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A new proof of the McKinsey-Tarski Theorem

Abstract. It is a landmark theorem of McKinsey and Tarski that if we interpret modal diamond as closure (and hence modal box as interior), then S4 is the logic of any dense-in-itself metrizable space. The McKinsey-Tarski Theorem relies heavily on a metric that gives rise to the topology. We give a new and more topological proof of the theorem, utilizing Bing's Metrization Theorem.

Keywords: Modal logic, topological semantics, metrizable space, Bing's metrization theorem

1. Introduction

It is a famous result of McKinsey and Tarski [14] that the modal system S4 is the logic of any dense-in-itself separable metrizable space when interpreting $\Diamond$ as closure (and hence $\Box$ as interior). Rasiowa and Sikorski [17] proved that the McKinsey-Tarski Theorem remains true for an arbitrary dense-in-itself metrizable space (that is, the separability condition can be dropped harmlessly). On the other hand, dropping the dense-in-itself condition results in new logics, and a complete classification of them can be found in [6].

Both the original proof of McKinsey and Tarski [14, Sec. 3] and the proof of Rasiowa and Sikorski [17, Sec. III.7 and III.8] rely heavily on a metric generating the topology of a given dense-in-itself (separable) metrizable space X to show that every finite subdirectly irreducible closure algebra is embeddable in the closure algebra of X. The result follows since S4 has the finite model property.

In the recent literature many simplified proofs of the McKinsey-Tarski Theorem have been produced for specific dense-in-itself metrizable spaces such as the real line [1, 7, 16], the rational line [3], and the Cantor discontinuum [15, 1]. These new proofs utilize relational semantics of S4 that was not available to McKinsey and Tarski. The proof technique of [7] produces an interior mapping of the real line onto any finite quasi-tree (such mappings correspond to the isomorphic embeddings used in the original proofs

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Presented by **Name of Editor**; *Received* December 1, 2005

Studia Logica (2006) 82: 1–22

of the McKinsey-Tarski Theorem), which is obtained by iteratively removing a copy of the Cantor discontinuum from the corresponding real intervals. This technique does not utilize the usual metric of the real line.

The aim of the present paper is to give a new proof of the McKinsey-Tarski Theorem, which makes the aforementioned idea work for an arbitrary dense-in-itself metrizable space. The new proof is more topological in that a metric is never used explicitly. Such is possible because of Bing's Metrization Theorem which characterizes metrizable spaces as exactly those spaces that admit a σ-discrete basis [8, 10]. It is this σ-discrete basis that encodes a blueprint for the interior mapping onto any finite rooted $\mathsf{S4}$-frame.

The paper concludes with a brief discussion about how the new proof of the McKinsey-Tarski Theorem could be utilized for better understanding of modal logics of space.

2. Classical approach

We start by outlining the original proof that $\mathsf{S4}$ is the logic of any dense-in-itself metrizable space. We recall that $\mathsf{S4}$ is the least set of formulas in the basic propositional modal language (with $\Box$) that contains the classical tautologies, the axioms:

- $\Box(p \to q) \to (\Box p \to \Box q)$,
- $\Box p \to p$,
- $\Box\Box p \to \Box p$,

and is closed under Modus Ponens $\frac{\varphi,\ \varphi \to \psi}{\psi}$, substitution $\frac{\varphi(p_1,\dots,p_n)}{\varphi(\psi_1,\dots,\psi_n)}$, and necessitation $\frac{\varphi}{\Box\varphi}$. As usual, we use the standard abbreviation $\Diamond\varphi := \neg\Box\neg\varphi$.

An $\mathsf{S4}$-*algebra* is a pair $\mathfrak{A} = (B, \Box)$, where B is a Boolean algebra and $\Box : B \to B$ satisfies Kuratowski's axioms for interior:

- $\Box(a \wedge b) = \Box a \wedge \Box b$,
- $\Box 1 = 1$,
- $\Box a \leq a$,
- $\Box a \leq \Box\Box a$.

