V)]$, which is equivalent to the condition $[\forall U, V \in \mathsf{Clop}(X_2)\ (R_1[f^{-1}(U)] \subseteq f^{-1}(V) \;\Rightarrow\; R_2[U] \subseteq V)]$, according to the definition of the duality that we provided above.

Now we recall some results that we will use later:

Lemma 4.2.9. *Suppose that X is a Stone space, and $C \subseteq X$, $D \subseteq X$ are closed. If the condition $\forall V \in \mathsf{Clop}(X)\ (C \subseteq V \;\Leftrightarrow\; D \subseteq V)$ holds, then $C = D$.*

Proof. Notice that C closed $\Rightarrow X \setminus C$ open $\Rightarrow X \setminus C = \bigcup_{\tilde{V} \subseteq X \text{ clopen s. t. } \tilde{V} \subseteq X \setminus C} \tilde{V} \Rightarrow C = \bigcap_{\tilde{V} \subseteq X \text{ clopen s. t. } \tilde{V} \subseteq X \setminus C} (X \setminus \tilde{V})$, because every Stone space has a basis of clopens. Then, since ($\tilde{V}$ clopen $\Leftrightarrow X \setminus \tilde{V}$ clopen) and ($\tilde{V} \subseteq X \setminus C \Leftrightarrow X \setminus \tilde{V} \supseteq C$), it holds that $C = \bigcap_{\tilde{V} \subseteq X \text{ clopen s. t. } \tilde{V} \subseteq X \setminus C} (X \setminus \tilde{V}) = \bigcap_{V \subseteq X \text{ clopen s. t. } C \subseteq V} V$, and similarly $D = \bigcap_{U \subseteq X \text{ clopen s. t. } D \subseteq U} U$. Hence by hypothesis $C = \bigcap_{V \subseteq X \text{ clopen s. t. } C \subseteq V} V = \bigcap_{U \subseteq X \text{ clopen s. t. } D \subseteq U} U = D$, as required. $\square$

Lemma 4.2.10 (Closed Map Lemma)**.** *Every continuous function $f : X \to Y$ from a compact space X to a Hausdorff space Y is closed.*[9]

Theorem 4.2.11. *Let (X, τ) be a topological space. Then (X, τ) is compact if and only if, for every collection of closed sets $\mathcal{F}$ from X, we have that, if $\mathcal{F}$ has the finite intersection property*[10]*, then $\bigcap_{F \in \mathcal{F}} F \neq \emptyset$.*[11]

Lemma 4.2.12. *If X is a Stone space, R is a closed binary relation on X and $U \subseteq X$ is closed, then $R[U]$ is closed.*

Proof. It is easy to see that, $\forall U \subseteq X$ closed, it holds that $R[U] = \{x \in X \mid \exists y \in U \text{ s. t. } yRx\} = \pi_2((U \times X) \cap R)$, where $\pi_2 : X \times X \to X$ is a continuous function from a compact space to a Hausdorff space, hence it is a closed map thanks to the the Closed

[9] This is a well-known result of topology: it isn't difficult to provide or to find a proof of it.

[10] Let Y be a set and let $\mathcal{A} = \{A_i\}_{i \in I}$ be a family of subsets of Y. Then $\mathcal{A}$ has the *finite intersection property* if, for any finite $J \subseteq I$, $\bigcap_{j \in J} A_j \neq 0$.

[11] This is a well-known result of topology: it isn't difficult to provide or to find a proof of it.

Map Lemma. Since $U \times X$ is closed being U closed, and $R \subseteq X \times X$ is closed, it holds that $(U \times X) \cap R$ is closed, and so $R[U]$ is closed.

□

We will now prove a version of Esakia's lemma [30, Lemma 3.3.12] for our spaces. Esakia's lemma normally speaks about the inverse of a relation R, but here we need a version which holds for R-images because our relation is symmetric.

Lemma 4.2.13 (Esakia's Lemma)**.** *Let X be a compact Hausdorff space, and R a point-closed*[12] *symmetric binary relation on X. Then for each downward directed*[13] *family $\mathcal{C} = \{C_i\}_{i\in I}$ of nonempty closed subsets of X, $R[\bigcap_{i\in I} C_i] = \bigcap_{i\in I} R[C_i]$.*

Proof. Obviously, $\bigcap_{i\in I} C_i \subseteq C_i \ \forall i \in I$. So $R[\bigcap_{i\in I} C_i] \subseteq R[C_i] \ \forall i \in I$, and thus $R[\bigcap_{i\in I} C_i] \subseteq \bigcap_{i\in I} R[C_i]$.
Now suppose $x \in \bigcap_{i\in I} R[C_i]$. Then $x \in R[C_i]$ for each C_i and, by symmetry, $R[x] \cap C_i$ is nonempty for each $i \in I$. But as C_i-s are donward directed, all the finite intersections $R[x] \cap C_{i_1} \cap ... \cap C_{i_n}$ (with $i_j \in I$ for $j \in \{1, ..., n\}$) are nonempty. By compactness, the infinite intersection (which equals $R[x] \cap \bigcap_{i\in I} C_i$) is nonempty and so, by symmetry, $x \in R[\bigcap_{i\in I} C_i]$.

□

Remark 4.2.14. If X is a Stone space and $x \in X$, then $\{x\} \subseteq X$ is closed, being X Hausdorff. Therefore, as we observed in the proof of the Lemma 4.2.9, $\{x\} = \bigcap_{V\subseteq X \text{ clopen s. t. } x\in V} V$. Moreover, the family $\mathcal{C} := \{V \subseteq X \text{ clopen s. t. } x \in V\}$ is a downward directed family of nonempty closed subsets of X, because $\forall U, V$ clopens in that family, $\{x\} \subseteq U \cap V$. Also, thanks to the Lemma 4.2.12, we have that every closed binary relation $R \subseteq X \times X$ is point-closed. Hence we can apply the Esakia's Lemma in our context, obtaining that $R^{-1}[\{x\}] = R^{-1}[\bigcap_{V\in\mathcal{C}} V] = \bigcap_{V\in\mathcal{C}} R^{-1}[V]$. Since we are considering closed binary relations R on X that are also symmetric, we have that $R[S] = R^{-1}[S] \ \forall S \subseteq X$. Hence we have that $R[\{x\}] = \bigcap_{V\in\mathcal{C}} R[V] \ \forall x \in X$, where $\mathcal{C} := \{V \subseteq X \text{ clopen s. t. } x \in V\}$.

Remark 4.2.15. Let (X_1, R_1) and (X_2, R_2) be two Stone spaces, both equipped with a closed, reflexive and symmetric binary relation. Let $f : X_1 \to X_2$ be a continuous function, and define the binary relation $R_f \subseteq X_2 \times X_2$ in this way: $\forall x, y \in X_2 \ xR_f y \overset{def}{\Longleftrightarrow} y \in f(R_1[f^{-1}(x)])$. Observe that R_f is symmetric. In fact, given $x, y \in X_2$, since R_1 is symmetric we have that $xR_f y \Leftrightarrow y \in f(R_1[f^{-1}(x)]) \Leftrightarrow \exists z, z' \in X_1$ s. t. $f(z) = y, \ f(z') = x \ \& \ z'R_1 z \Leftrightarrow \exists z, z' \in X_1$ s. t. $f(z) = y, \ f(z') = x \ \& \ zR_1 z' \Leftrightarrow x \in f(R_1[f^{-1}(y)]) \Leftrightarrow yR_f x$. Moreover, R_f is point-closed. In fact, given $x \in X_2$, $R_f[x] = \{y \in X_2 \mid xR_f y\} = \{y \in X_2 \mid$

[12]A binary relation R on a topological space X is said to be *point-closed* if $\forall x \in X \ R[x]$ is closed in X.
[13]A subset $\mathcal{C} \subseteq \mathcal{P}(X)$ is called *directed* if, for every $F, G \in \mathcal{C}$, $\exists H \in \mathcal{C}$ s. t. $H \subseteq F \cap G$.

$y \in f(R_1[f^{-1}(x)])\} = f(R_1[f^{-1}(x)])$. Since X_2 is Hausdorff, being a Stone space, we have that $\forall x \in X_2$ $\{x\} \subseteq X_2$ is closed. So, since f is continuous, $f^{-1}(\{x\})$ is closed $\forall x \in X_2$. Then, by the Lemma 4.2.12, $R_1[f^{-1}(x)]$ is closed $\forall x \in X_2$. Therefore, thanks to the Closed Map Lemma, $R_f[x] = f(R_1[f^{-1}(x)])$ is closed $\forall x \in X_2$. Hence, by Esakia's Lemma, as we also observed in the Remark 4.2.14 it holds that $f(R_1[f^{-1}(\{x\})]) = R_f[\{x\}] = \bigcap_{V \in \mathcal{C}} R_f[V] = \bigcap_{V \in \mathcal{C}} f(R_1[f^{-1}(V)])$ $\forall x \in X$, where $\mathcal{C} :=\{V \subseteq X$ clopen s. t. $x \in V\}$.

We can use the previous results in order to prove the following lemma, which will be useful also in the future:

Lemma 4.2.16. *Given a continuous stable morphism $f : (X_1, R_1) \to (X_2, R_2)$, the following conditions are equivalent:*

1. $(R_1[f^{-1}(U)] \subseteq f^{-1}(V) \;\Leftrightarrow\; R_2[U] \subseteq V)$ $\forall U, V \in \mathsf{Clop}(X_2)$
2. $(f(R_1[f^{-1}(U)]) \subseteq V \;\Leftrightarrow\; R_2[U] \subseteq V)$ $\forall U, V \in \mathsf{Clop}(X_2)$
3. $f(R_1[f^{-1}(U)]) = R_2[U]$ $\forall U \in \mathsf{Clop}(X_2)$
4. $f(R_1[f^{-1}(\{x\})]) = R_2[\{x\}]$ $\forall x \in X_2$
5. $\forall x, y \in X_2$ $[xR_2y \;\Leftrightarrow\; \exists y', \tilde{y} \in X_1$ *s. t.* $f(\tilde{y}) = x,\; f(y') = y \;\&\; \tilde{y}R_1y']$

Proof. ***(1. $\Leftrightarrow$ 2.)*** This equivalence follows from the fact that $R_1[f^{-1}(U)] \subseteq f^{-1}(V) \;\Leftrightarrow\; f(R_1[f^{-1}(U)]) \subseteq V$. This last equivalence holds because we have that:
$(\Rightarrow)$ $R_1[f^{-1}(U)] \subseteq f^{-1}(V) \;\Rightarrow\; f(R_1[f^{-1}(U)]) \subseteq f(f^{-1}(V)) \subseteq V$.
$(\Leftarrow)$ $f(R_1[f^{-1}(U)]) \subseteq V \;\Rightarrow\; R_1[f^{-1}(U)] \subseteq f^{-1}(f(R_1[f^{-1}(U)])) \subseteq f^{-1}(V)$.
(2. $\Rightarrow$ 3.) Notice that, being f continuous, U closed $\Rightarrow f^{-1}(U)$ closed. Therefore, thanks to the Lemma 4.2.12, $R_1[f^{-1}(U)]$ is closed. Then, thanks to the Closed Map Lemma, $f(R_1[f^{-1}(U)])$ is closed. So the statement follows immediately if we apply the Lemma 4.2.9.
(3. $\Rightarrow$ 2.) Obvious.
(3. $\Rightarrow$ 4.) By applying (in this order) the Remark 4.2.15, the hypothesis and the Remark 4.2.14, we obtain the following equalities: $f(R_1[f^{-1}(\{x\})]) = \bigcap_{V \in \mathcal{C}} f(R_1[f^{-1}(V)]) = \bigcap_{V \in \mathcal{C}} R_2[U] = R_2[\{x\}]$ $\forall x \in X$, where $\mathcal{C} :=\{V \subseteq X$ clopen s. t. $x \in V\}$.
(4. $\Rightarrow$ 3.) Using the hypothesis, I can deduce that $R_2[U] = \bigcup_{x \in U} R_2[\{x\}] = \bigcup_{x \in U} f(R_1[f^{-1}(\{x\})])$ $= f(\bigcup_{x \in U} R_1[f^{-1}(\{x\})]) \overset{(*)}{=} f(R_1[\bigcup_{x \in U} f^{-1}(\{x\}))] = f(R_1[f^{-1}(U)])$. The equality denoted by $(*)$ holds because $z \in \bigcup_{x \in U} R_1[f^{-1}(\{x\})] \Leftrightarrow \exists x \in U$ s. t. $\exists y \in f^{-1}(\{x\})$ with $yR_1z \Leftrightarrow \exists y \in f^{-1}(U)$ s. t. $yR_1z \Leftrightarrow z \in R_1[f^{-1}(U)]$.
(4. $\Rightarrow$ 5.) Fix $x \in X_2$. Then, given $y \in X_2$, by hypothesis it holds that $xR_2y \Leftrightarrow y \in R_2[\{x\}] = f(R_1[f^{-1}(\{x\})]) \Leftrightarrow \exists y' \in R_1[f^{-1}(\{x\})]$ s. t. $f(y') = y \Leftrightarrow \exists y' \in X_1$ s. t. $(f(y') =$

y & $\exists \tilde{y} \in f^{-1}(\{x\})$ s. t. $\tilde{y} R_1 y') \Leftrightarrow \exists y', \tilde{y} \in X_1$ s. t. $f(y') = y,\ f(\tilde{y}) = x$ & $\tilde{y} R_1 y'$, as required.
(*5.* $\Rightarrow$ *4.*) Fix $x \in X_2$, and let $y \in X_2$. Then, by hypothesis, it holds that $y \in R_2[\{x\}] \Leftrightarrow x R_2 y \Leftrightarrow \exists y', \tilde{y} \in X_1$ s. t. $f(\tilde{y}) = x,\ f(y') = y$ & $\tilde{y} R_1 y' \Leftrightarrow \exists y' \in R_1[f^{-1}(\{x\})]$ s. t. $f(y') = y \Leftrightarrow y \in f(R_1[f^{-1}(\{x\})])$, so $R_2[\{x\}] = f(R_1[f^{-1}(\{x\})])$.

$\square$

Corollary 4.2.16.1. *Given a surjective continuous stable morphism $f : (X_1, R_1) \twoheadrightarrow (X_2, R_2)$, where X_2 has the discrete topology and R_1, R_2 are reflexive and symmetric binary relations, the injective morphism of contact algebras $f^* = f^{-1}(-) : (\mathsf{Clop}(X_2), \prec_2) = (\mathcal{P}(X_2), \prec_2) \hookrightarrow (\mathsf{Clop}(X_1), \prec_1)$ is an embedding if and only if f satisfies the following condition: $\forall x, y \in X_1\ [f(x) R_2 f(y)\ \Rightarrow\ \exists x', y' \in X_1$ s. t. $f(x') = f(x),\ f(y') = f(y)$ and $x' R_1 y']$.*

Proof. ($\Leftarrow$) According to the Remark 4.2.8, we have to prove that $[\forall U, V \in \mathsf{Clop}(X_2)$ $(R_1[f^{-1}(U)] \subseteq f^{-1}(V) \Rightarrow R_2[U] \subseteq V)]$. So suppose that $x \in R_2[U]$. Then $\exists y \in U$ s. t. $y R_2 x$. Since f is surjective, $\exists \tilde{x} \in f^{-1}(x)$ and $\exists \tilde{y} \in f^{-1}(y) \subseteq f^{-1}(U)$. So $f(\tilde{y}) = y R_2 x = f(\tilde{x})$ and then, by hypothesis, $\exists x', y' \in X_1$ s. t. $f(x') = f(\tilde{x}) = x,\ f(y') = f(\tilde{y}) = y$ and $x' R_1 y'$. $f(y') = y \in U \Rightarrow y' \in f^{-1}(U)$. Hence ($x' R_1 y'$ & R_1 symmetric) $\Rightarrow$ $x' \in R_1[f^{-1}(U)] \subseteq f^{-1}(V)$. So $x = f(x') \in V$, i. e., $R_2[U] \subseteq V$.
($\Rightarrow$) Suppose that $f(x) R_2 f(y)$ and let $U = \{f(x)\} \in \mathsf{Clop}(X_2) = \mathcal{P}(X_2)$. Then, thanks to the hypothesis and to the Lemma 4.2.16, $f(R_1[f^{-1}(\{f(x)\})]) = R_2[\{f(x)\}] \ni f(y)$. So $f(y) \in f(R_1[f^{-1}(\{f(x)\})])$, i. e., $\exists y' \in R_1[f^{-1}(\{f(x)\})]$ such that $f(y') = f(y)$. $y' \in R_1[f^{-1}(\{f(x)\})] \Rightarrow \exists x' \in f^{-1}(\{f(x)\})$ s. t. $x' R_1 y'$. $x' \in f^{-1}(\{f(x)\}) \Rightarrow f(x') = f(x)$. So x' and y' satisfy the required condition.

$\square$

So now we recall the fact that (X_A, R_A), (X_B, R_B) and (X_C, R_C) are dual to the finite contact algebras $(A, \prec_A)$, $(B, \prec_B)$ and $(C, \prec_C)$, and so they all have the discrete topology. This allows us to use the Corollary 4.2.16.1, in order to show the following:

Lemma 4.2.17. *The injective morphisms of* **Sub** *which are the dual of π_1 and π_2 are embeddings.*

Proof. Consider $\pi_1 : (X_D, R_D) \twoheadrightarrow (X_B, R_B)$. According to the Corollary 4.2.16.1, it is sufficient to show that, given $(x, y), (\tilde{x}, \tilde{y}) \in X_D$, $[x = \pi_1((x, y)) R_B \pi_1((\tilde{x}, \tilde{y})) = \tilde{x} \Rightarrow \exists (x, y'), (\tilde{x}, \tilde{y}') \in X_D$ s. t. $\pi_1((x, y')) = x = \pi_1((x, y)),\ \pi_1((\tilde{x}, \tilde{y}')) = \tilde{x} = \pi_1((\tilde{x}, \tilde{y}))$ & $(x, y') R_D (\tilde{x}, \tilde{y}')]$. So suppose that $x = \pi_1((x, y)) R_B \pi_1((\tilde{x}, \tilde{y})) = \tilde{x}$. Then, since $\bar{f}$ is stable, we have that $\bar{f}(x) R_A \bar{f}(\tilde{x})$. Hence, since $(x, y), (\tilde{x}, \tilde{y}) \in X_D$, it holds that $\bar{g}(y) = \bar{f}(x) R_A \bar{f}(\tilde{x}) = \bar{g}(\tilde{y})$. Since the dual of $\bar{g}$ is an embedding, again by Corollary 4.2.16.1 we have that $\exists y', \tilde{y}' \in X_C$ such that $\bar{g}(y') = \bar{g}(y)$, $\bar{g}(\tilde{y}') = \bar{g}(\tilde{y})$ and $y' R_C \tilde{y}'$. Therefore we can consider (x, y') and $(\tilde{x}, \tilde{y}')$: since $\bar{g}(y') = \bar{g}(y) = \bar{f}(x)$ and $\bar{g}(\tilde{y}') = \bar{g}(\tilde{y}) = \bar{f}(\tilde{x})$, it holds that $(x, y), (\tilde{x}, \tilde{y}') \in X_D$. Clearly, $\pi_1((x, y')) = x = \pi_1((x, y))$ and $\pi_1((\tilde{x}, \tilde{y}')) = \tilde{x} = \pi_1((\tilde{x}, \tilde{y}))$.

Moreover, $xR_B\tilde{x}$ & $y'R_C\tilde{y}' \;\Rightarrow\; (x,y')R_D(\tilde{x},\tilde{y}')$, as required.

$\square$

Moreover, because on the definition of X_D, we have that $\forall (x,y) \in X_D\ \bar{f}(\bar{f}'((x,y))) = \bar{f}(\pi_1((x,y))) = \bar{f}(x) = \bar{g}(y) = \bar{g}(\pi_2((x,y))) = \bar{g}(\bar{g'}((x,y)))$, i. e., $\bar{f} \circ \bar{f}' = \bar{g} \circ \bar{g'}$. Hence, we can conclude that **Con** has the amalgamation property. Therefore, as a consequence of what we have proved in this section, we can conclude that the theory of **Con** has a model completion.

We conclude this section by observing that the continuous functions which satisfy the condition *5.* of Lemma 4.2.16 i. e., the morphisms of **StR** which are dual to the embeddings on **Con** (cf. Appendix A) coincide with the regular epimorphisms of the category **StR**:

Lemma 4.2.18. *The continuous functions which satisfy the condition 5. of Lemma 4.2.16 coincide with the regular epimorphisms of the category* **StR**.