Every such interior operator has its dual closure operator $\Diamond : B \to B$, defined by $\Diamond a = \neg\Box\neg a$. Fixpoints of $\Box$ are called *open elements* and fixpoints of $\Diamond$ are called *closed elements* of $\mathfrak{A}$.

Remark 2.1. $\mathsf{S4}$-algebras were introduced by McKinsey and Tarski [14] in the $\Diamond$-signature under the name of *closure algebras*. Rasiowa and Sikorski [17] call them *topological Boolean algebras*, and Blok [9] calls them *interior algebras*. In the modern modal logic literature it is common to call them $\mathsf{S4}$-algebras. We follow McKinsey and Tarski in working with $\mathsf{S4}$-algebras in the $\Diamond$-signature.

Typical examples of $\mathsf{S4}$-algebras come from topology: If X is a topological space, then $\mathfrak{A}_X := (\wp(X), \mathbf{c})$ is an $\mathsf{S4}$-algebra, where $\wp(X)$ is the powerset of X and $\mathbf{c}$ is the closure operator of X. By the McKinsey-Tarski Representation Theorem [14, Thm. 2.4], each $\mathsf{S4}$-algebra is isomorphic to a subalgebra of $\mathfrak{A}_X$ for some topological space X.

The modal language is interpreted in an $\mathsf{S4}$-algebra $\mathfrak{A} = (B, \Diamond)$ by evaluating propositional letters as elements of B, the classical connectives as the corresponding Boolean operations, and the modal box as the interior operator and hence modal diamond as the closure operator of $\mathfrak{A}$. A formula φ is *valid* in $\mathfrak{A}$, written $\mathfrak{A} \models \varphi$, provided it evaluates to 1 under all interpretations. It is well known (see, e.g., [17, Sec. XI.7]) that $\mathsf{S4} \vdash \varphi$ iff φ is valid in every $\mathsf{S4}$-algebra. In this notation, the McKinsey-Tarski Theorem can be stated as follows:

Theorem 2.2 (**McKinsey and Tarski**). *$\mathsf{S4} \vdash \varphi$ iff $\mathfrak{A}_X \models \varphi$ for every dense-in-itself metrizable space X.*

Proof. The left to right implication is obvious. For the right to left implication, if $\mathsf{S4} \nvdash \varphi$, then we must find a valuation on $\mathfrak{A}_X$ refuting φ. This can be done in three steps. We recall (see [14, Def. 1.10]) that an $\mathsf{S4}$-algebra is *well-connected* if $\Diamond a \wedge \Diamond b = 0$ implies $a = 0$ or $b = 0$.

Step 1 (Finite Model Property): If $\mathsf{S4} \nvdash \varphi$, then there is a finite well-connected $\mathsf{S4}$-algebra $\mathfrak{A}$ refuting φ (see [14, Thm. 4.16]).

For Step 2, we require the key notion of a dissectable $\mathsf{S4}$-algebra. For two elements x, y of a Boolean algebra B, write $a - b := a \wedge \neg b$, and say that $x_1, \dots, x_k \in B$ are *disjoint* provided $x_i \wedge x_j = 0$ for each $i \neq j$.

Definition 2.3. [14, Def. 3.4] An $\mathsf{S4}$-algebra $\mathfrak{A} = (B, \Diamond)$ is *dissectable* if for every open $a \in B \setminus \{0\}$ and every pair of integers $n \geq 0$ and $m > 0$, there are disjoint $u_1, \dots, u_n, a_1, \dots, a_m \in B \setminus \{0\}$ such that

- The elements $u_1, \dots, u_n$ are open;
- $\Diamond a_1 = \cdots = \Diamond a_m$;
- $u_1 \vee \cdots \vee u_n \vee a_1 \vee \cdots \vee a_m = a$;

- $\Diamond a - a \leq \Diamond a_i \leq \Diamond u_j$ for each $i \leq m$ and $j \leq n$.

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Step 2 (Dissection Lemma): If X is a dense-in-itself metrizable space, then $\mathfrak{A}_X$ is dissectable (see [14, Thm. 3.5] for the separable case, and [17, III.7.1] for the general case).

Step 3 (Embedding Lemma): Every finite well-connected $\mathsf{S4}$-algebra is embedded into every dissectable $\mathsf{S4}$-algebra (see [14, Thm. 3.7]).

Now, suppose $\mathsf{S4} \nvdash \varphi$. By Step 1, there is a finite well-connected $\mathsf{S4}$-algebra $\mathfrak{A}$ refuting φ. By Step 2, $\mathfrak{A}_X$ is dissectable. Therefore, by Step 3, $\mathfrak{A}$ is isomorphic to a subalgebra of $\mathfrak{A}_X$. Thus, since φ is refuted on $\mathfrak{A}$, it is refuted on $\mathfrak{A}_X$, and the proof of the McKinsey-Tarski Theorem is complete. ■

The proof of the Dissection Lemma makes nontrivial use of a metric that generates the topology on X. In the next section, we will discuss how this can be avoided using the modern approach.

3. Modern approach

The modern approach utilizes the relational semantics of modal logic. This semantics has its roots in the work of Jónsson and Tarski [12], and became the dominant semantics after the work of Kripke [13].

An $\mathsf{S4}$-*frame* is a pair $\mathfrak{F} = (W, R)$, where W is a nonempty set and R is a reflexive and transitive binary relation on W. As usual, for $w \in W$ and $A \subseteq W$, we write:

- $R[w] := \{v \in W \mid wRv\}$ and $R^{-1}[w] := \{v \in W \mid vRw\}$;
- $R[A] := \{v \in W \mid \exists w \in A : wRv\}$ and $R^{-1}[A] := \{v \in W \mid \exists w \in A : vRw\}$.

Each $\mathsf{S4}$-frame $\mathfrak{F}$ gives rise to the $\mathsf{S4}$-algebra $\mathfrak{A}_{\mathfrak{F}} := (\wp(W), R^{-1})$. By the Jónsson-Tarski Representation Theorem [12, Thm. 3.14], every $\mathsf{S4}$-algebra $\mathfrak{A}$ is isomorphic to a subalgebra of $\mathfrak{A}_{\mathfrak{F}}$ for some $\mathsf{S4}$-frame $\mathfrak{F}$. In fact, if $\mathfrak{A}$ is finite, then $\mathfrak{A}$ is isomorphic to $\mathfrak{A}_{\mathfrak{F}}$.

The modal language is interpreted in $\mathfrak{F}$ by interpreting formulas in $\mathfrak{A}_{\mathfrak{F}}$. A formula φ is *valid* in $\mathfrak{F}$, written $\mathfrak{F} \models \varphi$, provided $\mathfrak{A}_{\mathfrak{F}} \models \varphi$. The completeness of $\mathsf{S4}$ with respect to the algebraic semantics together with the Jónsson-Tarski Representation Theorem yields that $\mathsf{S4} \vdash \varphi$ iff φ is valid in every $\mathsf{S4}$-frame.

The relational semantics of $\mathsf{S4}$ is a particular case of its topological semantics (see, e.g., [2, Sec. 2.4.1]). For an $\mathsf{S4}$-frame $\mathfrak{F} = (W, R)$, call $A \subseteq W$

an *R-cone* if $A = R[A]$, and let τ_R be the set of all R-cones. Then τ_R is a topology on W such that R^{-1} is the closure operator, and each $w \in W$ has the least open neighborhood $R[w]$. Such spaces are usually referred to as *Alexandroff spaces.*

Let $\mathfrak{F} = (W, R)$ be an $\mathsf{S4}$-frame. We call $\mathfrak{F}$ *rooted* if there is $r \in W$ such that $R[r] = W$; such r is called a *root* of $\mathfrak{F}$. For finite $\mathfrak{F}$, it is well known (and easy to see) that $\mathfrak{A}_{\mathfrak{F}}$ is well-connected iff $\mathfrak{F}$ is rooted.

A map $f : X \to Y$ between topological spaces is called *interior* if it is continuous (V open in Y implies $f^{-1}(V)$ is open in X) and open (U open in X implies $f(U)$ is open in Y). Equivalently, f is interior iff $\mathbf{c}f^{-1}(A) = f^{-1}(\mathbf{c}A)$ for each $A \subseteq Y$.