Proof. Let f be a continuous function that satisfies the condition *5.* of Lemma 4.2.16. We now show that it is the coequalizer of its kernel pair (such a kernel pair exists, because we proved that the pullbacks exist in **StR** while proving that **Con** has the amalgamation property).

$$\begin{tikzcd}
(Z,R_Z) \arrow[r, "g"] \arrow[d, "h"'] & (X,R_X) \arrow[d, "f"] \\
(X,R_X) \arrow[r, "f"'] & (Y,R_Y)
\end{tikzcd}$$

By definition of kernel pair, $f \circ g = f \circ h$, so it remains to prove that the universal property of coequalizer holds: if k is such that $k \circ g = k \circ h$, then there exists a unique φ such that $\varphi \circ f = k$.

$$\begin{tikzcd}
(Z,R_Z) \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & (X,R_X) \arrow[r, "f"] \arrow[dr, "k"'] & (Y,R_Y) \arrow[d, dashed, "\varphi"] \\
 & & (W,R_W)
\end{tikzcd}$$

So let $y \in Y$. Since f is surjective (by the condition *5.* of Lemma 4.2.16, being R_Y reflexive), there exists $x \in X$ such that $f(x) = y$: we define $\varphi(y) := k(x)$. It is clear that $\varphi \circ f = k$. We now prove that φ is stable.

Observe that, since (Z,R_Z) is obtained as a pullback, we have that $Z = \{(x_1,x_2) \in X \times X \mid f(x_1) = f(x_2)\}$, $g = \pi_1$ and $h = \pi_2$, being π_1 and π_2 the projections. Therefore, if $k \circ g = k \circ h$, we have that $f(x_1) = f(x_2) \Rightarrow (x_1,x_2) \in Z \Rightarrow k(x_1) = k(g(x_1,x_2)) = k(h(x_1,x_2)) = k(x_2)$.

So suppose that $y_1, y_2 \in Y$ are such that $y_1 R_Y y_2$. Then, by the condition *5.* of Lemma 4.2.16, there exist $x_1, x_2 \in X$ such that $x_1 R_X x_2$, $f(x_1) = y_1$ and $f(x_2) = y_2$. Hence it follows that $\varphi(y_1) = k(x_1)$ and $\varphi(y_2) = k(x_2)$. Since $x_1 R_X x_2$ and k is stable (being a morphism of **StR**), it holds that $\varphi(y_1) = k(x_1) R_W k(x_2) = \varphi(y_2)$, i. e., φ is stable. It also holds that φ is continuous: in order to prove that, it is sufficient to show that, for every closed set $C \subseteq W$, $\varphi^{-1}(C) \subseteq Y$ is closed. Given $C \subseteq W$ closed, $f^{-1}(\varphi^{-1}(C)) = k^{-1}(C)$ is closed, being k continuous. Therefore $\varphi^{-1}(C) = f(f^{-1}(\varphi^{-1}(C))) = f(k^{-1}(C))$ is closed, because f is closed by the Closed Map Lemma. Hence φ is a morphism of **StR**. The uniqueness of φ follows from the surjectivity of f: if there exists another χ such that $\chi \circ f = k$, then $\varphi \circ f = k = \chi \circ f \Rightarrow \varphi = \chi$. Therefore we can conclude that f is the coequalizer of its kernel pair.

Now we prove that, if q is a regular epimorphism (i. e., a coequalizer) in **StR**, then it satisfies the condition *5.* of Lemma 4.2.16. By duality, this is equivalent to show that, if f is an equalizer in **Con**, then it is an embedding. We first observe that, given two morphisms $g, h : (A, \prec_A) \to (B, \prec_B)$ in **Con**, we can find an equalizer $\iota : (E, \prec_E) \to (A, \prec_A)$ of g and h in this way: $E := \{a \in A \mid g(a) = h(a)\}$ (it is easy to show that E so defined is a Boolean algebra), $\prec_E$ is the restriction of the relation $\prec_A \subseteq A \times A$ to E, and ι is the inclusion. By definition of $\prec_E$, we have that, given $a, b \in E$, $a \prec b \Leftrightarrow \iota(a) \prec \iota(b)$, i. e., ι is an embedding of contact algebras. We now prove that $((E, \prec_E), \iota) = eq(g, h)$.

$$\begin{tikzcd}
(E,\prec_E) \arrow[r, "\iota"] & (A,\prec_A) \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & (B,\prec_B) \\
(C,\prec_C) \arrow[u, dashed, "\chi"] \arrow[ur, "k"'] & &
\end{tikzcd}$$

Suppose that $k : (C, \prec_C) \to (A, \prec_A)$ is such that $g \circ k = h \circ k$. Then, for every $c \in C$, $k(c) \in E$. So $\chi = k$ is the unique morphism of contact algebras such that $\iota \circ \chi = k$. This implies that $((E, \prec_E), \iota) = eq(g, h)$. Therefore, if $((Q, \prec_Q), f) = eq(g, h)$ in **Con**, by the universal property of the equalizer there exists a unique $\varphi : (Q, \prec_Q) \to (E, \prec_E)$ (which is an isomorphism) such that $\iota \circ \varphi = f$. So f is an *injective* morphism of contact algebras, being composition of injective maps. Moreover, f is an embedding. In fact, by the universal property of the equalizer, there exists a unique $\psi : E \to Q$ such that $f \circ \psi = \iota$.

$$\begin{tikzcd}
(Q,\prec_Q) \arrow[r, "f"] & (A,\prec_A) \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & (B,\prec_B) \\
(E,\prec_E) \arrow[u, dashed, "\psi"] \arrow[ur, "\iota"'] & &
\end{tikzcd}$$

So, if we consider $a, b \in Q$ such that $f(a) \prec_A f(b)$, we have that $f(a) \prec_E f(b)$ (since $g \circ f = h \circ f$, we have that $f(a), f(b) \in E$). Since $f \circ \psi = \iota$ and ψ is a morphism in **Con**, it holds that $a = \psi(f(a)) \prec_Q \psi(f(b)) = b$, as required.
So we can conclude that the coequalizers in **StR** are the continuous functions which satisfy the condition *5.* of Lemma 4.2.16.

□

4.3 Axiomatisation of the model completion of the theory of contact algebras

We are now interested in finding a nice axiomatisation of the model completion of the theory of contact algebras. Hence, according to the Proposition 2.1.25, we are interested in studying the existentially closed contact algebras. The following result about existentially closed contact algebras holds (cf. [11, Proposition 2.16]):

Theorem 4.3.1. *Let $(B, \prec)$ be a contact algebra. $(B, \prec)$ is existentially closed iff for any finite subalgebra $(B_0, \prec) \subseteq (B, \prec)$ and for any finite extension $(C, \prec) \supseteq (B_0, \prec)$ there exists an embedding $(C, \prec) \hookrightarrow (B, \prec)$ fixing $(B_0, \prec)$ pointwise.*

$$\begin{tikzcd} (B_0, \prec) \arrow[r, hook] \arrow[d, hook] & (B, \prec) \\ (C, \prec) \arrow[ur, hook] & \end{tikzcd}$$

Proof. ($\Leftarrow$) Let $(D, \prec)$ be an extension of $(B, \prec)$ and $\exists x_1, ..., x_m \phi(x_1, ..., x_m, a_1, ..., a_n)$ an existential $\mathcal{L}_{(\mathcal{B}, \prec)}$-sentence, where $\phi(x_1, ..., x_m, a_1, ..., a_n)$ is quantifier-free and $a_1, ..., a_n \in B$. Suppose that $(D, \prec) \vDash \exists x_1, ..., x_m \phi(x_1, ..., x_m, a_1, ..., a_n)$. Let $d_1, ..., d_m$ be elements of D such that $(D, \prec) \vDash \phi(d_1, ..., d_m, a_1, ..., a_n)$. Consider the subalgebra $(B_0, \prec)$ of $(B, \prec)$ generated by $a_1, ..., a_n$ and the subalgebra $(C, \prec) \subseteq (D, \prec)$ generated by $d_1, ..., d_m, a_1, ..., a_n$. They are both finite because they are finitely generated and the variety generated by **Con** is locally finite. By hypothesis there exists an embedding $(C, \prec) \hookrightarrow (B, \prec)$ fixing $(B_0, \prec)$ pointwise. Let $d'_1, ..., d'_m$ be the images of $d_1, ..., d_m$ by this embedding. Thus $(B, \prec) \vDash \phi(d'_1, ..., d'_m, a_1, ..., a_n)$ because ϕ is quantifier-free.

Therefore $(B, \prec) \vDash \exists x_1, ..., x_m \phi(x_1, ..., x_m, a_1, ..., a_n)$: it follows that $(B, \prec)$ is existentially closed.

($\Rightarrow$) Now suppose that $(B, \prec)$ is existentially closed, and let $(C, \prec) \supseteq (B_0, \prec)$ be a finite extension of a finite subalgebra $(B_0, \prec) \subseteq (B, \prec)$. Since **Con** has the amalgamation property, there exists a contact algebra $(D, \prec)$ amalgamating $(C, \prec)$ and $(B, \prec)$ over $(B_0, \prec)$:

$$\begin{tikzcd} (B_0, \prec) \arrow[r, hook, "f"] \arrow[d, hook, "g"'] & (B, \prec) \arrow[d, hook, "f'"] \\ (C, \prec) \arrow[r, hook, "g'"'] & (D, \prec) \end{tikzcd}$$

Let Σ be the set of quantifier-free $\mathcal{L}_{(C, \prec)}$-sentences of the form $c \star c' = c''$ or $\neg c = c'$ true in $(C, \prec)$, where $c, c', c'' \in C$ and $\star$ is $\wedge$, $\vee$ or $\prec$. Now let $c_1, ..., c_r, a_1, ..., a_n$ be an enumeration of elements in C where the a_i's are elements in B. We obtain the quantifier-free $\mathcal{L}_{(C, \prec)}$-sentence $\sigma(c_1, ..., c_r, a_1, ..., a_n)$ by taking the conjunction of all the sentences in Σ and all the sentences of the form $\neg(c = c')$ for every $c, c' \in C$ such that $c \neq c'$. Clearly,

$\exists x_1, ..., x_r \sigma(x_1, ..., x_r, a_1, ..., a_n)$ is an existential $\mathcal{L}_{(\mathcal{B},\prec)}$-sentence true in $(D, \prec)$. Since $(B, \prec)$ is existentially closed, $(B, \prec) \vDash \exists x_1, ..., x_r \sigma(x_1, ..., x_r, a_1, ..., a_n)$. Let $c'_1, ..., c'_r \in B$ be such that $(B, \prec) \vDash \sigma(c'_1, ..., c'_r, a_1, ..., a_n)$. The map $(C, \prec) \hookrightarrow (B, \prec)$ fixing $(B_0, \prec)$ pointwise and mapping c_i to c'_i is the required embedding, because it is injective and it is a homomorphism (by definition of the sentence σ).

□

As in the previous chapter, since every finite extension of a finite contact algebra $(B_0, \prec)$ is composition of minimal extensions, we are interested in characterizing the minimal extensions of a finite contact algebra $(B_0, \prec)$. In fact, we have the following:

Corollary 4.3.1.1. *Let $(B, \prec)$ be a contact algebra. $(B, \prec)$ is existentially closed iff for any finite subalgebra $(B_0, \prec) \subseteq (B, \prec)$ and for any finite* minimal *extension $(C, \prec) \supseteq (B_0, \prec)$ there exists an embedding $(C, \prec) \hookrightarrow (B, \prec)$ fixing $(B_0, \prec)$ pointwise.*

Proof. ($\Rightarrow$) Direct application of the previous theorem: any finite minimal extension $(C, \prec) \supseteq (B_0, \prec)$ is a finite extension.
($\Leftarrow$) Given a finite extension $(B_0, \prec) \overset{\iota}{\hookrightarrow} (C, \prec)$, there exists a chain of minimal extensions $(B_0, \prec) \overset{\iota_1}{\hookrightarrow} (B_1, \prec) \overset{\iota_2}{\hookrightarrow} ... \overset{\iota_n}{\hookrightarrow} (B_n, \prec) = (C, \prec)$ such that $\iota = \iota_n \circ \iota_{n-1} \circ ... \circ \iota_1$: this chain is finite because ι is a finite extension. By hypothesis, for each of these inclusions there exits an embedding $(B_k, \prec) \overset{g_k}{\hookrightarrow} (C, \prec)$ fixing $(B_{k-1}, \prec)$ pointwise. Hence we can consider $(B_n, \prec) = (C, \prec) \overset{g_n}{\hookrightarrow} (B, \prec)$: this is an embedding which fixes $(B_{n-1}, \prec)$ pointwise. This implies that g_n fixes also $(B_0, \prec)$ pointwise, being $(B_0, \prec) \cong (\iota_{n-1} \circ \iota_{n-2} \circ ... \circ \iota_1)((B_0, \prec)) \subseteq (B_{n-1}, \prec)$. Hence, thanks to the previous theorem, $(B, \prec)$ is existentially closed.

□

Therefore, in order to study the minimal extensions of the finite contact algebras, we use the duality which is recalled above. Let $(B_0, \prec)$ be a finite subalgebra of a contact algebra $(B, \prec)$, let $(B_0, \prec) \hookrightarrow (C, \prec)$ be a finite minimal extension and let (X_{B_0}, R_{B_0}) be the dual of $(B_0, \prec)$. As we observed in the previous section, since B_0 is finite, X_{B_0} has the discrete topology. Hence the product topology on $X_{B_0} \times X_{B_0}$ coincides with the discrete topology, and then the condition of R_{B_0} being a closed relation is trivially satisfied. So we can regard (X_{B_0}, R_{B_0}) just as a set equipped with a reflexive and symmetric relation, and the same holds for (X_C, R_C). Clearly, since both X_C and X_{B_0} have the discrete topology, every function $f : X_C \to X_{B_0}$ is trivially continuous.

Now, if we dualize the diagram that we find in the statement of Theorem 4.3.1, we obtain the following diagram:

$$\begin{tikzcd}
(X_{B_0}, R_{B_0}) & (X_B, R_B) \arrow[l, two heads] \arrow[dl, two heads] \\
(X_C, R_C) \arrow[u, two heads] &
\end{tikzcd}$$

Hence, according to the duality that we explained at the beginning of this section and to Corollary 4.3.1.1, we are interested in studying the stable morphisms which are dual to the minimal extensions of finite contact algebras. So we define:

Definition 4.3.2. A stable morphism $f : (X_C, R_C) \twoheadrightarrow (X_{B_0}, R_{B_0})$ between finite discrete Stone spaces equipped with a reflexive and symmetric relation is *minimal* if $X_C = X_{B_0} \uplus \{x'\}$, $f \restriction_{X_{B_0}} = id_{X_{B_0}}$, $f(x') = x$ for some $x \in X_{B_0}$, and $R_C[y]$ $\forall y \in X_C \setminus \{x, x'\}$, $R_C[x]$, $R_C[x']$ are such that the following conditions hold: $f(R_C[x]) \cup f(R_C[x']) = f(R_C[x] \cup R_C[x']) = f(R_C[\{x, x'\}]) = R_{B_0}[x]$, $x \in R_C[x]$, $x' \in R_C[x']$ (so that R_C is reflexive), $\forall y \in X_C \setminus \{x, x'\}$ $R_{B_0}[y] \subseteq R_C[y] \subseteq R_{B_0}[y] \cup \{x, x'\}$ so that $(y \in R_C[x'] \Leftrightarrow x' \in R_C[y])$ and similarly $(y \in R_C[x] \Leftrightarrow x \in R_C[y])$ (so that R_C is symmetric)[14].

Lemma 4.3.3. *The morphism between finite contact algebras $(B_0, \prec) \hookrightarrow (C, \prec)$, which is dual to a minimal stable morphism, is a minimal extension.*

Proof. The morphism which is dual to a minimal stable morphism f is given by $\iota = f^* = f^{-1}(-) : (B_0, \prec) \cong (\mathcal{P}(X_{B_0}), \prec_{R_{B_0}}) \hookrightarrow (\mathcal{P}(X_C), \prec_{R_C}) \cong (C, \prec)$. This is a minimal extension. In order to show this, we first prove that ι is an embedding. It is clearly an injective homomorphism of contact algebras (being dual to a surjective stable morphism). In order to show that it is an embedding, according to the Lemma 4.2.16 and to the Remark 4.2.8, we have to prove that $\forall y, z \in X_{B_0}$ $(yR_{B_0}z \Leftrightarrow \exists y', z' \in X_C$ s. t. $f(y') = y$, $f(z') = z$ & $y'R_Cz')$. The implication $(\Leftarrow)$ follows from the fact that f is stable. So we are going to show that also the implication $(\Rightarrow)$ holds: let $y, z \in X_{B_0}$ be such that $yR_{B_0}z$. Suppose first that $y, z \in X_{B_0} \setminus \{x\}$. Then, since $f \restriction_{X_{B_0}} = id_{X_{B_0}}$ and $R_C[w] \supseteq R_{B_0}[w]$ $\forall w \in X_C \setminus \{x, x'\}$, it holds that yR_Cz (so the required condition holds by considering $y' = y$ and $z' = z$). Now suppose that $y = x \in X_{B_0}$, and $z \in X_{B_0} \setminus \{x\}$. Then $yR_{B_0}z \Rightarrow z \in R_{B_0}[y] = R_{B_0}[x] = f(R_C[x]) \cup f(R_C[x'])$, so either $z \in f(R_C[x])$ or $z \in f(R_C[x'])$. In both cases, the required condition is satisfied (by considering $y' = y = x$, $z' = z$ in the first case, and $y' = x'$, $z' = z$ in the second case). Being R_{B_0} and R_C symmetric, the case $z = x \in X_{B_0}$, and $y \in X_{B_0} \setminus \{x\}$ is similar to the previous one. Finally, suppose that $y = x = z$. Then the required condition is satisfied by $y' = x = z'$, being $f \restriction_{X_{B_0}} = id_{B_0}$ and $x \in R_C[x]$.

Moreover, $\iota = f^*$ is minimal. In fact, suppose that there is an intermediate extension $(B_0, \prec) \xhookrightarrow{\iota_1} (A, \prec) \xhookrightarrow{\iota_2} (C, \prec)$. Dually, we get two stable morphisms $(X_{B_0}, R_{B_0}) \xleftarrow{f_1} (X_A, R_A) \xleftarrow{f_2} (X_C, R_C)$. Since f_1 and f_2 are both surjective, we have that $|X_{B_0}| \leq$

[14] It is easy to check that the defined f is stable, so it really is a morphism of the category **StR**. Moreover, it is obviously surjective.

$|X_A| \leq |X_C| = (|X_{B_0}| + 1)$, so either $|X_A| = |X_{B_0}|$ or $|X_A| = |X_C| = (|X_{B_0}| + 1)$.
In the first case, f_1 is a surjective function between two finite non-empty sets having the same cardinality, and so it is bijective. Since ι_1 is an embedding, we know that f_1 satisfies the condition provided by the Corollary 4.2.16.1: $\forall z, y \in X_A\ [f_1(z) R_{B_0} f_1(y) \Rightarrow \exists z', y' \in X_A$ s. t. $f_1(z') = f_1(z),\ f_1(y') = f_1(y)$ and $z' R_A y']$. But since f_1 is bijective, I can rewrite this condition in this way: $\forall z, y \in X_A\ [f_1(z) R_{B_0} f_1(y) \Rightarrow zR_A y]$. Hence, since f_1 is also stable, we have that $\forall z, y \in X_A\ [f_1(z) R_{B_0} f_1(y) \Leftrightarrow zR_A y]$. So f_1 is a stable isomorphism, i. e., $(X_A, R_A) \overset{f_1}{\cong} (X_{B_0}, R_{B_0})$.

In the second case (i. e., if $|X_A| = |X_C| = (|X_{B_0}| + 1)$), we can reason as in the previous case replacing f_1 with f_2: we deduce that f_2 is a stable isomorphism (i. e., $(X_C, R_C) \overset{f_2}{\cong} (X_A, R_A)$) and then f is minimal also in this case.

□

Remark 4.3.4. It is easy to see that there can't exist other kinds of minimal finite extensions. In fact, suppose that $B_0 \subseteq C$ is a proper finite extension of B_0, and let $X_C \overset{f}{\twoheadrightarrow} X_{B_0}$ be the dual continuous stable morphism. Since f is surjective, it holds that $|X_{B_0}| \leq |X_C|$.

Suppose first that $|X_{B_0}| = |X_C|$. Then f is a surjective function between two finite non-empty sets having the same cardinality, and so it is bijective. Since f is the dual of an embedding, as we also observed in the proof of the Lemma 4.3.3, thanks to the Corollary 4.2.16.1 it holds that $\forall z, y \in X_C\ [f(z) R_{B_0} f(y) \Leftrightarrow zR_C y]$. So f is a stable isomorphism, i. e., $(X_C, R_C) \overset{f}{\cong} (X_{B_0}, R_{B_0})$, and then $(B_0, \prec_{B_0}) \cong (C, \prec_C)$, i. e., $B_0 \subseteq C$ isn't a proper extension.

Suppose now that $|X_{B_0}| < |X_C|$, and consider $x' \in X_C$ such that $f(x') = f(x)$ for some $x \in X_C \backslash \{x'\}$. Then $f = f_1 \circ f_2$, where $X_C \overset{f_2}{\twoheadrightarrow} X_A$ (with eventually $X_A = X_{B_0}$) is a minimal stable morphism such that $X_C = X_A \uplus \{x'\}$, $f_2 \restriction_{X_A} = id_{X_A}$, $f_2(x') = x \in X_A$, and $R_C[y]\ \forall y \in X_C \setminus \{x, x'\}$, $R_C[x]$, $R_C[x']$ are such that the following conditions hold: $f_2(R_C[x]) \cup f_2(R_C[x']) = R_A[x]$, $x \in R_C[x]$, $x' \in R_C[x']$ (being R_C reflexive), $\forall y \in X_C \setminus \{x, x'\}\ R_A[y] \subseteq R_C[y] \subseteq R_A[y] \cup \{x, x'\}$ so that $(y \in R_C[x'] \Leftrightarrow x' \in R_C[y])$ and similarly $(y \in R_C[x] \Leftrightarrow x \in R_C[y])$ (being R_C symmetric). Hence f is the composition of f_2 (which is a minimal stable morphism as we defined above) and another stable morphism f_1.

Now we prove the following useful result about the minimal extensions that we have just considered:

Proposition 4.3.5. *Let $(B_0, \prec)$ be a finite contact algebra, and let $(B_0, \prec_{B_0}) \overset{\iota}{\hookrightarrow} (C, \prec_C)$ be a finite minimal extension. Then there exists a unique atom $b \in B_0$ with the property that there exist two atoms $c_1, c_2 \in C$ such that $c_1 \vee c_2 = \iota(b)$ and $c_1 \wedge c_2 = 0$.*

Proof. Since ι is a finite minimal extension, we know that the stable morphism $f : (X_C, R_C) \twoheadrightarrow (X_{B_0}, R_{B_0})$ which is dual to ι is minimal, according to the Definition 4.3.2. Therefore we can consider the atom $b \cong \{x\} \in \mathcal{P}(X_{B_0}) \cong B_0$: it is such that $\iota(b) \cong f^{-1}(\{x\}) = \{x, x'\}$. So, if we consider the two atoms $c_1 \cong \{x\} \in \mathcal{P}(X_C) \cong C$ and $c_2 \cong \{x'\} \in \mathcal{P}(X_C) \cong C$, they are such that $c_1 \vee c_2 \cong \{x\} \cup \{x'\} = \{x, x'\} \cong \iota(b)$ and $c_1 \wedge c_2 \cong \{x\} \cap \{x'\} = \emptyset \cong 0$.

Now we prove that the atom $b \in B_0$ that satisfies the statement is unique. Suppose by contradiction that there exists another atom $\tilde{b} \in B_0$ satisfying it. Then, $\tilde{b} \cong \{z\} \in \mathcal{P}(X_{B_0}) \cong B_0$ for a certain $z \in X_{B_0} \setminus \{x\}$. But then $\iota(\tilde{b}) \cong f^{-1}(\{z\}) = \{z\} \in \mathcal{P}(X_C)$, being $f \restriction_{X_{B_0}} = id_{X_{B_0}}$. So we have obtained that $\iota(\tilde{b}) \in C$ is an atom, and then there can't be two different atoms $c_1, c_2 \in C$ such that $c_1 \vee c_2 = \iota(\tilde{b})$.

$\square$

The following remark gives the motivation for the definition of signature that we are going to provide later:

Remark 4.3.6. Suppose that (X_{B_0}, R_{B_0}) is such that X_{B_0} is a finite discrete Stone space, and R_{B_0} is a symmetric and reflexive binary relation on X_{B_0}. Consider a triple (x, C_1, C_2), where $x \in X_{B_0}$ and $C_1, C_2 \subseteq X_{B_0}$ are such that $C_1 \cup C_2 \cup \{x\} = R_{B_0}[x]$ and $x \notin C_1 \cup C_2$. Then we can recover two minimal stable morphisms $f_1 : (X^1_C, R^1_C) \twoheadrightarrow (X_{B_0}, R_{B_0})$ and $f_2 : (X^2_C, R^2_C) \twoheadrightarrow (X_{B_0}, R_{B_0})$ by defining:

- $R^1_C[x] = C_1 \cup \{x\}$, $R^1_C[x'] = C_2 \cup \{x'\}$ (so $x \cancel{R^1_C} x'$)
- $R^2_C[x] = C_1 \cup \{x, x'\}$, $R^2_C[x'] = C_2 \cup \{x, x'\}$ (so $x R^2_C x'$)
- for $i \in \{1, 2\}$, $X^i_C := X_{B_0} \uplus \{x'\}$, $f_i \restriction_{X_{B_0}} := id_{X_{B_0}}$, $f_i(x') := x$ and $\forall y \in X^i_C \setminus \{x, x'\}$ $R_{B_0}[y] \subseteq R^i_C[y] \subseteq R_{B_0}[y] \cup \{x, x'\}$ in such a way that $(y \in R^i_C[x'] \Leftrightarrow x' \in R^i_C[y])$ and $(y \in R^i_C[x] \Leftrightarrow x \in R^i_C[y])$

It is easy to see that both f_1 and f_2 are minimal stable morphisms, because the following conditions hold for $i \in \{1, 2\}$: $f_i(R_C[x]) \cup f_i(R_C[x']) = R_{B_0}[x]$, $x \in R^i_C[x]$, $x' \in R^i_C[x']$. Moreover, it is clear that $(X^1_C, R^1_C) \ncong (X^2_C, R^2_C)$, being $x \cancel{R^1_C} x'$ and $x R^2_C x'$, while the objects of **StR** that we obtain by switching C_1 and C_2 in the previous definition are isomorphic to the ones defined above (for instance, $\tilde{f} : (X^1_C, R^1_C) \to (\tilde{X}^1_C, \tilde{R}^1_C)$ such that $\tilde{f} \restriction_{X^1_C \setminus \{x, x'\}} = id_{X^1_C \setminus \{x, x'\}}$, $\tilde{f}(x) = x'$ and $\tilde{f}(x') = x$ is an isomorphism).

Therefore we can associate the tuple $(x, C_1, C_2, 0)$ to f_1, and $(x, C_1, C_2, 1)$ to f_2 (so that 0 stands for $x \cancel{R^1_C} x'$, while 1 stands for $x R^2_C x'$).

Also, given a minimal stable morphism $f : (X_C, R_C) \twoheadrightarrow (X_{B_0}, R_{B_0})$, we can get a tuple $(x, C_1, C_2, 0)$ or $(x, C_1, C_2, 1)$, where $x \in X_{B_0}$, $C_1, C_2 \subseteq X_{B_0}$ are such that $x \notin C_1 \cup C_2$ and $C_1 \cup C_2 \cup \{x\} = R_{B_0}[x]$, in the following way: $x := f(x')$ (where x' is such that $X_C := X_{B_0} \uplus \{x'\}$), $C_1 := f(R_C[x]) \setminus \{x\}$, $C_2 := f(R_C[x']) \setminus \{x\}$, and we add 0 if $x \cancel{R_C} x'$, while we add 1 if $x R_C x'$.

What we have observed in the Remark 4.3.6 suggests that we can distinguish two kinds of minimal extensions, in the following way:

Definition 4.3.7. A minimal stable morphism $f : (X_C, R_C) \to (X_{B_0}, R_{B_0})$ is *of the first kind* if xR_Cx', where $x, x' \in X_C$ are such that $f(x) = f(x')$.

A minimal stable morphism $f : (X_C, R_C) \to (X_{B_0}, R_{B_0})$ is *of the second kind* if $x\not{R_C}x'$, where $x, x' \in X_C$ are such that $f(x) = f(x')$.

Definition 4.3.8. A minimal extension is *of the first kind* if it is dual to a minimal stable morphism of the first kind.

A minimal extension is *of the second kind* if it is dual to a minimal stable morphism of the second kind.

We can distinguish the two kinds of minimal extensions by means of the following proposition:

Proposition 4.3.9. *Let $(B_0, \prec)$ be a finite contact algebra, and let $(B_0, \prec_{B_0}) \overset{\iota}{\hookrightarrow} (C, \prec_C)$ be a finite minimal extension. Then there exist $\tilde{c_1}, \tilde{c_2} \in B_0$ such that the following condition holds:*

- *if ι is a minimal extension of the first kind, then, for $i \in \{1, 2\}$, $\tilde{c_i} \wedge b = 0$, $c_i \prec_C \iota(\tilde{c_i} \vee b)$ and, $\forall \bar{c} \in C$ such that $\bar{c} < \iota(\tilde{c_i} \vee b)$, $c_i \not\prec_C \bar{c}$, where $b, c_1, c_2 \in C$ are as in the statement of the Proposition 4.3.5*

- *if ι is a minimal extension of the second kind, then, for $i \in \{1, 2\}$, $\tilde{c_i} \wedge b = 0$, $c_i \prec_C \iota(\tilde{c_i}) \vee c_i$ and, $\forall \bar{c} \in C$ such that $\bar{c} < \iota(\tilde{c_i}) \vee c_i$, $c_i \not\prec_C \bar{c}$, where $b, c_1, c_2 \in C$ are as in the statement of the Proposition 4.3.5*

Proof. ι is a minimal extension, so it is dual to a minimal stable morphism $f : (X_C, R_C) \twoheadrightarrow (X_{B_0}, R_{B_0})$ which satisfies the conditions given by the Definition 4.3.2. Hence we can consider $\tilde{c_1} \cong f(R_C[x]) \setminus \{x\} \in \mathcal{P}(X_C) \cong C$ and $\tilde{c_2} \cong f(R_C[x']) \setminus \{x\} \in \mathcal{P}(X_C) \cong C$: they are such that $\tilde{c_1} \wedge b \cong (f(R_C[x]) \setminus \{x\}) \cap \{x\} = \emptyset \cong 0$, $\tilde{c_2} \wedge b \cong (f(R_C[x']) \setminus \{x\}) \cap \{x\} = \emptyset \cong 0$, $\iota(\tilde{c_1}) \cong f^{-1}(f(R_C[x]) \setminus \{x\}) = R_C[x] \setminus \{x, x'\}$ and $\iota(\tilde{c_2}) \cong f^{-1}(f(R_C[x']) \setminus \{x\}) = R_C[x'] \setminus \{x, x'\}$.

Therefore, if ι is a minimal extension of the first kind (i. e., dually xR_Cx'), then $c_1 \cong \{x\} \prec_C R_C[x] = f^{-1}(f(R_C[x])) = f^{-1}(f((R_C[x] \setminus \{x\}) \cup \{x\})) \cong \iota(\tilde{c_1} \vee b)$, and similarly $c_2 \cong \{x'\} \prec_C R_C[x'] = f^{-1}(f(R_C[x'])) = f^{-1}(f((R_C[x'] \setminus \{x\}) \cup \{x\})) \cong \iota(\tilde{c_2} \vee b)$. Suppose now that $\bar{c} \in C$ is such that $\bar{c} < \iota(\tilde{c_1} \vee b)$. This dually means that $U_{\bar{c}} \subsetneq f^{-1}(f((R_C[x] \setminus \{x\}) \cup \{x\})) = R_C[x]$. Hence $R_C[x] \not\subseteq U_{\bar{c}}$, i. e., $c_1 \not\prec_C \bar{c}$. Similarly, if $\bar{c} \in C$ is such that $\bar{c} < \iota(\tilde{c_2} \vee b)$, we get that $c_2 \not\prec_C \bar{c}$.

Suppose now that ι is a minimal extension of the second kind (i. e., dually $x\not{R_C}x'$). Then $c_1 \cong \{x\} \prec_C R_C[x] = (R_C[x] \setminus \{x, x'\}) \cup \{x\} \cong \iota(\tilde{c_1}) \vee c_1$, and similarly $c_2 \cong \{x'\} \prec_C R_C[x'] = (R_C[x'] \setminus \{x, x'\}) \cup \{x'\} \cong \iota(\tilde{c_2}) \vee c_2$. Suppose now that $\bar{c} \in C$ is such that

$\bar{c} < \iota(\tilde{c_1}) \vee c_1$. This dually means that $U_{\bar{c}} \subsetneq (R_C[x] \setminus \{x, x'\}) \cup \{x\} = R_C[x]$. So $R_C[x] \not\subseteq U_{\bar{c}}$, i. e., $c_1 \not\prec_C \bar{c}$. Similarly, if $\bar{c} \in C$ is such that $\bar{c} < \iota(\tilde{c_2}) \vee c_2$, we get that $c_2 \not\prec_C \bar{c}$.

□

The Remark 4.3.6 suggests to define the following:

Definition 4.3.10. Let $(B_0, \prec_{B_0})$ be a finite contact algebra. We call a *signature of the first kind* in $(B_0, \prec_{B_0})$ a tuple $(b, \tilde{c_1}, \tilde{c_2}, 1)$, where $b \in B_0$ is an atom, and $\tilde{c_1}, \tilde{c_2} \in B_0$ are such that, for $i \in \{1, 2\}$, $\tilde{c_i} \wedge b = 0$, $b \prec_{B_0} \tilde{c_1} \vee \tilde{c_2} \vee b$ and, $\forall \bar{b} \in B_0$ such that $\bar{b} < \tilde{c_1} \vee \tilde{c_2} \vee b$, $b \not\prec_{B_0} \bar{b}$. We call a *signature of the second kind* in $(B_0, \prec_{B_0})$ a tuple $(b, \tilde{c_1}, \tilde{c_2}, 0)$, where $b \in B_0$ is an atom, and $\tilde{c_1}, \tilde{c_2} \in B_0$ are such that, for $i \in \{1, 2\}$, $\tilde{c_i} \wedge b = 0$, $b \prec_{B_0} \tilde{c_1} \vee \tilde{c_2} \vee b$ and, $\forall \bar{b} \in B_0$ such that $\bar{b} < \tilde{c_1} \vee \tilde{c_2} \vee b$, $b \not\prec_{B_0} \bar{b}$.

We can define the following:

Definition 4.3.11. Let $(B_0, \prec_{B_0})$ be a finite subalgebra of a contact algebra $(B, \prec_B)$. Then, $\forall b \in B$, we can define $[b]^{\prec_{B_0}} = \bigwedge\{x \in B_0 \mid b \prec_B x\}$[15].

So we can now provide an equivalent definition of signature:

Definition 4.3.12. Let $(B_0, \prec_{B_0})$ be a finite contact algebra. We call a *signature of the first kind* in $(B_0, \prec_{B_0})$ a tuple $(b, \tilde{c_1}, \tilde{c_2}, 1)$, where $b \in B_0$ is an atom, and $\tilde{c_1}, \tilde{c_2} \in B_0$ are such that $[b]^{\prec_{B_0}} \wedge \neg b = \tilde{c_1} \vee \tilde{c_2}$. We call a *signature of the second kind* in $(B_0, \prec_{B_0})$ a tuple $(b, \tilde{c_1}, \tilde{c_2}, 0)$, where $b \in B_0$ is an atom, and $\tilde{c_1}, \tilde{c_2} \in B_0$ are such that $[b]^{\prec_{B_0}} \wedge \neg b = \tilde{c_1} \vee \tilde{c_2}$.

Lemma 4.3.13. *The two definitions of signature either of the first or of the second kind that we provided above are equivalent.*

Proof. It is sufficient to prove that, $\forall b, \tilde{c_1}, \tilde{c_2} \in B_0$, $[b]^{\prec_{B_0}} \wedge \neg b = \tilde{c_1} \vee \tilde{c_2}$ if and only if the following conditions hold: for $i \in \{1, 2\}$, $\tilde{c_i} \wedge b = 0$, $b \prec_{B_0} \tilde{c_1} \vee \tilde{c_2} \vee b$ and, $\forall \bar{b} \in B_0$ such that $\bar{b} < \tilde{c_1} \vee \tilde{c_2} \vee b$, $b \not\prec_{B_0} \bar{b}$.

Suppose first that $[b]^{\prec_{B_0}} \wedge \neg b = \tilde{c_1} \vee \tilde{c_2}$. Hence, for $i \in \{1, 2\}$, we have that $\tilde{c_i} = \tilde{c_i} \wedge (\tilde{c_1} \vee \tilde{c_2}) = \tilde{c_i} \wedge [b]^{\prec_{B_0}} \wedge \neg b \leq \neg b$. So $\tilde{c_i} \wedge b \leq \neg b \wedge b = 0$. Moreover, $\tilde{c_1} \vee \tilde{c_2} \vee b = ([b]^{\prec_{B_0}} \wedge \neg b) \vee b = ([b]^{\prec_{B_0}} \vee b) \wedge (\neg b \vee b) = ([b]^{\prec_{B_0}} \vee b) \wedge 1 = [b]^{\prec_{B_0}} \vee b$, and $b \prec_{B_0} [b]^{\prec_{B_0}}$ thanks to the axiom *(S2)*. So it holds that $b \prec_{B_0} [b]^{\prec_{B_0}} \leq [b]^{\prec_{B_0}} \vee b = \tilde{c_1} \vee \tilde{c_2} \vee b$ and then, by axiom *(S4)*, $b \prec_{B_0} \tilde{c_1} \vee \tilde{c_2} \vee b$. Suppose now that $\bar{b} < \tilde{c_1} \vee \tilde{c_2} \vee b = [b]^{\prec_{B_0}} \vee b$, and suppose by contradiction that $b \prec_{B_0} \bar{b}$. Then, by definition of $[b]^{\prec_{B_0}}$, we have that $[b]^{\prec_{B_0}} \leq \bar{b}$. Therefore $[b]^{\prec_{B_0}} \leq \bar{b} < [b]^{\prec_{B_0}} \vee b$, i. e., $[b]^{\prec_{B_0}} < [b]^{\prec_{B_0}} \vee b$, so $b \not\leq [b]^{\prec_{B_0}}$. But we observed that, by axiom *(S2)*, $b \prec_{B_0} [b]^{\prec_{B_0}}$ and so, by axiom *(S5)*, $b \leq [b]^{\prec_{B_0}}$, which leads to a contradiction.

Suppose now that for $i \in \{1, 2\}$, $\tilde{c_i} \wedge b = 0$, $b \prec_{B_0} \tilde{c_1} \vee \tilde{c_2} \vee b$ and, $\forall \bar{b} \in B_0$ such that $\bar{b} < \tilde{c_1} \vee \tilde{c_2} \vee b$, $b \not\prec_{B_0} \bar{b}$. Since $b \prec_{B_0} \tilde{c_1} \vee \tilde{c_2} \vee b$, by definition of $[b]^{\prec_{B_0}}$ we have that

[15] We observe that such a $[b]^{\prec_{B_0}}$ exists for every $b \in B$, because B_0 is finite.

$[b]^{\prec_{B_0}} \leq \tilde{c_1} \vee \tilde{c_2} \vee b$. So $[b]^{\prec_{B_0}} \wedge \neg b \leq (\tilde{c_1} \vee \tilde{c_2} \vee b) \wedge \neg b = [(\tilde{c_1} \vee \tilde{c_2}) \wedge \neg b] \vee (b \wedge \neg b) = (\tilde{c_1} \vee \tilde{c_2}) \vee 0 = \tilde{c_1} \vee \tilde{c_2}$, being $\tilde{c_i} \wedge b = 0$ for $i \in \{1,2\}$[16]. Hence $[b]^{\prec_{B_0}} \wedge \neg b \leq \tilde{c_1} \vee \tilde{c_2}$. Moreover, in order to show that $\tilde{c_1} \vee \tilde{c_2} \leq [b]^{\prec_{B_0}} \wedge \neg b$, it is sufficient to prove that $\tilde{c_1} \vee \tilde{c_2} \leq [b]^{\prec_{B_0}}$, being $\tilde{c_1} \vee \tilde{c_2} = (\tilde{c_1} \vee \tilde{c_2}) \wedge \neg b$. So suppose that $x \in B_0$ is such that $b \prec_{B_0} x$. Since we also have that $b \prec_{B_0} \tilde{c_1} \vee \tilde{c_2} \vee b$, by axiom *(S2)* we have that $b \prec_{B_0} (\tilde{c_1} \vee \tilde{c_2} \vee b) \wedge x$. So, by hypothesis, $(\tilde{c_1} \vee \tilde{c_2} \vee b) \wedge x \not< \tilde{c_1} \vee \tilde{c_2} \vee b$. Therefore, since $(\tilde{c_1} \vee \tilde{c_2} \vee b) \wedge x \leq \tilde{c_1} \vee \tilde{c_2} \vee b$, we have that $(\tilde{c_1} \vee \tilde{c_2} \vee b) \wedge x = \tilde{c_1} \vee \tilde{c_2} \vee b$. So $\tilde{c_1} \vee \tilde{c_2} \vee b \leq x$. Therefore $\tilde{c_1} \vee \tilde{c_2} = [(\tilde{c_1} \vee \tilde{c_2}) \wedge \neg b] \vee 0 = [(\tilde{c_1} \vee \tilde{c_2}) \wedge \neg b] \vee (b \wedge \neg b) = (\tilde{c_1} \vee \tilde{c_2} \vee b) \wedge \neg b \leq x \wedge \neg b \leq x$, and this implies the thesis, by definition of $[b]^{\prec_{B_0}}$.