Remark 3.1. It is well known (and easy to see) that if X and Y are Alexandroff spaces, then $f : X \to Y$ is an interior map iff it is a p-morphism ($R^{-1}[f^{-1}(x)] = f^{-1}(R^{-1}[x])$ for each $x \in X$).

We say Y is an *interior image* of X provided there is an interior mapping from X onto Y. Interior images will play an important role in our story since $\mathfrak{A}_Y$ is isomorphic to a subalgebra of $\mathfrak{A}_X$ iff Y is an interior image of X. In particular, $\mathfrak{A}_{\mathfrak{F}}$ is isomorphic to a subalgebra of $\mathfrak{A}_X$ iff $\mathfrak{F}$ viewed as an Alexandroff space is an interior image of X. Thus, proving the McKinsey-Tarski Theorem amounts to showing that every finite rooted $\mathsf{S4}$-frame $\mathfrak{F}$ is an interior image of every dense-in-itself metrizable space X.

We can further restrict the class of finite rooted $\mathsf{S4}$-frames. Let $\mathfrak{F} = (W, R)$ be an $\mathsf{S4}$-frame. The equivalence classes of the equivalence relation $\{(w, v) \mid wRv \text{ and } vRw\}$ on W are called *clusters.* A *quasi-chain* is a subset Q of W such that wRv or vRw for $w, v \in Q$. We call $\mathfrak{F}$ a *quasi-tree* if $\mathfrak{F}$ is rooted and $R^{-1}[w]$ is a quasi-chain for each $w \in W$.

Remark 3.2. The relation R induces a partial ordering on the set of clusters such that quasi-chains in $\mathfrak{F}$ correspond to chains in the poset P of clusters, and $\mathfrak{F}$ is a quasi-tree iff P is a tree (see Figure 1).

It is well known (see, e.g., [7, Cor. 6]) that $\mathsf{S4}$ is complete with respect to the class of finite quasi-trees. Therefore, if $\mathsf{S4} \nvdash \varphi$, then there is a finite quasi-tree $\mathcal{T}$ refuting φ. Thus, to prove the McKinsey-Tarski Theorem, it is sufficient to show that every finite quasi-tree $\mathcal{T}$ is an interior image of every dense-in-itself metrizable space X.

Let us examine what it takes for an onto interior map $f : X \to \mathcal{T}$ to exist. We recall that the *depth* of $\mathcal{T}$, denoted $\operatorname{depth}(\mathcal{T})$, is the greatest $n \geq 1$ such that there are $w_1, \ldots, w_n \in W$ satisfying $w_i R w_{i+1}$ but not $w_{i+1} R w_i$ for each $i \in \{1, \ldots, n-1\}$.

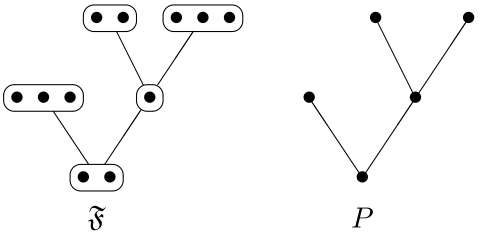

Figure 1. A quasi-tree $\mathfrak{F}$ and its poset of clusters P.

If $\operatorname{depth}(\mathcal{T}) = 1$, then $\mathcal{T}$ is a single cluster, consisting say of m points (see Figure 2).

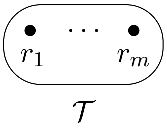

Figure 2. A single cluster quasi-tree $\mathcal{T}$.

We recall that a space X is *m-resolvable* provided there is a partition $\{A_1, \dots, A_m\}$ of X such that each A_i is dense in X; such partitions are called *dense*. By [5, Lem. 5.9], $\mathcal{T}$ is an interior image of X iff X is m-resolvable. It follows from Hewitt's theory of resolvability (see [11]) that every dense-in-itself metrizable space is m-resolvable. Therefore, if $\operatorname{depth}(\mathcal{T}) = 1$, then it is a consequence of Hewitt's theory of resolvability that $\mathcal{T}$ is an interior image of X.