□

The definition of signature is useful because we have the following:

Theorem 4.3.14. *Let $(B_0, \prec_{B_0})$ be a finite contact algebra. To give a finite minimal extension either of the first or of the second kind of $(B_0, \prec_{B_0})$ (up to isomorphism) is equivalent to give respectively:*

1. *a signature of the first kind $(b, \tilde{c_1}, \tilde{c_2}, 1)$ in $(B_0, \prec_{B_0})$*
2. *a signature of the second kind $(b, \tilde{c_1}, \tilde{c_2}, 0)$ in $(B_0, \prec_{B_0})$*

Proof. In this proof, we refer to the first definition of signature that we provided above. Let $\iota : (B_0, \prec) \hookrightarrow (C, \prec)$ be a finite minimal extension of $(B_0, \prec_{B_0})$. Then, by the Proposition 4.3.5, there exists a unique atom $b \in B_0$ with the following property: there exist two atoms $c_1, c_2 \in C$ such that $c_1 \vee c_2 = \iota(b)$ and $c_1 \wedge c_2 = 0$. Moreover, by Proposition 4.3.9, there exist $\tilde{c_1}, \tilde{c_2} \in B_0$ such that, if ι is a minimal extension of the first kind, then, for $i \in \{1,2\}$, $\tilde{c_i} \wedge b = 0$, $c_i \prec_C \iota(\tilde{c_i} \vee b)$ and, $\forall \bar{c} \in C$ such that $\bar{c} < \iota(\tilde{c_i} \vee b)$, $c_i \not\prec_C \bar{c}$, while if ι is a minimal extension of the second kind, then, for $i \in \{1,2\}$, $\tilde{c_i} \wedge b = 0$, $c_i \prec_C \iota(\tilde{c_i}) \vee c_i$ and, $\forall \bar{c} \in C$ such that $\bar{c} < \iota(\tilde{c_i}) \vee c_i$, $c_i \not\prec_C \bar{c}$. So suppose that ι is a minimal extension of the first kind. Then $c_i \prec_C \iota(\tilde{c_i} \vee b) \leq \iota(\tilde{c_1} \vee \tilde{c_2} \vee b)$ and so, by the axiom *(S4)* of the definition of contact algebras, $c_i \prec_C \iota(\tilde{c_1} \vee \tilde{c_2} \vee b)$. Hence, by the axiom *(S3)* of the definition of contact algebras, $\iota(b) = c_1 \vee c_2 \prec_C \iota(\tilde{c_1} \vee \tilde{c_2} \vee b)$. Since ι is an embedding, we can then deduce that $b \prec_{B_0} \tilde{c_1} \vee \tilde{c_2} \vee b$.

Now consider $\bar{b} \in B_0$ such that $\bar{b} < \tilde{c_1} \vee \tilde{c_2} \vee b$, and suppose by contradiction that $b \prec_{B_0} \bar{b}$. Then $c_1 \vee c_2 = \iota(b) \prec_C \iota(\bar{b})$, and so, by axiom *(S4)*, $c_1 \prec_C \iota(\bar{b})$ and $c_2 \prec_C \iota(\bar{b})$. Moreover, by Proposition 4.3.9, we know that $c_1 \prec_C \iota(\tilde{c_1} \vee b)$ and $c_2 \prec_C \iota(\tilde{c_2} \vee b)$. Therefore, by axiom *(S2)*, $c_1 \prec_C \iota(\bar{b} \wedge (\tilde{c_1} \vee b))$ and $c_2 \prec_C \iota(\bar{b} \wedge (\tilde{c_2} \vee b))$. But we are supposing that $\bar{b} < \tilde{c_1} \vee \tilde{c_2} \vee b$, so either $\bar{b} \wedge (\tilde{c_1} \vee b) < \tilde{c_1} \vee b$ (and so $\iota(\bar{b} \wedge (\tilde{c_1} \vee b)) < \iota(\tilde{c_1} \vee b)$) or $\bar{b} \wedge (\tilde{c_2} \vee b) < \tilde{c_2} \vee b$ (and so $\iota(\bar{b} \wedge (\tilde{c_2} \vee b)) < \iota(\tilde{c_2} \vee b)$). In any case, we get a contradiction with the statement of the Proposition 4.3.9. Hence we have obtained a signature of the first kind $(b, \tilde{c_1}, \tilde{c_2}, 1)$ in $(B_0, \prec_{B_0})$. Suppose now that ι is a minimal extension of the second kind. Then, for

[16] $\tilde{c_i} \wedge \neg b = (\tilde{c_i} \wedge \neg b) \vee 0 = (\tilde{c_i} \wedge \neg b) \vee (\tilde{c_i} \wedge b) = \tilde{c_i} \wedge (\neg b \vee b) = \tilde{c_i} \wedge 1 = \tilde{c_i}$.

$i \in \{1,2\}$, $c_i \prec_C \iota(\tilde{c_i}) \vee c_i \leq (\iota(\tilde{c_1}) \vee c_1) \vee (\iota(\tilde{c_2}) \vee c_2) = \iota(\tilde{c_1}) \vee \iota(\tilde{c_2}) \vee (c_1 \vee c_2) = \iota(\tilde{c_1} \vee \tilde{c_2} \vee b)$. Hence, by axiom *(S4)*, $c_i \prec_C \iota(\tilde{c_1} \vee \tilde{c_2} \vee b)$ and so, by axiom *(S3)*, $\iota(b) = c_1 \vee c_2 \prec_C \iota(\tilde{c_1} \vee \tilde{c_2} \vee b)$. Since ι is an embedding, we can deduce that $b \prec_{B_0} \tilde{c_1} \vee \tilde{c_2} \vee b$. Now consider $\bar{b} \in B_0$ such that $\bar{b} < \tilde{c_1} \vee \tilde{c_2} \vee b$, and suppose by contradiction that $b \prec_{B_0} \bar{b}$. Then $c_1 \vee c_2 = \iota(b) \prec_C \iota(\bar{b})$, and so, by axiom *(S4)*, $c_1 \prec_C \iota(\bar{b})$ and $c_2 \prec_C \iota(\bar{b})$. Moreover, by Proposition 4.3.9, we know that, for $i \in \{1,2\}$, $c_i \prec_C \iota(\tilde{c_i}) \vee c_i$. Hence, by axiom *(S2)*, $c_1 \prec_C \iota(\bar{b}) \wedge (\iota(\tilde{c_1}) \vee c_1)$ and $c_2 \prec_C \iota(\bar{b}) \wedge (\iota(\tilde{c_2}) \vee c_1)$. But we are supposing that $\bar{b} < \tilde{c_1} \vee \tilde{c_2} \vee b$, so $\iota(\bar{b}) < \iota(\tilde{c_1} \vee \tilde{c_2} \vee b) = (\iota(\tilde{c_1}) \vee c_1) \vee (\iota(\tilde{c_2}) \vee c_2)$. So either $\iota(\bar{b}) \wedge (\tilde{c_1}) \vee c_1) < \tilde{c_1}) \vee c_1$, or $\iota(\bar{b}) \wedge (\tilde{c_2}) \vee c_2) < \tilde{c_2}) \vee c_2$. In any case, we get a contradiction with the statement of the Proposition 4.3.9. Hence we have obtained a signature of the second kind $(b, \tilde{c_1}, \tilde{c_2}, 0)$ in $(B_0, \prec_{B_0})$.

Conversely, let $(b, \tilde{c_1}, \tilde{c_2}, 1)$ be a signature of the first kind in $(B_0, \prec_{B_0})$. Then b dually corresponds to a singleton $\{x\}$ of X_{B_0}, being an atom of B_0. Moreover, $\tilde{c_1} \cong C_1 \in \mathcal{P}(X_{B_0}) \cong B_0$ and $\tilde{c_2} \cong C_2 \in \mathcal{P}(X_{B_0}) \cong B_0$, where C_1 and C_2 are such that $x \notin C_1 \cup C_2$ (being $\tilde{c_1} \wedge b = 0 = \tilde{c_2} \wedge b$), $R_{B_0}[x] \subseteq C_1 \cup C_2 \cup \{x\}$ and, $\forall U_{\bar{b}} \subsetneq C_1 \cup C_2 \cup \{x\}$, $R_{B_0}[x] \not\subseteq U_{\bar{b}}$ (so $R_{B_0}[x] = C_1 \cup C_2 \cup \{x\}$). Therefore, thanks to the Remark 4.3.6, we can obtain a finite minimal extension of the first kind of $(B_0, \prec_{B_0})$ from the given signature (it is sufficient to consider the finite minimal extension of $(B_0, \prec_{B_0})$ which is dual to the stable morphism f_2 that is defined in the Remark). In a similar way, if $(b, \tilde{c_1}, \tilde{c_2}, 0)$ is a signature of the second kind in $(B_0, \prec_{B_0})$, then we can obtain a finite minimal extension of the second kind of $(B_0, \prec_{B_0})$ from the given signature (it is sufficient to consider the finite minimal extension of $(B_0, \prec_{B_0})$ which is dual to the stable morphism f_1 that is defined in the Remark).

□

Now we are ready to prove this important result:

Theorem 4.3.15. *A contact algebra is existentially closed if and only if, for any finite subalgebra* $(B_0, \prec_{B_0}) \overset{\bar{\iota}}{\hookrightarrow} (B, \prec_B)$*, the following conditions hold:*

1. *for every signature of the first kind* $(b, \tilde{c_1}, \tilde{c_2}, 1)$ *in* $(B_0, \prec_{B_0})$*, there exist* $b_1, b_2 \in B \setminus \{0\}$ *such that* $\bar{\iota}(b) = b_1 \vee b_2$*,* $b_1 \wedge b_2 = 0$*,* $b_i \prec_B \bar{\iota}(\tilde{c_i} \vee b)$ *for* $i \in \{1,2\}$ *and* $b_1 \not\prec_B \neg b_2$[17]*. Moreover, for every* $a \in B_0 \setminus \{0\}$ *such that* $a \leq \tilde{c_i}$ *(*$i \in \{1,2\}$*), it holds that* $b_i \not\prec_B \neg\bar{\iota}(a)$*.*
2. *for every signature of the second kind* $(b, \tilde{c_1}, \tilde{c_2}, 0)$ *in* $(B_0, \prec_{B_0})$*, there exist* $b_1, b_2 \in B \setminus \{0\}$ *such that* $\bar{\iota}(b) = b_1 \vee b_2$*,* $b_1 \wedge b_2 = 0$*,* $b_1 \prec_B \bar{\iota}(\tilde{c_1}) \vee b_1$ *and* $b_2 \prec_B \bar{\iota}(\tilde{c_2}) \vee b_2$*. Moreover, for every* $a \in B_0 \setminus \{0\}$ *such that* $a \leq \tilde{c_i}$ *(*$i \in \{1,2\}$*), it holds that* $b_i \not\prec_B \neg\bar{\iota}(a)$*.*

Proof. ($\Rightarrow$) Let $(B_0, \prec_{B_0}) \overset{\bar{\iota}}{\hookrightarrow} (B, \prec_B)$ be a finite subalgebra, and let $(b, \tilde{c_1}, \tilde{c_2}, 1)$ be a signature of the first kind in $(B_0, \prec_{B_0})$. Let $(B_0, \prec_{B_0}) \overset{\iota}{\hookrightarrow} (C, \prec_C)$ be the finite minimal extension associated to that signature (according to the Theorem 4.3.14). Thanks to the Theorem 4.3.1, we know that there exists an embedding $(C, \prec_C) \overset{\bar{\iota}}{\hookrightarrow} (B, \prec_B)$ that fixes

[17]Observe that, by axiom *(S6)*, $[b_1 \prec_B \neg b_2 \Leftrightarrow b_2 \prec_B \neg b_1]$, so $[b_1 \not\prec_B \neg b_2 \Leftrightarrow b_2 \not\prec_B \neg b_1]$.

$(B_0, \prec_{B_0})$ pointwise, i. e., such that $\tilde{\iota} \circ \iota = \bar{\iota}$. Hence we can consider $b_1 := \tilde{\iota}(c_1)$ and $b_2 := \tilde{\iota}(c_2)$, where c_1, c_2 are as in the statement of the Proposition 4.3.5. So we have that: $\bar{\iota}(b) = \tilde{\iota}(\iota(b)) = \tilde{\iota}(c_1 \vee c_2) = \tilde{\iota}(c_1) \vee \tilde{\iota}(c_2) = b_1 \vee b_2$, $b_1 \wedge b_2 = \tilde{\iota}(c_1) \wedge \tilde{\iota}(c_2) = \tilde{\iota}(c_1 \wedge c_2) = \tilde{\iota}(0) = 0$, $b_i = \tilde{\iota}(c_i) \prec_B \tilde{\iota}(\iota(\tilde{c_i} \vee b)) = \bar{\iota}(\tilde{c_i} \vee b)$ for $i \in \{1,2\}$ (being $c_i \prec_C \iota(\tilde{c_i} \vee b)$, according to the Proposition 4.3.9), and $b_1 = \tilde{\iota}(c_1) \not\prec_B \tilde{\iota}(\neg c_2) = \neg\tilde{\iota}(c_2) = \neg b_2$ (because ι is a minimal extension of the first kind, so xR_Cx', then $R_C[x] \nsubseteq X_C \setminus \{x'\}$, i. e., $c_1 \cong \{x\} \not\prec_C X_C \setminus \{x'\} \cong \neg c_2$, i. e., $c_1 \not\prec_C \neg c_2$). Moreover, $\forall a \in B_0 \setminus \{0\}$ such that $a \leq \tilde{c_i}$, it holds that $c_i \not\prec_C \neg\iota(a)$. In fact, the condition $0 \neq a \leq \tilde{c_1}$ dually corresponds to the condition $\emptyset \neq U_a \subseteq f(R_C[x]) \setminus \{x\}$. This implies that $f(f^{-1}(U_a) \cap (R_C[x] \setminus \{x, x'\})) = U_a$[18]. Since $U_a \neq \emptyset$, this implies that $f^{-1}(U_a) \cap (R_C[x] \setminus \{x, x'\}) \neq \emptyset$. So $R_C[x] \nsubseteq X_C \setminus f^{-1}(U_a)$: this dually corresponds to the fact that $c_1 \not\prec_C \neg\iota(a)$. Similarly, it holds that $0 \neq a \leq \tilde{c_2} \Rightarrow c_2 \not\prec_C \neg\iota(a)$. Therefore, if $0 \neq a \leq \tilde{c_i}$, we get that $b_i = \tilde{\iota}(c_i) \not\prec_B \tilde{\iota}(\neg\iota(a)) = \neg\bar{\iota}(a)$, as required, being $\tilde{\iota}$ an embedding.

Suppose now that $(b, \tilde{c_1}, \tilde{c_2}, 0)$ is a signature of the second kind in $(B_0, \prec_{B_0})$, and let $(B_0, \prec_{B_0}) \overset{\iota}{\hookrightarrow} (C, \prec_C)$ be the finite minimal extension associated to that signature (according to the Theorem 4.3.14). Again thanks to the Theorem 4.3.1, we know that there exists an embedding $(C, \prec_C) \overset{\tilde{\iota}}{\hookrightarrow} (B, \prec_B)$ that fixes $(B_0, \prec_{B_0})$ pointwise, i. e., such that $\tilde{\iota} \circ \iota = \bar{\iota}$. Hence we can consider, as above, $b_1 := \tilde{\iota}(c_1)$ and $b_2 := \tilde{\iota}(c_2)$, where c_1, c_2 are as in the statement of the Proposition 4.3.5. So we have that: $\bar{\iota}(b) = \tilde{\iota}(\iota(b)) = \tilde{\iota}(c_1 \vee c_2) = \tilde{\iota}(c_1) \vee \tilde{\iota}(c_2) = b_1 \vee b_2$, $b_1 \wedge b_2 = \tilde{\iota}(c_1) \wedge \tilde{\iota}(c_2) = \tilde{\iota}(c_1 \wedge c_2) = \tilde{\iota}(0) = 0$ and $b_i = \tilde{\iota}(c_i) \prec_B \tilde{\iota}(\iota(\tilde{c_i}) \vee c_i) = \tilde{\iota}(\iota(\tilde{c_i})) \vee \tilde{\iota}(c_i) = \bar{\iota}(\tilde{c_i}) \vee b_i$ for $i \in \{1,2\}$ (being $c_i \prec_C \iota(\tilde{c_i}) \vee c_i$, according to the Proposition 4.3.9). The fact that for every atom $a \in B_0$ such that $a \leq \tilde{c_i}$, it holds that $b_i \not\prec_B \neg\bar{\iota}(a)$, can be shown as in the previous case.

($\Leftarrow$) We show that $(B, \prec_B)$ is existentially closed by proving that the condition given by Corollary 4.3.1.1 is satisfied. Let $(B_0, \prec_{B_0}) \overset{\iota}{\hookrightarrow} (C, \prec_C)$ be a finite minimal extension. We need to provide an embedding $(C, \prec_C) \overset{\tilde{\iota}}{\hookrightarrow} (B, \prec_B)$ such that $\tilde{\iota} \circ \iota = \bar{\iota}$. In order to do that, we use the duality: ι dually corresponds to a continuous stable morphism $X_C = X_{B_0} \uplus \{x'\}$, $f \upharpoonright_{X_{B_0}} = id_{X_{B_0}}$, $f(x') = x$ for some $x \in X_{B_0}$, $R_C[y] = R_{B_0}[y]$ $\forall y \in X_C \setminus \{x, x'\}$, and $R_C[x]$, $R_C[x']$ are such that the following conditions hold: $f(R_C[x]) \cup f(R_C[x']) = f(R_C[x] \cup R_C[x']) = f(R_C[\{x, x'\}]) = R_{B_0}[x]$, $x \in R_C[x]$, $x' \in R_C[x']$, $\forall z \in X_C$ ($z \in R_C[x'] \Leftrightarrow x' \in R_C[z]$) and similarly $\forall z \in X_C$ ($z \in R_C[x] \Leftrightarrow x \in R_C[z]$). We also have that the embedding $(B_0, \prec_{B_0}) \overset{\bar{\iota}}{\hookrightarrow} (B, \prec_B)$ dually corresponds to a continuous stable morphism $X_{B_0} \overset{\bar{f}}{\twoheadleftarrow} X_B$, which satisfies the condition $\forall z_1, z_2 \in X_{B_0}$ $[z_1 R_{B_0} z_2 \Leftrightarrow \exists y_1, y_2 \in X_B$ s. t. $\bar{f}(y_1) = z_1,\ \bar{f}(y_2) = z_2\ \&\ y_1 R_B y_2]$. So, in order to provide the required $\tilde{\iota}$, we provide a continuous stable morphism $X_B \overset{\tilde{f}}{\twoheadrightarrow} X_C$ such that $\forall z_1, z_2 \in X_C$ $[z_1 R_C z_2 \Leftrightarrow \exists y_1, y_2 \in X_B$ s. t. $\tilde{f}(y_1) = z_1,\ \tilde{f}(y_2) = z_2\ \&\ y_1 R_B y_2]$ and $f \circ \tilde{f} = \bar{f}$:

[18] In fact, $f(f^{-1}(U_a) \cap (R_C[x] \setminus \{x, x'\})) \subseteq f(f^{-1}(U_a)) = U_a$. Moreover, $z \in U_a \subseteq f(R_C[x]) \setminus \{x\} \Rightarrow \exists \tilde{z} \in R_C[x] \setminus \{x, x'\}$ s. t. $f(\tilde{z}) = z \in U_a \Rightarrow \tilde{z} \in f^{-1}(U_a) \cap (R_C[x] \setminus \{x, x'\}) \Rightarrow z = f(\tilde{z}) \in f(f^{-1}(U_a) \cap (R_C[x] \setminus \{x, x'\}))$, so $U_a \subseteq f(f^{-1}(U_a) \cap (R_C[x] \setminus \{x, x'\}))$.

$$\begin{tikzcd}
(X_{B_0}, R_{B_0}) & (X_B, R_B) \arrow[l, two heads, "\bar{f}"'] \arrow[dl, dashed, two heads, "\tilde{f}"] \\
(X_C, R_C) \arrow[u, two heads, "f"] &
\end{tikzcd}$$

So we now define such a stable morphism $\tilde{f}$. By hypothesis, we know that $\exists U_{b_1}, U_{b_2} \in \mathsf{Clop}(X_B)\backslash\{\emptyset\} \cong B\backslash\{0\}$ such that $b_1 \cong U_{b_1}$, $b_2 \cong U_{b_2}$, $U_{b_1} \cap U_{b_2} = \emptyset$ and $\bar{\iota}(b) \cong \bar{f}^{-1}(\{x\}) = U_{b_1} \cup U_{b_2}$. From the proof of the Proposition 4.3.5, we know that the elements that satisfy the statement of this proposition are such that $b \cong \{x\} \in \mathcal{P}(X_{B_0}) \cong B_0$, $\iota(b) \cong \{x, x'\}$, $c_1 \cong \{x\} \in \mathcal{P}(X_C) \cong C$, $c_2 \cong \{x'\} \in \mathcal{P}(X_C) \cong C$, $\tilde{c_1} \cong f(R_C[x])$ and $\tilde{c_2} \cong f(R_C[x'])$. So now we use what we have in order to define the continuous stable morphism $\tilde{f}$. Then let $y \in X_B$. We distinguish two cases: either $\bar{f}(y) = x$ or $\bar{f}(y) \neq x$. In the first case, $y \in \bar{f}^{-1}(\{x\}) = U_{b_1} \cup U_{b_2}$ with $U_{b_1} \cap U_{b_2} = \emptyset$. If $y \in U_{b_1}$, we define $\tilde{f}(y) := x$, so that $f(\tilde{f}(y)) = f(x) = x = \bar{f}(y)$, while if $y \in U_{b_2}$, we define $\tilde{f}(y) := x'$, so that $f(\tilde{f}(y)) = f(x') = x = \bar{f}(y)$. In the second case (i. e., if $\bar{f}(y) \neq x$), we define $\tilde{f}(y) := f^{-1}(\bar{f}(y))$: it is well-defined because $f \restriction_{X_{B_0}} = id_{X_{B_0}}$, so $f^{-1}(\bar{f}(y))$ is a singleton, being $\bar{f}(y) \in X_{B_0}\backslash\{x\}$. By definition of $\tilde{f}$, it is clear that $\tilde{f}$ is surjective and that $f \circ \tilde{f} = \bar{f}$.

Now we prove that $\tilde{f}$ is a continuous stable morphism, whose dual morphism is an embedding of contact algebras. Since X_C has the discrete topology, in order to show that $\tilde{f}$ is continuous it is sufficient to show that $\tilde{f}^{-1}(\{z\}) = \tilde{f}^{-1}(z) \subseteq X_B$ is open $\forall z \in X_C$. So suppose that $z \in X_C\backslash\{x, x'\}$. Then $\tilde{f}^{-1}(z) = \{y \in X_B \mid f^{-1}(\bar{f}(y)) = \tilde{f}(y) = z\} = \{y \in X_B \mid \bar{f}(y) = f(z) = z\} = \bar{f}^{-1}(z)$, which is open because X_{B_0} has the discrete topology and $\bar{f}$ is continuous. Now suppose that $z = x \in X_C$. Then $\tilde{f}^{-1}(z) = \tilde{f}^{-1}(x) = U_{b_1} \subseteq X_B$[19], which is open because it is clopen. Similarly, if $z = x' \in X_C$, it holds that $\tilde{f}^{-1}(z) = \tilde{f}^{-1}(x') = U_{b_2} \subseteq X_B$, which is also open. Hence $\tilde{f}$ is continuous. According to the Lemma 4.2.16, it remains to prove that $\forall z_1, z_2 \in X_C\ [z_1 R_C z_2 \iff \exists y_1, y_2 \in X_B \text{ s. t. } \tilde{f}(y_1) = z_1,\ \tilde{f}(y_2) = z_2\ \&\ y_1 R_B y_2]$.

So suppose first that $y_1 R_B y_2$: we have to show that then $\tilde{f}(y_1) R_C \tilde{f}(y_2)$. We distinguish some cases:

1. $y_1, y_2 \in X_B \setminus (U_{b_1} \cup U_{b_2})$
2. $y_1 \in U_{b_1}$
3. $y_1 \in U_{b_2}$

In the first case, since both f and $\bar{f}$ dually correspond to morphisms of contact algebras and $f \restriction_{B_0} = id_{B_0}$, it holds that $y_1 R_B y_2 \Rightarrow \bar{f}(y_1) R_{B_0} \bar{f}(y_2) \Rightarrow \tilde{f}(y_1) = f^{-1}(\bar{f}(y_1)) R_C f^{-1}(\bar{f}(y_2)) = \tilde{f}(y_2)$, as required. Moreover, by the hypotheses we have that $b_i \prec_B \bar{\iota}(\tilde{c_i} \vee b)$ (this holds also if we deal with a signature of the second kind because, thanks to the axiom *(S4)*,

[19] We defined $\tilde{f}$ in such a way that $\tilde{f}(U_{b_1}) = x$, so $U_{b_1} \subseteq \tilde{f}^{-1}(x)$. If $w \in X_B \setminus U_{b_1}$, then either $w \in U_{b_2}$ (and so $\tilde{f}(x) = x'$, according to the definition of $\tilde{f}$), or $w \in X_B \setminus U_{b_1} \cup U_{b_2}$ (and then $\bar{f}(w) \in X_{B_0}\backslash\{x\}$, so $\tilde{f}(w) = f^{-1}(\bar{f}(w)) \neq x$). So $\tilde{f}^{-1}(x) \subseteq U_{b_1}$.

we have that $b_i \prec_B \bar{\iota}(\tilde{c_i}) \vee b_i \leq \bar{\iota}(\tilde{c_i}) \vee (b_1 \vee b_2) = \bar{\iota}(\tilde{c_i}) \vee \bar{\iota}(b) = \bar{\iota}(\tilde{c_i} \vee b)$, so $b_i \prec_B \bar{\iota}(\tilde{c_i} \vee b))$. Dually, this means that $R_B[U_{b_1}] \subseteq \bar{f}^{-1}((f(R_C[x]) \setminus \{x\}) \cup \{x\}) = \bar{f}^{-1}(f(R_C[x]))$ and similarly $R_B[U_{b_2}] \subseteq \bar{f}^{-1}(f(R_C[x']))$. Therefore, in the second case we have that $y_1 \in U_{b_1}$ & $y_1 R_B y_2 \Rightarrow y_2 \in R_B[U_{b_1}] \subseteq \bar{f}^{-1}(f(R_C[x])) \Rightarrow \bar{f}(y_2) \in f(R_C[x]) \Rightarrow \exists \bar{y_2} \in R_C[x]$ s. t. $f(\bar{y_2}) = \bar{f}(y_2)$, i. e. $\exists \bar{y_2} \in X_C$ s. t. $x R_C \bar{y_2}$ & $f(\bar{y_2}) = \bar{f}(y_2) = f(\tilde{f}(y_2))$. Since $f(\bar{y_2}) = f(\tilde{f}(y_2))$, exactly one of the following situations occur:

a) $\bar{y_2} = \tilde{f}(y_2)$

b) $\bar{y_2} = x'$ & $\tilde{f}(y_2) = x$

c) $\bar{y_2} = x$ & $\tilde{f}(y_2) = x'$

In the case a), we have that $\tilde{f}(y_1) = x R_C \bar{y_2} = \tilde{f}(y_2)$, as required. In the case b), since R_C is reflexive, we have that $\tilde{f}(y_1) = x R_C x = \tilde{f}(y_2)$, as required. In the case c), we have to distinguish two subcases. If we are dealing with a signature of the first kind, then $\tilde{f}(y_1) = x R_C x' \tilde{f}(y_2)$, as required. If we are dealing with a signature of the second kind, then the situation described in case c) can't occur. In fact, by definition of signature and by hypotheses we have that $0 = \bar{\iota}(0) = \bar{\iota}(\tilde{c_1} \wedge b) = \bar{\iota}(\tilde{c_1}) \wedge \bar{\iota}(b) = \bar{\iota}(\tilde{c_1}) \wedge (b_1 \vee b_2) = (\bar{\iota}(\tilde{c_1}) \wedge b_1) \vee (\bar{\iota}(\tilde{c_1}) \wedge b_2)$, so $\bar{\iota}(\tilde{c_1}) \wedge b_2 = 0$, and that $b_1 \wedge b_2 = 0$, so $b_2 \wedge (\bar{\iota}(\tilde{c_1}) \vee b_1) = (b_2 \wedge \bar{\iota}(\tilde{c_1})) \vee (b_2 \wedge b_1) = 0 \vee 0 = 0$. Moreover, by hypothesis we have that $b_1 \prec_B \bar{\iota}(\tilde{c_1}) \vee b_1$. These conditions dually correspond to $U_{b_2} \cap (\bar{f}^{-1}(U_{\tilde{c_1}}) \cup U_{b_1}) = \emptyset$ and $R_B[U_{b_1}] \subseteq \bar{f}^{-1}(U_{\tilde{c_1}}) \cup U_{b_1}$. So we have that $R_B[U_{b_1}] \cap U_{b_2} = \emptyset$. Recall that $\tilde{f}(y_2) = x' \Leftrightarrow y_2 \in U_{b_2}$. Therefore, $\tilde{f}(y_2) = x'$, $y_1 \in U_{b_1}$ & $y_1 R_B y_2 \Rightarrow y_2 \in R_B[U_{b_1}] \cap U_{b_2} = \emptyset$.

The situation in which $y_1 \in U_{b_2}$ can be studied in a similar way (this situation is symmetric to the one we have just considered).

Now suppose that $z_1, z_2 \in X_C$ are such that $z_1 R_C z_2$: we have to show that then $\exists y_1, y_2 \in X_B$ s. t. $\tilde{f}(y_1) = z_1$, $\tilde{f}(y_2) = z_2$ & $y_1 R_B y_2$. Suppose first that $z_1, z_2 \in X_C \setminus \{x, x'\}$. Then $z_1 = f(z_1) R_{B_0} f(z_2) = z_2$ with $z_1, z_2 \in X_{B_0} \setminus \{x\}$. Since $\bar{f}$ is dual to a morphism of contact algebras, it then holds that $\exists y_1, y_2 \in X_B \setminus (U_{b_1} \cup U_{b_2})$ s. t. $y_1 R_B y_2$, $f(\tilde{f}(y_1)) = \bar{f}(y_1) = z_1$ & $f(\tilde{f}(y_2)) = \bar{f}(y_2) = z_2$. Hence y_1, y_2 are such that $y_1 R_B y_2$, $\tilde{f}(y_1) = z_1$ & $\tilde{f}(y_2) = z_2$, as required. Suppose now that $z_1 = x \in X_C$ and $z_2 = x' \in X_C$. This situation is possible only if we are dealing with a signature of the first kind. By hypothesis, we know that $b_1 \not\prec_B \neg b_2$: this dually means that $R_B[U_{b_1}] \not\subseteq X_B \setminus U_{b_2}$. This implies that $\exists y_2 \in R_B[U_{b_1}] \cap U_{b_2}$, so $\exists y_1 \in U_{b_1}$ such that $y_1 R_B y_2$. Since $y_1 \in U_{b_1}$ and $y_2 \in U_{b_2}$, it holds that $\tilde{f}(y_1) = x = z_1$ and $\tilde{f}(y_2) = x' = z_2$, as required. Suppose then that $z_1 = x \in X_C$ and $z_2 \in X_C \setminus \{x, x'\}$. In this situation, $\{f(z_2)\}$ dually corresponds to an atom $a \in B_0 \setminus \{b\}$ such that $a \leq \tilde{c_1}$. In fact, $a \cong \{f(z_2)\} \subseteq f(R_C[x] \setminus \{x, x'\}) = f(R_C[x]) \setminus \{x\} \cong \tilde{c_1}$. Hence, by hypothesis, we get that $b_1 \not\prec_B \neg\bar{\iota}(a)$: this dually means that $R_B[U_{b_1}] \not\subseteq X_B \setminus \bar{f}^{-1}(f(z_2)) = X_B \setminus \tilde{f}^{-1}(z_2)$, being $\bar{f}^{-1}(f(z_2)) = \tilde{f}^{-1}(z_2)$[20]. Therefore $\exists y_2 \in R_B[U_{b_1}] \cap \tilde{f}^{-1}(z_2)$, so $\exists y_1 \in U_{b_1}$ such that $y_1 R_B y_2$. Hence it holds that $\exists y_1, y_2 \in X_B$ such that $\tilde{f}(y_1) = x$, $\tilde{f}(y_2) = z_2$ and $y_1 R_B y_2$, as required. The situation in which $z_1 = x' \in X_C$ and $z_2 \in X_C \setminus \{x, x'\}$ is similar to the one that we have just considered.

□

Corollary 4.3.15.1. *A contact algebra $(B, \prec_B)$ is existentially closed if and only if, for any finite subalgebra $(B_0, \prec_{B_0}) \subseteq (B, \prec_B)$, the following conditions hold:*

1. *for every signature of the first kind $(b, \tilde{c_1}, \tilde{c_2}, 1)$ in $(B_0, \prec_{B_0})$, there exist $b_1, b_2 \in B \setminus \{0\}$ such that $b = b_1 \vee b_2$, $b_1 \wedge b_2 = 0$, $[b_i]^{\prec_{B_0}} = \tilde{c_i} \vee b$ for $i \in \{1, 2\}$ and $b_1 \not\prec_B \neg b_2$*[21].

2. *for every signature of the second kind $(b, \tilde{c_1}, \tilde{c_2}, 0)$ in $(B_0, \prec_{B_0})$, there exist $b_1, b_2 \in B \setminus \{0\}$ such that $b = b_1 \vee b_2$, $b_1 \wedge b_2 = 0$, $[b_i]^{\prec_{B_0}} = \tilde{c_i} \vee b$ for $i \in \{1, 2\}$ and $b_1 \prec_B \neg b_2$.*

Proof. We prove that the statement of the corollary is equivalent to the statement of the Theorem 4.3.15. So let $(b, \tilde{c_1}, \tilde{c_2}, 1)$ be a signature of the first kind in $(B_0, \prec_{B_0})$, and suppose that there exist $b_1, b_2 \in B \setminus \{0\}$ such that $b = b_1 \vee b_2$, $b_1 \wedge b_2 = 0$, $b_i \prec_B \tilde{c_i} \vee b$ ($i \in \{1, 2\}$) and $b_1 \not\prec_B \neg b_2$. Moreover, suppose that for every $a \in B_0 \setminus \{0\}$ such that $a \leq \tilde{c_i}$ ($i \in \{1, 2\}$), it holds that $b_i \not\prec_B \neg a$. Since it holds that $b_i \prec_B \tilde{c_i} \vee b$ ($i \in \{1, 2\}$), we have that $[b_i]^{\prec_{B_0}} \leq \tilde{c_i} \vee b$, being $[b_i]^{\prec_{B_0}} = \bigwedge\{x \in B_0 \mid b_i \prec_B x\}$. Suppose by contradiction that $[b_i]^{\prec_{B_0}} < \tilde{c_i} \vee b$. Then, being $[b_i]^{\prec_{B_0}}, \tilde{c_i}, b \in B_0$ and being b an atom of B_0, we distinguish two cases: either $[b_i]^{\prec_{B_0}} \leq \tilde{c_i}$ or $\exists a < \tilde{c_i}$ such that $[b_i]^{\prec_{B_0}} = a \vee b$. In the first case, by axiom *(S4)* we have that $b_i \prec_B [b_i]^{\prec_{B_0}} \leq \tilde{c_i} \Rightarrow b_i \prec_B \tilde{c_i}$. So, if $i = 1$, we have that $b_1 \prec_B \tilde{c_1}$, and $0 = \tilde{c_1} \wedge b = \tilde{c_1} \wedge (b_1 \vee b_2) = (\tilde{c_1} \wedge b_1) \vee (\tilde{c_1} \wedge b_2)$. So $\tilde{c_1} \wedge b_2 = 0$, i. e., $\tilde{c_1} \leq \neg b_2$. Hence $b_1 \prec_B \tilde{c_1} \leq \neg b_2 \Rightarrow b_1 \prec_B \neg b_2$ again by axiom *(S4)*, but this contradicts the hypotheses. Similarly if $i = 2$. In the second case, if $i = 1$, by axioms *(S2)* and *(S4)*, $b_1 \prec_B [b_i]^{\prec_{B_0}} = a \vee b \leq a \vee \tilde{c_2} \vee b \Rightarrow b_1 \prec_B a \vee \tilde{c_2} \vee b$, and we also have that $b_2 \prec_B \tilde{c_2} \vee b \leq a \vee \tilde{c_2} \vee b \Rightarrow b_2 \prec_B a \vee \tilde{c_2} \vee b$. Therefore, by axiom *(S3)*, $b = b_1 \vee b_2 \prec_B a \vee \tilde{c_2} \vee b < \tilde{c_1} \vee \tilde{c_2} \vee b$, but this contradicts the definition of signature. Similarly if $i = 2$. So we can conclude that $[b_i]^{\prec_{B_0}} = \tilde{c_i} \vee b$.

Now let $(b, \tilde{c_1}, \tilde{c_2}, 1)$ be a signature of the first kind in $(B_0, \prec_{B_0})$, and suppose that there exist $b_1, b_2 \in B \setminus \{0\}$ such that $b = b_1 \vee b_2$, $b_1 \wedge b_2 = 0$, $b_1 \not\prec_B \neg b_2$ and $[b_i]^{\prec_{B_0}} = \tilde{c_i} \vee b$ for $i \in \{1, 2\}$. Then, by definition of $[b_i]^{\prec_{B_0}}$ and by axiom *(S2)*, it follows that $b_i \prec_B [b_i]^{\prec_{B_0}} = \tilde{c_i} \vee b$. Moreover, suppose by contradiction that there exists $a \in B_0 \setminus \{0\}$ such that $a \leq \tilde{c_i}$ and $b_i \prec_B \neg a$. Hence, by definition of $[b_i]^{\prec_{B_0}}$, we have that $\tilde{c_i} \vee b = [b_i]^{\prec_{B_0}} \leq \neg a$. So $a \leq \tilde{c_i} \leq \tilde{c_i} \vee b \leq \neg a$, and then $a = a \wedge a \leq \neg a \wedge a = 0$, i. e., $a = 0$, which is a contradiction.

Let $(b, \tilde{c_1}, \tilde{c_2}, 0)$ be a signature of the second kind in $(B_0, \prec_{B_0})$, and suppose that there

[20] If $y \in \bar{f}^{-1}(f(z_2))$, then $f(\tilde{f}(y)) = \bar{f}(y) = f(z_2)$. Since $z_2 \in X_C \setminus \{x, x'\}$, it follows that $\tilde{f}(y) = z_2$, i. e., $y \in \tilde{f}^{-1}(z_2)$. So $\bar{f}^{-1}(f(z_2)) \subseteq \tilde{f}^{-1}(z_2)$. Conversely, if $y \in \tilde{f}^{-1}(z_2)$, then $\tilde{f}(y) = z_2$, so $\bar{f}(y) = f(\tilde{f}(y)) = f(z_2)$, i. e., $y \in \bar{f}^{-1}(f(z_2))$. Hence $\tilde{f}^{-1}(z_2) \subseteq \bar{f}^{-1}(f(z_2))$.

[21] Observe that, by axiom *(S6)*, $[b_1 \prec_B \neg b_2 \Leftrightarrow b_2 \prec_B \neg b_1]$, so $[b_1 \not\prec_B \neg b_2 \Leftrightarrow b_2 \not\prec_B \neg b_1]$.

exist $b_1, b_2 \in B \setminus \{0\}$ such that $b = b_1 \vee b_2$, $b_1 \wedge b_2 = 0$, $b_1 \prec_B \tilde{c_1} \vee b_1$ and $b_2 \prec_B \tilde{c_2} \vee b_2$. Moreover, suppose that for every $a \in B_0 \setminus \{0\}$ such that $a \leq \tilde{c_i}$ ($i \in \{1,2\}$), it holds that $b_i \not\prec_B \neg a$. Since for $i \in \{1,2\}$ $b_i \prec_B \tilde{c_i} \vee b_i \leq \tilde{c_i} \vee b$, by axiom *(S4)* we have that $b_i \prec_B \tilde{c_i} \vee b \in B_0$. Hence, by definition of $[b_i]^{\prec_{B_0}}$, we have that $[b_i]^{\prec_{B_0}} \leq \tilde{c_i} \vee b$. Suppose by contradiction that $[b_i]^{\prec_{B_0}} < \tilde{c_i} \vee b$. Then, since b is an atom of B_0, either $[b_i]^{\prec_{B_0}} \leq \tilde{c_i}$ or $\exists a < \tilde{c_i}$ such that $[b_i]^{\prec_{B_0}} = a \vee b$. In the first case, by axioms *(S4)* and *(S3)* we have that $b_i \prec_B \tilde{c_i} \leq \tilde{c_1} \vee \tilde{c_2} \Rightarrow b_i \prec_B \tilde{c_1} \vee \tilde{c_2} \Rightarrow b = b_1 \vee b_2 \prec_B \tilde{c_1} \vee \tilde{c_2}$. However, $\tilde{c_1} \vee \tilde{c_2} < \tilde{c_1} \vee \tilde{c_2} \vee b$, being $\tilde{c_i} \wedge b = 0$ by definition of signature, so the fact that $b \prec_B \tilde{c_1} \vee \tilde{c_2}$ contradicts the definition of signature. In the second case, if $i = 1$, by axioms *(S2)* and *(S4)*, $b_1 \prec_B [b_i]^{\prec_{B_0}} = a \vee b \leq a \vee \tilde{c_2} \vee b \Rightarrow b_1 \prec_B a \vee \tilde{c_2} \vee b$, and we also have that $b_2 \prec_B \tilde{c_2} \vee b \leq a \vee \tilde{c_2} \vee b \Rightarrow b_2 \prec_B a \vee \tilde{c_2} \vee b$. Therefore, by axiom *(S3)*, $b = b_1 \vee b_2 \prec_B a \vee \tilde{c_2} \vee b < \tilde{c_1} \vee \tilde{c_2} \vee b$, but this contradicts the definition of signature. Similarly if $i = 2$. So we can conclude that $[b_i]^{\prec_{B_0}} = \tilde{c_i} \vee b$. Moreover, since we are assuming that $b_1 \prec_B \tilde{c_1} \vee b_1$ and $\tilde{c_1} \vee b_1 \leq \neg b_2$ (being $b_1 \wedge b_2 = 0$ and $b = b_1 \vee b_2$ by assumption, and $\tilde{c_i} \wedge b = 0$ by definition of signature), by axiom *(S4)* we have that $b_1 \prec_B \neg b_2$.

Let $(b, \tilde{c_1}, \tilde{c_2}, 0)$ be a signature of the second kind in $(B_0, \prec_{B_0})$, and suppose that there exist $b_1, b_2 \in B \setminus \{0\}$ such that $b = b_1 \vee b_2$, $b_1 \wedge b_2 = 0$, $[b_i]^{\prec_{B_0}} = \tilde{c_i} \vee b$ for $i \in \{1,2\}$ and $b_1 \prec_B \neg b_2$. Since by axiom *(S2)* we have that $b_i \prec_B [b_i]^{\prec_{B_0}}$, again by axiom *(S2)* we have that $b_1 \prec_B (\tilde{c_1} \vee b) \wedge \neg b_2 = (\tilde{c_1} \wedge \neg b_2) \vee (b_1 \wedge \neg b_2) \vee (b_2 \wedge \neg b_2) = \tilde{c_1} \vee b_1 \vee 0 = \tilde{c_1} \vee b_1$, being $b_1 \wedge b_2 = 0$ and $b = b_1 \vee b_2$ by assumption, and $\tilde{c_i} \wedge b = 0$ by definition of signature. Similarly, for $i = 2$, we have that $b_2 \prec_B \tilde{c_2} \vee b_2$, because $b_1 \prec_B \neg b_2 \Rightarrow b_2 \prec_B \neg b_1$ by axiom *(S6)*. Moreover, suppose by contradiction that there exists $a \in B_0 \setminus \{0\}$ such that $a \leq \tilde{c_i}$ and $b_i \prec_B \neg a$. Hence, by definition of $[b_i]^{\prec_{B_0}}$, we have that $\tilde{c_i} \vee b = [b_i]^{\prec_{B_0}} \leq \neg a$. So $a \leq \tilde{c_i} \leq \tilde{c_i} \vee b \leq \neg a$, and then $a = a \wedge a \leq \neg a \wedge a = 0$, i. e., $a = 0$, which is a contradiction. □

Remark 4.3.16. The Corollary 4.3.15.1 provides an infinite axiomatization of the model completion of the theory of the contact algebras. In fact, we can obtain an axiom (A_n) for every $n \in \mathbb{N}$ in the following way: given a contact algebra $(B, \prec_B)$, we consider all its subalgebras of a fixed cardinality $n \in \mathbb{N}$. Observe that the number of subalgebras of $(B, \prec_B)$ having cardinality n is finite: by definition of contact algebra, B is a Boolean algebra, so B is isomorphic (as a Boolean algebra) to $\mathcal{P}(X)$ for a certain finite set X, and moreover, since the cardinality of B is finite, there are finitely many ways to define the relation $\prec_B \subseteq B \times B$. We denote with $B_0^n, B_1^n, ..., B_m^n$ these subalgebras, and with $\Delta_{B_0^n}, ..., \Delta_{B_m^n}$ the conjuction of the formulas $\varphi(x_1, ..., x_n)$ contained in their diagrams. We can then consider the following:

(A_n) $\forall x_1, ..., x_n \bigwedge_{i=1,...,m} [[(\Delta_{B_i^n} \wedge \bigwedge_{j=1,...,n} (0 \leq x_j \leq x_1 \to (x_j = 0) \vee (x_j = x_1)) \wedge ([(x_1 \prec x_k) \wedge \bigwedge_{i=1,...,n} (x_1 \prec x_i \to x_k \leq x_i)] \to x_k \wedge \neg x_1 = x_2 \vee x_3)] \to \exists y_1, y_2, y_3, y_4[(x_1 = y_1 \vee y_2) \wedge (y_1 \wedge y_2 = 0) \wedge ((y_1 \prec x_2 \vee x_1) \wedge \bigwedge_{i=1,...,n} (y_1 \prec x_i \to x_2 \vee x_1 \leq x_i)) \wedge ((y_2 \prec$

$x_3 \vee x_1) \wedge \bigwedge_{i=1,\ldots,n} (y_2 \prec x_i \rightarrow x_3 \vee x_1 \leq x_i)) \wedge \neg(y_1 \prec \neg y_2) \wedge (x_1 = y_3 \vee y_4) \wedge (y_3 \wedge y_4 = 0) \wedge ((y_3 \prec x_2 \vee x_1) \wedge \bigwedge_{i=1,\ldots,n} (y_3 \prec x_i \rightarrow x_2 \vee x_1 \leq x_i)) \wedge ((y_4 \prec x_3 \vee x_1) \wedge \bigwedge_{i=1,\ldots,n} (y_4 \prec x_i \rightarrow x_3 \vee x_1 \leq x_i)) \wedge (y_3 \prec \neg y_4)]]$

so that $x_1 = b, x_2 = \tilde{c_1}, x_2 = \tilde{c_2},\ y_1 = b_1,\ y_2 = b_2$ (considering the part of the statement about the signature of the first kind), and $y_3 = b_1,\ y_4 = b_2$ (considering the part of the statement about the signature of the second kind).

Observe that some axioms are trivially tautologies. In fact, the finite subalgebras of $(B, \prec_B)$ can only have cardinalities of the kind 2^n for a certain $n \in \mathbb{N}$, being every contact algebra a Boolean algebra. Therefore, if $k \neq 2^n$ for a certain $n \in \mathbb{N}$, the conjunction $\bigwedge_{i=1,\ldots,m}$ at the beginning of the axiom (A_k) is a conjuction of tautologies (because $(B, \prec_B)$ doesn't have subalgebras of cardinality k), and so (A_k) is a tautology.

Chapter 5

Relation between the model completion and the admissible rules

In this chapter, we first recall from [5] the deductive system called *symmetric strict implication calculus*. Then, in section 5.3, we present our contribution: we specify the relation between the model completion of the theory of contact algebras and the rules that are admissible in this system.

5.1 The strict implication calculus

In this section, we introduce the strict implication calculus. We will consider the language of classical propositional logic: we will regard $\wedge$, $\neg$ and $\rightsquigarrow$ as primitive connectives, where $\rightsquigarrow$ is a binary connective called *strict implication*. $\top$, $\bot$, $\vee$, $\rightarrow$, $\leftrightarrow$ are the usual abbreviations, while $\Box\varphi$ is the abbreviation for $\top \rightsquigarrow \varphi$. We provide the following definitions:

Definition 5.1.1. Given a strict implication algebra $(B, \rightsquigarrow)$, a *valuation* on $(B, \rightsquigarrow)$ is an assignment of elements of B to propositional letters of the language $\mathcal{L}$ that we are working with: this assignment can be extended to all formulas of $\mathcal{L}$ in the usual way. A valuation v on $(B, \rightsquigarrow)$ *satisfies* a formula φ if $v(\varphi) = 1$: in this case, we write $(B, \rightsquigarrow, v) \models \varphi$. If all the valuations on $(B, \rightsquigarrow)$ satisfy φ, then we say that $(B, \rightsquigarrow)$ *validates* φ, and write $(B, \rightsquigarrow) \models \varphi$. If Γ is a set of formulas, we write $(B, \rightsquigarrow) \models \Gamma$ if $(B, \rightsquigarrow) \models \varphi$ for every $\varphi \in \Gamma$.

Definition 5.1.2. Suppose that $\mathcal{U} \subseteq$**SIA**, φ is a formula and Γ is a set of formulas. We say that φ is a *semantic consequence* of Γ over $\mathcal{U}$, and write $\Gamma \models_{\mathcal{U}} \varphi$, if for each $(B, \rightsquigarrow) \in \mathcal{U}$ and each valuation v on $(B, \rightsquigarrow)$, if $v(\gamma) = 1$ for each $\gamma \in \Gamma$, then $v(\varphi) = 1$.

Then we can finally define:

Definition 5.1.3. The *strict implication calculus* SIC is the derivation system containing:

- all the theorems of the classical propositional calculus CPC

- the following axiom schemes:

  (A1) $(\bot \rightsquigarrow \varphi) \wedge (\varphi \rightsquigarrow \top)$

  (A2) $[(\varphi \vee \psi) \rightsquigarrow \chi] \leftrightarrow [(\varphi \rightsquigarrow \chi) \wedge (\psi \rightsquigarrow \chi)]$

  (A3) $[\varphi \rightsquigarrow (\psi \wedge \chi)] \leftrightarrow [(\varphi \rightsquigarrow \psi) \wedge (\varphi \rightsquigarrow \chi)]$

  (A4) $(\varphi \rightsquigarrow \psi) \rightarrow (\varphi \rightarrow \psi)$

  (A8) $\Box\varphi \rightarrow \Box\Box\varphi$

  (A9) $\neg\Box\varphi \rightarrow \Box\neg\Box\varphi$

  (A10) $(\varphi \rightsquigarrow \psi) \leftrightarrow \Box(\varphi \rightsquigarrow \psi)$

  (A11) $\Box\varphi \rightarrow (\neg\Box\varphi \rightsquigarrow \bot)$

and is closed under the following inference rules:

(MP) $\dfrac{\varphi \quad \varphi \rightarrow \psi}{\psi}$

(N) $\dfrac{\varphi}{\Box\varphi}$

So now we can also define the following:

Definition 5.1.4. A *proof* of a formula φ from a set of formulas Γ is a finite sequence $\psi_1, ..., \psi_n$ such that $\psi_n = \varphi$ and each ψ_i is in Γ, or is an instance of an axiom of SIC, or is obtained from ψ_j, ψ_k for some $j, k < i$ by applying *(MP)*, or is obtained from ψ_j for some $j < i$ by applying *(N)*. The elements of Γ are called *assumptions*.

If there is a proof of φ from Γ, then we say that φ is *derivable in* SIC *from* Γ, and write $\Gamma \vdash_{\mathsf{SIC}} \varphi$.

If $\Gamma = \emptyset$, then we say that φ is *derivable in* SIC and write $\vdash_{\mathsf{SIC}} \varphi$.

We now state the following remarkable results about SIC, which will be useful later (see [5, Theorem 4.4, Proposition 4.5, Proposition 4.11]):

Theorem 5.1.5. *For any set of formulas Γ and for any formulas φ, ψ, we have that:*

$$\Gamma \cup \{\varphi\} \vdash_{\mathsf{SIC}} \psi \Leftrightarrow \Gamma \vdash_{\mathsf{SIC}} \Box\varphi \rightarrow \psi$$

Proposition 5.1.6. *SIC is strongly sound and complete with respect to* ***SIA****. That is, for a set of formulas Γ and a formula φ, $\Gamma \vdash_{\mathsf{SIC}} \varphi$ if and only if $\Gamma \models_{\boldsymbol{SIA}} \varphi$.*

Theorem 5.1.7. *For a set of formulas Γ and a formula φ, the following holds:*

$$\Gamma \vdash_{\mathsf{SIC}} \varphi \Leftrightarrow \Gamma \models_{\boldsymbol{SIA}} \varphi \Leftrightarrow \Gamma \models_{\boldsymbol{RSub}} \varphi$$

5.2 The symmetric strict implication calculus

We now modify the strict implication calculus that we have introduced above. We start from the following definition:

Definition 5.2.1. We call a strict implication algebra $(B, \rightsquigarrow)$ *symmetric* if it satisfies the axiom *(I5)*[1]. We denote with $\mathsf{S^2IA}$ the variety of symmetric strict implication algebras.

Definition 5.2.2. The *symmetric strict implication calculus* $\mathsf{S^2IC}$ is obtained from the strict implication calculus SIC by postulating the following axiom:

(A5) $(\varphi \rightsquigarrow \psi) \leftrightarrow (\neg\psi \rightsquigarrow \neg\varphi)$

We have the following important results about $\mathsf{S^2IC}$ (see [5, Theorem 5.2, Theorem 5.8, Theorem 5.10]):

Theorem 5.2.3. *For a set of formulas* Γ *and a formula* φ*,* $\Gamma \vdash_{\mathsf{S^2IC}} \varphi \Leftrightarrow \Gamma \models_{\mathsf{S^2IA}} \varphi \Leftrightarrow \Gamma \models_{\mathbf{Con}} \varphi$*, where* **Con** *is the class of contact algebras.*

Theorem 5.2.4. $\mathsf{S^2IC}$ *is strongly sound and complete with respect to* **Com***, which is the class of compingent algebras, i. e., for a set of formulas* Γ *and a formula* φ*, we have that* $\Gamma \vdash_{\mathsf{S^2IC}} \varphi \Leftrightarrow \Gamma \models_{\mathbf{Com}} \varphi$*.*

Theorem 5.2.5. *The system* $\mathsf{S^2IC}$ *is strongly sound and complete with respect to* **DeV***, which is the class of de Vries algebras.*

Moreover, by de Vries duality [17], every de Vries algebra is isomorphic to the de Vries algebra of some compact Hausdorff space, being the de Vries algebra of a compact Hausdorff space X the pair $(\mathcal{RO}(X), \prec)$, where $\mathcal{RO}(X)$ is the complete Boolean algebra of regular open subsets of X and $U \prec V$ iff $\mathsf{Cl}(U) \subseteq V$. This allows us to define topological semantics for our language, in the following way:

Definition 5.2.6. A *compact Hausdorff model* is a pair (X, v), where X is a compact Hausdorff space and v is a valuation assigning a regular open set to each propositional letter.

If $\rightsquigarrow$ is the strict implication corresponding to $\prec$, then the formulas of our language are interpreted in $(\mathcal{RO}(X), \rightsquigarrow) \in \mathbf{DeV}$. So we also have the following result, which follows from Theorem 5.2.5 and de Vries duality (see [5, Theorem 5.10]):

Theorem 5.2.7. *The system* $\mathsf{S^2IC}$ *is strongly sound and complete with respect to compact Hausdorff models.*

[1] We introduced it in the Remark 4.1.15.

Moreover, from Theorems 5.2.3 and 5.2.5, we know that $\mathsf{S^2IC}$ is strongly sound and complete with respect to both **Con** and **DeV**. Hence neither the axiom *(I6)* nor *(I7)*[2] is expressible in our logic. We are then going to show that we can express *(I6)* and *(I7)* in our propositional language by means of Π_2-rules. In order to do that, we first rewrite *(I6)* and *(I7)* in the following form:

(Π6) $\forall x_1, x_2, y\Big(x_1 \rightsquigarrow x_2 \not\leq y \to \exists z\ (x_1 \rightsquigarrow z) \wedge (z \rightsquigarrow x_2) \not\leq y\Big)$

(Π7) $\forall x, y\Big(x \not\leq y \to \exists z\ z \wedge (z \rightsquigarrow x) \not\leq y\Big)$

(Π6) and (Π7) are equivalent respectively to *(I6)* and *(I7)*, in the sense that the following lemma holds (see [5, Lemma 6.1]):

Lemma 5.2.8. *Let* $(B, \rightsquigarrow) \in$**RSub**.

1. $(B, \rightsquigarrow) \models (I6)$ *iff* $(B, \rightsquigarrow) \models (\Pi6)$.
2. $(B, \rightsquigarrow) \models (I7)$ *iff* $(B, \rightsquigarrow) \models (\Pi7)$.

Now we are going to show that, in this calculus, $\forall\exists$-statements can be expressed by means of non-standard rules, called Π_2-rules. So we first define the following:

Definition 5.2.9. A Π_2-*rule* is a rule of the form

$$(\rho)\ \frac{F(\bar{\varphi}, \bar{p}) \to \chi}{G(\bar{\varphi}) \to \chi}$$

where F, G are formulas, $\bar{\varphi}$ is a tuple of formulas, χ is a formula, and $\bar{p}$ is a tuple of propositional letters which do not occur in $\bar{\varphi}$ and χ.

To each Π_2-rule ρ, we associate the following $\forall\exists$-statement:

$$\Pi(\rho) := \forall \bar{x}, z(G(\bar{x}) \not\leq z \to \exists \bar{y}(F(\bar{x}, \bar{y}) \not\leq z))$$

Definition 5.2.10. A strict implication algebra $(B, \rightsquigarrow)$ *validates* a Π_2-rule ρ if $(B, \rightsquigarrow)$ satisfies $\Pi(\rho)$. We denote this situation with $(B, \rightsquigarrow) \models \rho$.

Consider for example the two following Π_2-rules:

$$(\rho 6)\ \frac{(\varphi \rightsquigarrow p) \wedge (p \rightsquigarrow \psi) \to \chi}{(\varphi \rightsquigarrow \psi) \to \chi} \text{ and } (\rho 7)\ \frac{p \wedge (p \rightsquigarrow \varphi) \to \chi}{\varphi \to \chi}$$

It is easy to see that $\Pi(\rho 6) = (\Pi 6)$ and $\Pi(\rho 7) = (\Pi 7)$. Hence, by Lemma 5.2.8, for each $(B, \rightsquigarrow) \in$**RSub** we have that $(B, \rightsquigarrow) \models (\rho 6)$ iff $(B, \rightsquigarrow) \models (I6)$, and $(B, \rightsquigarrow) \models (\rho 7)$ iff $(B, \rightsquigarrow) \models (I7)$.

[2] These two axioms correspond to *(S7)* and *(S8)*, as specified in the Remark 4.1.15.

Definition 5.2.11. Let Σ be a set of Π_2-rules. For a set of formulas Γ and a formula φ, we say that φ is *derivable* from Γ in SIC using the Π_2-rules in Σ, and we write $\Gamma \vdash_\Sigma \varphi$, if there is a proof $\psi_1, ..., \psi_n$ such that $\psi_n = \varphi$ and each ψ_i is in Γ, an instance of an axiom of SIC, obtained either by (MP) or (N) from some previous ψ_j's, or there is $j < i$ such that ψ_i is obtained from ψ_j by an application of one of the Π_2-rules $\rho \in \Sigma$; that is, $\psi_j = F(\bar{\xi}, \bar{p}) \to \chi$ and $\psi_i = G(\bar{\xi}) \to \chi$, where F, G are formulas, $\bar{\xi}$ is a tuple of formulas, χ is a formula, and $\bar{p}$ is a tuple of propositional letters not occurring in $\bar{\xi}$, χ or any of the formulas from Γ that are used in $\psi_1, ..., \psi_{i-1}$ as assumptions.

Now let $\mathcal{S}$ be the system obtained by adding the Π_2-rules $\{\rho_n \mid n \in \mathbb{N}\}$ to SIC, and let $\mathcal{U}$ be the inductive[3] subclass of **RSub** defined by the $\forall\exists$-statements $\{\Pi(\rho_n) \mid n \in \mathbb{N}\}$. We have that $\mathcal{S}$ is strongly sound and complete with respect to $\mathcal{U}$ (for a proof of this result, see [5, Theorem 6.6]):

Theorem 5.2.12. *Let $\mathcal{S} = \mathsf{SIC} + \{\rho_n \mid n \in \mathbb{N}\}$, let $\mathcal{U}$ be the inductive subclass of* **RSub** *defined by $\{\Pi(\rho_n) \mid n \in \mathbb{N}\}$, and let $\mathcal{V}$ be the variety generated by $\mathcal{U}$. For a set of formulas Γ and a formula φ, we have:*

1. $\Gamma \vdash_{\mathcal{S}} \varphi \Leftrightarrow \Gamma \models_{\mathcal{U}} \varphi$.
2. $\vdash_{\mathcal{S}} \varphi \Leftrightarrow \models_{\mathcal{V}} \varphi$.

It follows that the class of subdirectly irreducible algebras in $\mathsf{S^2IA}$ validating a set of Π_2-rules is an inductive subclass of **RSub**. The converse is also true: for every inductive subclass $\mathcal{U}$ of **RSub**, there is a set of Π_2-rules $\{\rho_i \mid i \in I\}$ such that $\mathcal{S} = \mathsf{SIC} + \{\rho_i \mid i \in I\}$ is strongly sound and complete with respect to $\mathcal{U}$. To obtain such a set of Π_2-rules, it is sufficient to show that every $\forall\exists$-statement is equivalent to a statement of the form $\Pi(\rho)$ for some Π_2-rule ρ. Without loss of generality we may assume that all atomic formulas $\Phi(\bar{x}, \bar{y})$ are of the form $t(\bar{x}, \bar{y}) = 1$ for some term t. We then define:

Definition 5.2.13. Given a quantifier-free first-order formula $\Phi(\bar{x}, \bar{y})$, we associate with the tuples of variables $\bar{x}$, $\bar{y}$ the tuples of propositional letters $\bar{p}$, $\bar{q}$, and define the formula $\Phi^*(\bar{p}, \bar{q})$ in the language of SIC as follows:

$$(t(\bar{x}, \bar{y}) = 1)^* = \Box t(\bar{p}, \bar{q})$$

$$(\neg\Psi)^*(\bar{x}, \bar{y}) = \neg\Psi^*(\bar{p}, \bar{q})$$

$$(\Psi_1(\bar{x}, \bar{y}) \wedge \Psi_2(\bar{x}, \bar{y}))^* = \Psi_1^*(\bar{p}, \bar{q}) \wedge \Psi_2^*(\bar{p}, \bar{q})$$

So we now have the following result (see [5, Lemma 6.8]):

[3] A class is *inductive* if it is closed under unions of chains. By the Chang-Łoś-Suzko Theorem, which we recalled in the chapter about preliminaries, the elementary classes corresponding to $\forall\exists$-statements are inductive classes.

Lemma 5.2.14. *Let* $(B, \rightsquigarrow) \in$**RSub** *and let* $\Phi(\bar{x}, \bar{y})$ *be a quantifier-free formula. It holds that:*

1. $(B, \rightsquigarrow)$ *satisfies* $\Phi(\bar{x}, \bar{y})$ *if and only if* $(B, \rightsquigarrow)$ *satisfies the formula* $\Phi^*(\bar{p}, \bar{q})$.
2. $(B, \rightsquigarrow)$ *satisfies* $\forall\bar{x}\exists\bar{y}\Phi(\bar{x}, \bar{y})$ *if and only if* $(B, \rightsquigarrow)$ *satisfies* $\forall\bar{x}, z(1 \not\leq z \to \exists\bar{y}(\Phi^*(\bar{x}, \bar{y}) \not\leq z))$.

As a consequence, an arbitrary Π_2-statement $\forall\bar{x}\exists\bar{y}\Phi(\bar{x}, \bar{y})$ is equivalent to the Π_2-statement associated to the Π_2-rule

$$(\rho_\Phi) \quad \frac{\Phi^*(\bar{\varphi}, \bar{p}) \to \chi}{\chi}$$

Therefore, by Theorem 5.2.12, we obtain the following result (see [5, Theorem 6.9]):

Theorem 5.2.15. *If* T *is a* Π_2*-theory of first-order logic axiomatizing an inductive subclass* $\mathcal{U}$ *of* **RSub***, then the system* $\mathcal{S} = \mathsf{SIC} + \{\rho_\Phi \mid \Phi \in T\}$ *is strongly sound and complete with respect to* $\mathcal{U}$*; that is, for a set of formulas* Γ *and a formula* φ*, we have:*

$$\Gamma \vdash_{\mathcal{S}} \varphi \Leftrightarrow \Gamma \models_{\mathcal{U}} \varphi.$$

Now we can define the following:

Definition 5.2.16. A rule ρ is *admissible* in a system $\mathcal{S}$ if, for each formula φ, $\vdash_{\mathcal{S}+\rho} \varphi$ implies $\vdash_{\mathcal{S}} \varphi$.

We have the following important results about the admissible rules (see [5, Lemma 6.11, Theorem 6.12, Corollary 6.13]):

Lemma 5.2.17. *If a* Π_2*-rule*

$$(\rho) \quad \frac{F(\bar{\varphi}, \bar{p}) \to \chi}{G(\bar{\varphi}) \to \chi}$$

is admissible in $\mathcal{S} \supseteq \mathsf{SIC}$*, then* $\Gamma \vdash_{\mathcal{S}+\rho} \varphi \Leftrightarrow \Gamma \vdash_{\mathcal{S}} \varphi$.

Theorem 5.2.18 (Admissibility Criterion)**.** *The following statements hold:*

1. *A* Π_2*-rule* ρ *is admissible in* SIC *if and only if, for each* $(B, \rightsquigarrow) \in$**RSub***, there is* $(C, \rightsquigarrow) \in$**RSub** *such that* $(B, \rightsquigarrow)$ *is a substructure of* $(C, \rightsquigarrow)$ *and* $(C, \rightsquigarrow) \models \Pi(\rho)$.
2. *A* Π_2*-rule* ρ *is admissible in* $\mathsf{S^2IC}$ *if and only if, for each* $(B, \rightsquigarrow) \in$**Con***, there is* $(C, \rightsquigarrow) \in$**Con** *such that* $(B, \rightsquigarrow)$ *is a substructure of* $(C, \rightsquigarrow)$ *and* $(C, \rightsquigarrow) \models \Pi(\rho)$.

Corollary 5.2.18.1. *The following statements hold:*

1. $(\rho6)$ *is admissible in* SIC *and in* $\mathsf{S^2IC}$.
2. $(\rho7)$ *is admissible in* SIC *and in* $\mathsf{S^2IC}$.

5.3 Model completion and admissible rules

Now, in the Remark 2.1.26 we observed that the model completion of any universal theory can be axiomatized by means of $\forall\exists$-axioms, provided it exists: in the previous chapter we found such an axiomatization for the model completion of the theory of the contact algebras. Therefore it is natural to ask if there is any relation between the axioms of such a model completion and the Π_2-rules which are admissible in $\mathsf{S^2IC}$. So let SCON be the theory of symmetric strict implication algebras which are also simple: these algebras correspond to contact algebras $(B, \prec)$, according to the Lemma 4.1.10, the Remark 4.1.15 and the Proposition 4.2.1.[4] Let SCON* denote the model completion of SCON. The following result answers our question:

Theorem 5.3.1. *A Π_2-rule is admissible in $\mathsf{S^2IC}$ if and only if* SCON$^* \models \Pi(\rho)$.

Proof. According to Theorem 5.2.18 (part *2.*), we have to show that SCON$^* \models \Pi(\rho)$ holds if and only if every algebra $\mathcal{B}$ which is a model of CON can be embedded into some $\mathcal{C}$ which is also a model of CON and satisfies $\Pi(\rho)$. This can be shown by using the fact that $\Pi(\rho)$ is a Π_2-sentence, recalling that the models of SCON* are just the existentially closed models of SCON, according to the Proposition 2.1.25.

Suppose first that SCON$^* \models \Pi(\rho)$ holds and let $\mathcal{B}$ be any model of SCON. Then, by Proposition 2.1.21, $\mathcal{B}$ embeds into a model of SCON* as required.

Conversely, suppose that every algebra $\mathcal{B}$ which is a model of CON can be embedded into some $\mathcal{C}$ which is also a model of CON and satisfies $\Pi(\rho)$. Pick a model $\mathcal{B}$ of SCON* and let $\Pi(\rho)$ be $\forall \underline{x} \exists \underline{y} H(\underline{x}, \underline{y})$ where H is quantifier free. Let $\underline{b}$ be a tuple from the support of $\mathcal{B}$; then we have $\mathcal{C} \models \exists \underline{y} H(\underline{b}, \underline{y})$ for some extension $\mathcal{C}$ of $\mathcal{B}$. Being $\mathcal{B}$ existentially closed, this immediately entails $\mathcal{B} \models \exists \underline{y} H(\underline{b}, \underline{y})$. Since the $\underline{b}$ were arbitrary, we conclude $\mathcal{B} \models \Pi(\rho)$, as required.

□

Observe that then, by Theorem 5.3.1 and Corollary 5.2.18.1, every existentially closed contact algebra is a compingent algebra. Moreover, Theorem 5.3.1 implies that checking whether a Π_2-rule is admissible or not amounts to checking whether SCON$^* \models \Pi(\rho)$ holds or not. This can be done by means of the quantifier elimination in SCON*: we are now going to show that quantifier elimination is effective. In order to prove this result, we first recall the following definitions (see [10]):

Definition 5.3.2. A *language* (or *type*) of algebras is a set $\mathcal{F}$ of function symbols such that a nonnegative integer n is assigned to each member f of $\mathcal{F}$: this integer is called the *arity* (or *rank*) of f, and f is said to be an n-ary function symbol. The subset of n-ary function symbols in $\mathcal{F}$ is denoted by $\mathcal{F}_n$.

[4] According to the Theorem 5.2.3, $\mathsf{S^2IC}$ is strongly sound and complete with respect to these algebras.

Definition 5.3.3. If $\mathcal{F}$ is a language of algebras, then an *algebra* $\mathbf{A}$ of *type* $\mathcal{F}$ is an ordered pair $\langle A, F\rangle$ where A is a nonempty set and F is a family of finitary operations on A indexed by the language $\mathcal{F}$ such that corresponding to each n-ary function symbol f in $\mathcal{F}$ there is an n-ary operation $f^{\mathbf{A}}$ on A. The set A is called the *universe* (or *underlying set*) of $\mathbf{A} = \langle A, F\rangle$, and the $f^{\mathbf{A}}$'s are called the *fundamental operations* of $\mathbf{A}$.

Definition 5.3.4. Let X be a set of (distinct) objects called *variables*. Let $\mathcal{F}$ be a type of algebras. The set $T(X)$ of *terms* of type $\mathcal{F}$ over X is the smallest set such that

1. $X \cup \mathcal{F}_0 \subseteq T(X)$
2. if $p_1, ..., p_n \in T(X)$ and $f \in \mathcal{F}_n$, then the "string" $f(p_1, ..., p_n) \in T(X)$

Definition 5.3.5. Given $\mathcal{F}$ and X, if $T(X) \neq \emptyset$ then the *term algebra* of type $\mathcal{F}$ over X, written $\mathbf{T}(X)$, has as its universe the set $T(X)$, and the fundamental operations satisfy $f^{\mathbf{T}(X)} : \langle p_1, ..., p_n\rangle \longmapsto f(p_1, ..., p_n)$ for $f \in \mathcal{F}_n$ and $p_1 \in T(X)$, $1 \leq i \leq n$.

Definition 5.3.6. We introduce the following operators mapping classes of algebras to classes of algebras (all of the same type):
$\mathbf{A} \in I(K)$ iff $\mathbf{A}$ is isomorphic to some member of K
$\mathbf{A} \in S(K)$ iff $\mathbf{A}$ is a subalgebra of some member of K

Definition 5.3.7. Let K be a family of algebras of type $\mathcal{F}$. Given a set X of variables, define the congruence $\theta_K(X)$ on $\mathbf{T}(X)$ by $\theta_K(X) := \bigcap \Phi_K(X)$, where $\Phi_K(X) = \{\phi \in Con\mathbf{T}(X) \mid \mathbf{T}(X)/\phi \in IS(K)\}$. We also define the *$K$-free algebra over $\overline{X}$* by $F_K(\overline{X}) := \mathbf{T}(X)/\theta_K(X)$, where $\overline{X} = X/\theta_K(X)$.

We also have the following theorem (see [10, Theorem 10.15]):

Theorem 5.3.8. *A variety V is locally finite if and only if the following condition holds: $card(X) < \infty \Rightarrow card(F_V(\overline{X})) < \infty$.*

The following result provides an equivalent definition of locally finite variety:

Theorem 5.3.9. *A variety V is locally finite if and only if the theory T associated to that variety satisfies the following condition: T is finite and, for every finite set of free constants $\underline{a}$, there are finitely many ground terms $t_1, ..., t_{k_{\underline{a}}}$ in the language $\mathcal{L} \cup \{\underline{a}\}$ (where $\mathcal{L}$ is the language of T) such that for every further ground term u, we have that $T \models u = t_i$ (for some $i \in \{1, ..., k_{\underline{a}}\}$).*

Proof. Suppose first that V is locally finite. Then the thesis follows from Theorem 5.3.8. In fact, given two terms t and u, $T \models t(x_1, ..., x_n) = u(x_1, ..., x_n)$ if and only if the two terms are equal in the free algebra generated by $x_1, ..., x_n$, which is finite by hypothesis and by Theorem 5.3.8.

Conversely, if an algebra $\mathcal{B}$ of the variety V in finitely generated and its generators are

$x_1, ..., x_n$, then every element of $\mathcal{B}$ is of the kind $t(x_1, ..., x_n)$ for some term t. Hence $\mathcal{B}$ is finite by hypothesis.

□

So the Theorem 5.3.9 provides a new equivalent definition of locally finite variety: later we will refer to this new definition when considering locally finite varieties.

Going back to the quantifier elimination problem, we consider the following:

Definition 5.3.10. A *quantifier elimination procedure (QEP)* for a theory T and a set of formulas $\mathcal{G}$ is a function that computes for every $\varphi \in \mathcal{G}$ a quantifier-free formula which is T-equivalent to φ.

Fact 5.3.11. If a theory T has a (QEP) for all $\exists x\varphi$ where φ is quantifier-free, then T has a (QEP) for all formulas.

Indeed, given any formula ψ, we can find another formula of the kind $Q_1x_1...Q_nx_n\varphi$ (where Q_i is either $\forall$ or $\exists$, and φ is quantifier-free) that is logically equivalent to ψ, by using the following logical laws which allow us to shift the quantifiers:

$$\neg\forall yC \leftrightarrow \exists y\neg C$$

$$\neg\exists yC \leftrightarrow \forall y\neg C$$

$$\exists yA_1 \wedge A_2 \leftrightarrow \exists y(A_1 \wedge A_2)$$

$$\exists yA_1 \vee A_2 \leftrightarrow \exists y(A_1 \vee A_2)$$

$$\forall yA_1 \wedge A_2 \leftrightarrow \forall y(A_1 \wedge A_2)$$

$$\forall yA_1 \vee A_2 \leftrightarrow \forall y(A_1 \vee A_2)$$

where we are assuming that y doesn't occur free in A_2. If Q_n is $\exists$, we can apply the (QEP) we have in order to eliminate this quantifier, while if Q_n is $\forall$ we can observe that $\forall\varphi$ is logically equivalent to $\neg\exists\neg\varphi$ and then apply the (QEP). By induction, we can eliminate all the quantifiers $Q_1, ..., Q_n$.

Therefore, thanks to the fact which is recalled above, the following lemma is sufficient to prove that the quantifier elimination is effective in our situation, being the variety of contact algebras locally finite[5]:

Lemma 5.3.12. *The quantifier-free formula $R(\underline{x})$ provably equivalent in* SCON* *to an existential formula $\exists\underline{y}H(\underline{x},\underline{y})$ is the strongest quantifier free formula $G(\underline{x})$ implied (modulo* SCON*) by $H(\underline{x},\underline{y})$.*

[5]Referring to the statement of the lemma, since the variety of contact algebras is locally finite, we have to consider a *finite* number of quantifier free formulas $G(\underline{x})$ in order to find the right one.

Proof. Recall that SCon and SCon* prove the same quantifier-free formulae, by definition of model completion. Thus we have the following chain of equivalences:

$$\frac{\dfrac{\dfrac{\dfrac{\textsc{SCon} \vdash H(\underline{x}, \underline{y}) \to G(\underline{x})}{\textsc{SCon}^* \vdash H(\underline{x}, \underline{y}) \to G(\underline{x})}}{\textsc{SCon}^* \vdash \exists \underline{y} H(\underline{x}, \underline{y}) \to G(\underline{x})}}{\textsc{SCon}^* \vdash R(\underline{x}) \to G(\underline{x})}}{\textsc{SCon} \vdash R(\underline{x}) \to G(\underline{x})}$$

yieldying the claim.

□

The above argument yields decidability of admissibility of Π_2-rules. In order to prove this fact, it would actually have been sufficient to show that we can eliminate an existential quantifier from a *primitive* formula, which is a formula of the kind $\exists x\phi$, where ϕ is a conjunction of literals (i. e., atomic formulas or their negation). Indeed, it is a well-known fact that all logical formulas can be converted into an equivalent *disjunctive normal form*, i. e., into an equivalent formula which is a disjunction of one or more conjunctions of one or more literals. Moreover, the existential quantifier commutes with disjunction. Hence it follows that, if we can eliminate every existential quantifier from each primitive formula, then the quantifier eliminiation is effective.

5.4 A relevant example of admissible rule

We now consider the following property, which is linked to the zero-dimensionality of topological spaces:

(S9) $a \prec b$ implies $\exists c\ c \prec c$ and $a \prec c \prec b$

According to [3], we can call a de Vries algebra *zero-dimensional* if it satisfies ($S9$). If $(B, \rightsquigarrow)$ is a zero-dimensional de Vries algebra and X is the de Vries dual of $(B, \rightsquigarrow)$, it follows from de Vries duality and [3, Lemma 4.11] that X is zero-dimensional, and hence $(\mathcal{RO}(X), \rightsquigarrow)$ is a zero-dimensional de Vries algebra by [3, Lemma 4.1].

The $\forall\exists$-statement corresponding to ($S9$) is:

(Π9) $\forall x, y, z\Big(x \rightsquigarrow y \nleq z \to \exists u : (u \rightsquigarrow u) \wedge (x \rightsquigarrow u) \wedge (u \rightsquigarrow y) \nleq z\Big)$

In fact, we have the following:

Lemma 5.4.1. *Let* $(B, \rightsquigarrow) \in$**Com**. *Then* $(B, \rightsquigarrow) \models (S9)$ *if and only if* $(B, \rightsquigarrow) \models (\Pi9)$.[6]

[6]cfr [5, Lemma 6.14].

The Π_2-rule corresponding to (Π9) is:

$$(\rho 9) \quad \frac{(p \rightsquigarrow p) \wedge (\varphi \rightsquigarrow p) \wedge (p \rightsquigarrow \psi) \to \chi}{(\varphi \rightsquigarrow \psi) \to \chi}$$

We have the following result (see [5, Theorem 6.15]):

Theorem 5.4.2. *(ρ9) is admissible in* SIC *and* $\mathsf{S^2IC}$*.*

Proof. It is easy to see that the $(C, \prec)$ constructed in the proof of Lemma 4.1.13 satisfies (S9). Therefore, for admissibility in SIC, we can apply Theorem 5.2.18 (part *1.*), Lemma 5.4.1, and Lemma 4.1.13 (part *1.*); and for admissibility in $\mathsf{S^2IC}$, we can apply Theorem 5.2.18 (part *2.*), Lemma 5.4.1, and Lemma 4.1.13 (part *2.*).

□

Observe that the proof of Theorem 5.4.2 is based on the application of the Admissibility Criterion (i. e., Theorem 5.2.18). However, according to Theorem 5.3.1, for our purpose we could equivalently prove that (S9) is provable in $\mathrm{SC{\small ON}}^*$. In order to do that, according to the Lemma 5.3.12, we first compute the quantifier-free formula equivalent in $\mathrm{SC{\small ON}}^*$ to $\exists c\ c \prec c \wedge a \prec c \prec b$ by taking the conjunction of the (finitely many) quantifier-free first-order formulae $\phi(x, y)$ which are implied (modulo SCON) by $c \prec c \wedge a \prec c \prec b$: this is, up to equivalence, $a \prec b$. Now, in order to show the admissibility of (ρ9) is sufficient to observe that SCON $\models a \prec b \to a \prec b$.

As an alternative, relying on Theorem 4.3.1, we can show that (S9) is true in every existentially closed contact algebra. In order to do that, it is sufficient to enumerate all contact algebras $(B_0, \prec_{B_0})$ generated by two elements a, b such that $(B_0, \prec_{B_0}) \models a \prec b$, and to show that all such algebras can be embedded in a contact algebra $(C, \prec_C)$ generated by three elements a, b, c such that $(C, \prec_C) \models c \prec c \wedge a \prec c \prec b$ (this can be done automatically, for instance by means of a model finder tool).

Both of the above procedures are not elegant, but at least they are mechanical and do not require ingenious *ad hoc* constructions.

Open problems

We list below two of open problems related to this thesis.

At the end of Chapter 4, we gave an infinite axiomatization of the model completion of the theory of contact algebras. However, it is not clear whether there exists a *finite* axiomatization of this theory. Compactness Theorem could be useful while trying to prove that such a finite axiomatization does not exist. There are some other universal theories for which a finite axiomatization of their model completion is known. For instance, we can take into account the theory of S5-algebras, that is considered in Chapter 3. Other examples of theories having a model completion with a finite axiomatization are the theory of Brouwerian semilattices, which is studied in [11], and the theories of some varieties of Heyting algebras, which are investigated in [15]. In fact, it is known that exactly eight varieties of Heyting algebras have a model completion: an axiomatization of the model completion of the trivial variety (reduced to the one-point algebra) and of the variety of Boolean algebras was already known. In [15], an explicit and finite axiomatization of the model completion for four of the six remaining varieties (those which are locally finite) is provided.

The infinite axiomatization supplied by Corollary 4.3.15.1 should be naturally convertible into a basis for admissible Π_2-rules for the symmetric strict implication calculus, once the notion of a *basis* for admissible Π_2-rules is suitably defined. We leave this task for future research.

Appendix A

A modified duality for contact algebras

As we observed in the Remark 4.1.15, we can equivalently define the contact algebras (as objects) by using a binary operation $\rightsquigarrow$ instead of a binary relation $\prec$. However, if we modify the definition of contact algebras in this way, every injective homomorphism $h : (B_1, \rightsquigarrow_1) \to (B_2, \rightsquigarrow_2)$ has to be an embedding according to the first definition we introduced. Therefore we have to slightly modify the duality we have: this is the aim of this appendix. We then start with the following definitions:

Definition A.0.1. A *contact algebra* is a pair $(B, \rightsquigarrow)$, where B is a Boolean algebra and $\rightsquigarrow: B \times B \to B$ is a binary operation with values in $\{0, 1\}$ satisfying:

(I1) $0 \rightsquigarrow a = a \rightsquigarrow 1 = 1$

(I2) $(a \vee b) \rightsquigarrow c = (a \rightsquigarrow c) \wedge (b \rightsquigarrow c)$

(I3) $a \rightsquigarrow (b \wedge c) = (a \rightsquigarrow b) \wedge (a \rightsquigarrow c)$

(I4) $a \rightsquigarrow b \leq a \to b$

(I5) $a \rightsquigarrow b = \neg b \rightsquigarrow \neg a$

We denote with **Contact** the category whose objects are the contact algebras, and whose morphisms are the Boolean homomorphisms h satisfying the condition $h(a \rightsquigarrow b) = h(a) \rightsquigarrow h(b)$.

Remark A.0.2. Observe that, in terms of the relation $\prec$, the condition $h(a \rightsquigarrow b) = h(a) \rightsquigarrow h(b)$ can be written as $a \prec b \Leftrightarrow h(a) \prec h(b)$ (while in Chapter 4 we only required the implication $\Rightarrow$). Therefore, according to this new definition, every injective morphism is an embedding.

Definition A.0.3. We denote with **StRel** the category whose objects are pairs (X, R), where X is a Stone space and R is a reflexive and symmetric closed[1] relation on X, and

[1] A binary relation R on a topological space X is said to be *closed* if R is a closed set in the product topology on $X \times X$.

whose morphisms are either continuous (stable[2]) morphisms $f\colon (X_1, R_1) \to (X_2, R_2)$ which satisfy the condition $\forall x, y \in X_2\ [xR_2y \Leftrightarrow \exists y', \tilde{y} \in X_1$ s. t. $f(\tilde{y}) = x,\ f(y') = y\ \&\ \tilde{y}R_1y']$, or the empty function $\alpha : (\emptyset, R_\emptyset) \to (X, R)$, for every object (X, R) of **StRel**.

We can now define two controvariant functors between these categories (see [6, Definition 2.16, Definition 2.19]):

Definition A.0.4. Define $(-)^* : \mathbf{StRel} \to \mathbf{Contact}$ as follows. If (X, R) is an object of **StRel**, then $(X, R)^* := (\mathsf{Clop}(X), \rightsquigarrow)$, where $[U \rightsquigarrow V = X \Leftrightarrow R[U] \subseteq V]\ \forall U, V \in \mathsf{Clop}(X)$. If $f : (X_1, R_1) \to (X_2, R_2)$ is a morphism in **StRel**, then define $f^* := f^{-1}(-) : (\mathsf{Clop}(X_2), \rightsquigarrow_2) \to (\mathsf{Clop}(X_1), \rightsquigarrow_1)$ such that $\forall U \in \mathsf{Clop}(X_2)\ f^*(U) = f^{-1}(U)$.

The functor that we have just introduced is well-defined. In order to show this, we need the following lemma:

Lemma A.0.5. *Given a continuous stable morphism $f : (X_1, R_1) \to (X_2, R_2)$, the following conditions are equivalent:*[3]

1. $(R_1[f^{-1}(U)] \subseteq f^{-1}(V) \Leftrightarrow R_2[U] \subseteq V)\ \forall U, V \in \mathsf{Clop}(X_2)$
2. $(f(R_1[f^{-1}(U)]) \subseteq V \Leftrightarrow R_2[U] \subseteq V)\ \forall U, V \in \mathsf{Clop}(X_2)$
3. $f(R_1[f^{-1}(U)]) = R_2[U]\ \forall U \in \mathsf{Clop}(X_2)$
4. $f(R_1[f^{-1}(\{x\})]) = R_2[\{x\}]\ \forall x \in X_2$
5. $\forall x, y \in X_2\ [xR_2y \Leftrightarrow \exists y', \tilde{y} \in X_1$ *s. t.* $f(\tilde{y}) = x,\ f(y') = y\ \&\ \tilde{y}R_1y']$

So we can now prove the following:

Lemma A.0.6. *If (X, R) is an object of* **StRel**, *then $(X, R)^*$ is a contact algebra. If $f : (X_1, R_1) \to (X_2, R_2)$ is a morphism of* **StRel**, *then $f^* := f^{-1}(-) : (\mathsf{Clop}(X_2), \rightsquigarrow_2) \to (\mathsf{Clop}(X_1), \rightsquigarrow_1)$ is a morphism of* **Contact**.

Proof. Suppose that (X, R) is an object of **StRel**, and let $(X, R)^* := (\mathsf{Clop}(X), \rightsquigarrow)$, where $[U \rightsquigarrow V = X \Leftrightarrow R[U] \subseteq V]\ \forall U, V \in \mathsf{Clop}(X)$. Then we have that:

1. $R[\emptyset] = \emptyset \subseteq V$ and $R[V] \subseteq X\ \forall V \in \mathsf{Clop}(X)$, so $\emptyset \rightsquigarrow V = V \rightsquigarrow X = X\ \forall V \in \mathsf{Clop}(X)$
2. $\forall U, V, W \in \mathsf{Clop}(X)$, $R[U \cup V] = R[U] \cup R[V]$. Hence $R[U \cup V] \subseteq W \Leftrightarrow R[U] \subseteq W\ \&\ R[V] \subseteq W$. Therefore $U \cup V \rightsquigarrow W = X \Leftrightarrow (U \rightsquigarrow W = X\ \&\ V \rightsquigarrow W = X)$, i. e., $U \cup V \rightsquigarrow W = (U \rightsquigarrow W) \wedge (V \rightsquigarrow W)$

[2]A map $f\colon (X_1, R_1) \to (X_2, R_2)$ is said to be *stable* if it satisfies the condition $[xR_1y \Rightarrow f(x)R_2f(y)]$. In this case, the condition of f being stable is entailed by the condition that follows.

[3]For a proof of it, see Lemma 4.2.16.

3. $\forall U, V, W \in \mathsf{Clop}(\mathsf{X})$, $R[U] \subseteq V \cap W \iff R[U] \subseteq V \;\&\; R[U] \subseteq W$. Hence $U \rightsquigarrow V \cap W = X \iff (U \rightsquigarrow V = X \;\&\; U \rightsquigarrow W = X)$, i. e., $U \rightsquigarrow V \cap W = (U \rightsquigarrow V) \wedge (U \rightsquigarrow W)$

The fact that $(\mathsf{Clop}(X), \rightsquigarrow)$ satisfies the axioms *(I4)* and *(I5)* follows from the Lemma 4.1.10 and from the Remark 4.1.15. Hence $(X, R)^*$ is an object of **Contact**.

Now consider a morphism of **StRel** $f : (X_1, R_1) \to (X_2, R_2)$ different from the empty function α, and consider $f^* := f^{-1}(-) : (\mathsf{Clop}(X_2), \rightsquigarrow_2) \to (\mathsf{Clop}(X_1), \rightsquigarrow_1)$. It follows from Stone duality that f^* is a Boolean homomorphism. Now we have to show that $f^{-1}(U \rightsquigarrow_2 V) = f^{-1}(U) \rightsquigarrow_1 f^{-1}(V)$ $\forall U, V \in \mathsf{Clop}(X_2)$. Observe that $U \rightsquigarrow_2 V$ is either X_2 or $\emptyset$. Hence we have that $f^{-1}(U \rightsquigarrow_2 V) = X_1 \Leftrightarrow U \rightsquigarrow_2 V = X_2 \Leftrightarrow R_2[U] \subseteq V \overset{(*)}{\Leftrightarrow} R_1[f^{-1}(U)] \subseteq f^{-1}(V) \Leftrightarrow f^{-1}(U) \rightsquigarrow_1 f^{-1}(V) = X_1$, where the equivalence denoted by $(*)$ holds because f satisfies the condition *5.* of the Lemma A.0.5 (by definition of morphism of **StRel**), and so it also satisfies the condition *1.* of the same lemma. Therefore f^* is a morphism of **Contact**. Consider now the empty function $\alpha : (\emptyset, R_\emptyset) \to (X, R)$, for any object (X, R) of **StRel**. Then $\alpha^* : (\mathsf{Clop}(X), \rightsquigarrow_R) \to (\{\emptyset\}, \rightsquigarrow_{\{\emptyset\}})$ trivially satisfies the condition $\alpha^*(U \rightsquigarrow_R V) = \alpha^*(U) \rightsquigarrow_{\{\emptyset\}} \alpha^*(V)$, so it is a morphism of **Contact**.

$\square$

Definition A.0.7. Define $(-)_* : \textbf{Contact} \to \textbf{StRel}$ as follows. If $(B, \rightsquigarrow)$ is a subordination algebra, then $(B, \rightsquigarrow)_* := (X, R)$, where X is the Stone space dual to B and $[xRy \Leftrightarrow \uparrow x \subseteq y]$[4] $\forall x, y \in X$. If $h : (B_1, \rightsquigarrow_1) \to (B_2, \rightsquigarrow_2)$ is a morphism in **Contact**, then define $h_* := h^{-1}(-) : (X_2, R_2) \to (X_1, R_1)$ such that $\forall x \in X_2$ $h_*(x) = h^{-1}(x)$.

The functor that we have just introduced is well-defined. In order to prove this, we need the following lemma (see [6, Lemma 2.12]):

Lemma A.0.8. *Let X be a compact Hausdorff space and let R be a binary relation on X. The following conditions are equivalent:*

1. *R is a closed relation.*

2. *For each closed subset F of X, both $R[F]$ and $R^{-1}[F]$[5] are closed.*

3. *If A is an arbitrary subset of X, then $\overline{R[A]} \subseteq R[\bar{A}]$ and $\overline{R^{-1}[A]} \subseteq R^{-1}[\bar{A}]$.*

4. *If $(x, y) \notin R$, then there is an open neighborhood U of x and an open neighborhood V of y such that $R[U] \cap V = \emptyset$.*

Thanks to this lemma and to the Lemma A.0.5, we can now prove the following:

[4] If $(B, \rightsquigarrow)$ is a subordination algebra and $S \subseteq B$, then $\uparrow S := \{b \in B \mid \exists a \in S \text{ with } a \rightsquigarrow b = 1\}$, and $\downarrow S := \{b \in B \mid \exists a \in S \text{ with } b \rightsquigarrow a = 1\}$.

[5] $R[F] := \{x \in X \mid \exists y \in F \text{ with } yRx\}$ and $R^{-1}[F] := \{x \in X \mid \exists y \in F \text{ with } xRy\}$.

Lemma A.0.9. *If* $(B, \rightsquigarrow)$ *is a contact algebra, then* $(B, \rightsquigarrow)_* := (X, R)$ *is an object of* **StRel**. *If* $h : (B_1, \rightsquigarrow_1) \to (B_2, \rightsquigarrow_2)$ *is a morphism of* **Contact**, *then* $h_* := h^{-1}(-) : (X_2, R_2) \to (X_1, R_1)$ *is a morphism of* **StRel**.

Proof. To see that $(B, \rightsquigarrow)_* := (X, R)$ is an object of **StRel**, it is sufficient to show that R is a reflexive and symmetric closed reation on X. If $(x, y) \notin R$, then $\uparrow x \not\subseteq y$. Therefore, there are $a \in x$ and $b \notin y$ with $a \rightsquigarrow b = 1$. But $a \rightsquigarrow b = 1$ implies that $R[\varphi(a)] \subseteq \varphi(b)$, where $\varphi(a), \varphi(b)$ are elements of a basis for the topology on X, and they are clopens. In fact, $y \in R[\varphi(a)] \Rightarrow \exists x \in \varphi(a)$ with xRy. $x \in \varphi(a) \Rightarrow a \in x$, and $xRy \Rightarrow \uparrow x \subseteq y$. Hence $a \rightsquigarrow b = 1 \Rightarrow b \in \uparrow x \subseteq y \Rightarrow b \in y \Rightarrow y \in \varphi(b)$. So now set $U := \varphi(a)$ and $V := X \setminus \varphi(b)$. Then U is an open neighborhood of x, V is an open neighborhood of y (being $b \notin y$, so $y \notin \varphi(b)$), and $R[U] \cap V = \emptyset$. Thus, by Lemma A.0.8, R is a closed relation on X. The fact that R is reflexive and symmetric follows from the Lemma 4.1.10 and from the Remark 4.1.15.

Now consider $h_* := h^{-1}(-) : (X_2, R_2) \to (X_1, R_1)$, where $h : (B_1, \rightsquigarrow_1) \to (B_2, \rightsquigarrow_2)$ is a morphism in **Contact**. By Stone duality, h_* is a well-defined continuous map. It remains to show that $\forall x, y \in X_1$ $(xR_1y \Leftrightarrow \exists y', \tilde{y} \in X_2$ such that $h_*(\tilde{y}) = x,\ h_*(y') = y\ \&\ \tilde{y}R_2y')$. Suppose that $0 \neq 1$ in B_2 (i. e., that $B_2 \neq \{0 = 1\}$). Then $0 \neq 1$ also in B_1, being h a morphism of Boolean algebras. Since we have that $h(a \rightsquigarrow_1 b) = h(a) \rightsquigarrow_2 h(b)$ (being h a morphism of **Contact**), it holds that $a \rightsquigarrow_1 b = 1 \Leftrightarrow h(a) \rightsquigarrow_2 h(b) = 1$. Therefore, using the duality, we have the following chain of equivalences (where $a \cong U_a \in \mathsf{Clop}(X_1)$, $b \cong U_b \in \mathsf{Clop}(X_1)$): $R_1[U_a] \subseteq U_b \Leftrightarrow U_a \rightsquigarrow_1 U_b = X_1 \Leftrightarrow a \rightsquigarrow_1 b = 1 \Leftrightarrow h(a) \rightsquigarrow_2 h(b) = 1 \Leftrightarrow (h_*)^{-1}(U_a) \rightsquigarrow_2 (h_*)^{-1}(U_b) = X_2 \Leftrightarrow R_2[(h_*)^{-1}(U_a)] \subseteq (h_*)^{-1}(U_b)$. Hence the condition *1.* of the Lemma A.0.5 is satisfied, and so also the condition *5.* holds, i. e., $\forall x, y \in X_1$ $[xR_1y \Leftrightarrow \exists y', \tilde{y} \in X_2$ such that $h_*(\tilde{y}) = x,\ h_*(y') = y\ \&\ \tilde{y}R_2y']$, as required. Therefore h_* is a morphism of **StRel**. Suppose now that $0 = 1$ in B_2 (i. e., that $B_2 = \{0 = 1\}$). Then $X_2 = \emptyset$, because B_2 doesn't have any ultrafilter. Hence h_* is the empty function, which is a morphism of **StRel** too.

□

Moreover, we recall the two following lemmas, which will allow us to prove two additional results:

Lemma A.0.10 (Prime Filter Theorem)**.** *Let* F *be a filter of a Boolean algebra* B*, and let* I *be an ideal such that* $F \cap I = \emptyset$*. Then there is a prime filter* $F' \supseteq F$ *such that* $F' \cap I = \emptyset$.

Lemma A.0.11. *Let* B *be a Boolean algebra, and let* F *be a proper filter of* B*. Then the following are equivalent:*

1. *F is maximal*
2. *F is prime*

3. F is an ultrafilter

So we can now prove the following result (cf. [6, Lemma 2.20]):

Lemma A.0.12. *Let $(B, \rightsquigarrow)$ be a contact algebra, and let $\varphi : B \to (B_*)^*$ be the Stone map*[6]*. Then $a \rightsquigarrow b = 1 \Leftrightarrow \varphi(a) \rightsquigarrow \varphi(b) = X$.*

Proof. Suppose first that $0 \neq 1$ in B, and let $a, b \in B$. If $a \rightsquigarrow b = 1$, then $R[\varphi(a)] \subseteq \varphi(b)$, so $\varphi(a) \rightsquigarrow \varphi(b) = X$. If $a \rightsquigarrow b = 0$, then $b \notin \uparrow a$.
$\uparrow a$ is a filter. In fact, $1 \in \uparrow a$ by axiom *(I1)* of the definition of contact algebras, so $\uparrow a \neq \emptyset$. Moreover, if $c \in \uparrow a$, it holds that $a \rightsquigarrow c = 1$. So, if $c \leq d$, by axiom *(I3)* we have that $(a \rightsquigarrow c) \wedge (a \rightsquigarrow d) = a \rightsquigarrow (c \wedge d) = a \rightsquigarrow c = 1$, so $a \rightsquigarrow d = 1$, i. e., $d \in \uparrow a$. Also, if $c, d \in \uparrow a$, then $a \rightsquigarrow (c \wedge d) = (a \rightsquigarrow c) \wedge (a \rightsquigarrow d) = 1$, i. e., $c \wedge d \in \uparrow a$.

Therefore, by the Prime Filter Theorem and by the Lemma A.0.11, there is an ultrafilter x such that $\uparrow a \subseteq x$ and $b \notin x$. By Zorn's Lemma, there is an ultrafilter y such that $a \in y$ and $\uparrow y \subseteq x$. Thus, there is $y \in B_*$ such that $y \in \varphi(a)$ and yRx. This gives $x \in R[\varphi(a)]$. On the other hand, $x \notin \varphi(b)$. Consequently, $R[\varphi(a)] \nsubseteq \varphi(b)$, yielding $\varphi(a) \rightsquigarrow \varphi(b) = \emptyset$.

If $B = \{0 = 1\}$, then the condition in the statement is trivially satisfied, being $(B_*)^* = \{X = \emptyset\}$.

□

For a Stone space X, define $\psi : X \to (X^*)_*$ by $\psi(x) := \{U \in \mathsf{Clop}(X) \mid x \in U\}$. It follows from Stone duality that ψ is a homeomorphism. Moreover, we have the following lemma (see [6, Lemma 2.21]):

Lemma A.0.13. *Let (X, R) be an object of* **StRel** *and let $\psi : X \to (X^*)_*$ be given as above. Then $xRy \Leftrightarrow \psi(x)R\psi(y)$.*

Proof. First suppose that $X \neq \emptyset$ (so we also have that $(X^*)_* \neq \emptyset$) and that xRy. To see that $\psi(x)R\psi(y)$ we must show that $\uparrow \psi(x) \subseteq \psi(y)$. So let $V \in \uparrow \psi(x)$. Then there is $U \in \psi(x)$ with $U \rightsquigarrow V = X$. Therefore, $R[U] \subseteq V$, and $x \in U$ (by definition of $\psi(x)$). Thus, $y \in V$, being xRy. So $\uparrow \psi(x) \subseteq \psi(y)$, and hence $\psi(x)R\psi(y)$.

Conversely, suppose that suppose that $(x, y) \notin R$. Since X has a basis of clopens and R is a closed relation, by Lemma A.0.8, there exist a clopen neighborhood U of x and a clopen neighborhood W of y such that $R[U] \cap W = \emptyset$. set $V := X \setminus W$. Then $U \in \psi(x)$, $V \notin \psi(y)$ and $R[U] \subseteq V$ (being $R[U] \cap W = \emptyset$). Therefore $U \rightsquigarrow V = X$, so $V \in \uparrow \psi(x)$, but $V \notin \psi(y)$. Thus, $(\psi(x), \psi(y)) \notin R$ If $X = \emptyset$, then $(X^*)_* = \emptyset$, and the condition given by the statement is trivially satisfied.

□

[6] Recall that the Stone space which is dual to a Boolean algebra B is given by $X :=$ {ultrafilters of B}, with the topology whose basis is $\{\varphi(a) \mid a \in B\}$, where $\varphi(a) := \{x \in X \mid a \in x\}$. This is a basis of clopen sets for X. The Stone map is then $\varphi : B \to \mathsf{Clop}(X)$, which is an isomorphism of Boolean algebras.

As a consequence of what we have proved until now, we can state the following theorem (cf. [6, Theorem 2.22]):

Theorem A.0.14. *The categories* **Contact** *and* **StRel** *are dually equivalent.*

Proof. By Lemma A.0.6, $(-)^* : \mathbf{StRel} \to \mathbf{Contact}$ is a well-defined controvariant functor, and by Lemma A.0.9, $(-)_* : \mathbf{Contact} \to \mathbf{StRel}$ is a well-defined controvariant functor. By Stone duality and Lemmas A.0.12 and A.0.13, each $(B, \rightsquigarrow) \in \mathbf{Contact}$ is isomorphic in **Contact** to $((B, \rightsquigarrow)_*)^*$, and each $(X, R) \in \mathbf{StRel}$ is isomorphic in **StRel** to $((X, R)^*)_*$. It is easy to see that these isomorphisms are natural. Thus, **Contact** is dually equivalent to **StRel**.

$\square$

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