Suppose $\operatorname{depth}(\mathcal{T}) > 1$ and C is the root cluster of $\mathcal{T} = (W, R)$ consisting of m points. Then $W \setminus C \neq \varnothing$, and there are $w_1, \dots, w_n \in W$ such that $\{C, R[w_1], \dots, R[w_n]\}$ is a partition of W (see Figure 3).

If an onto interior map $f : X \to \mathcal{T}$ exists, then set $G = f^{-1}(C)$ and $U_i = f^{-1}(R[w_i])$ for each $i \in \{1, \dots, n\}$. A direct calculation shows that $\{G, U_1, \dots, U_n\}$ is a partition of X such that G is m-resolvable and nowhere dense ($\mathbf{ic}G = \varnothing$), U_i is open, and $G \subseteq \mathbf{c}U_i$ for each $i \in \{1, \dots, n\}$.

Thus, the existence of f amounts to the existence of such a partition of X, which is the simplified version of the dissectability of X. How can we build such a partition without using a metric generating the topology? We will see in the next section that this is achievable using Bing's Metrization Theorem. As our guiding example, we consider the case of the real line $\mathbf{R}$ as described in [7].

Since $\mathbf{R}$ is homeomorphic to any nonempty bounded open interval, it is

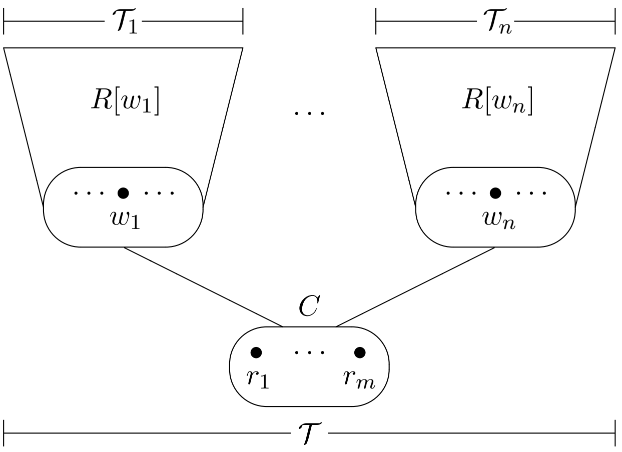

Figure 3. Partition of $\mathcal{T}$: the root cluster C and the sub-quasi-trees $\mathcal{T}_1, \dots, \mathcal{T}_n$.

sufficient to show for an arbitrary nonempty bounded open interval (a, b) that there is an onto interior map $f : (a, b) \to \mathcal{T}$. The proof is by induction on depth($\mathcal{T}$), and we only discuss the inductive step in which depth($\mathcal{T}$) > 1.

Construct the Cantor set $\mathbf{C}$ inside (a, b) by the usual process of taking away open "middle thirds". Let the root cluster C of $\mathcal{T}$ consist of m points. Since $\mathbf{C}$ is m-resolvable, there is an interior map f from $\mathbf{C}$ onto C. Our aim is to extend f to the entire (a, b). As depth($\mathcal{T}$) > 1, there are $w_1, \dots, w_n \in W$ such that $\{C, R[w_1], \dots, R[w_n]\}$ is a partition of W. By the inductive hypothesis, we may let f map each removed open "middle third" onto one of the quasi-trees, say $\mathcal{T}_i$ whose underlying set is $R[w_i]$. For the sake of illustration, suppose (c, d) is a removed open "middle third" and f sends it to $\mathcal{T}_1$. Then we construct the Cantor set inside (c, d) and proceed by induction (see Figure 4). Therefore, the mapping f is defined iteratively by moving "upward" through $\mathcal{T}$ and sending appropriately chosen "copies" of the Cantor set to the "lower" parts of $\mathcal{T}$. Notice that the role of G is played by the initial copy of the Cantor set in (a, b), and the roles of the U_i are played by the removed open middle thirds, which themselves contain a copy of the Cantor set containing $f^{-1}(w_i)$.

Our goal for the remainder of the paper is to mimic this proof in the setting of an arbitrary dense-in-itself metrizable space.

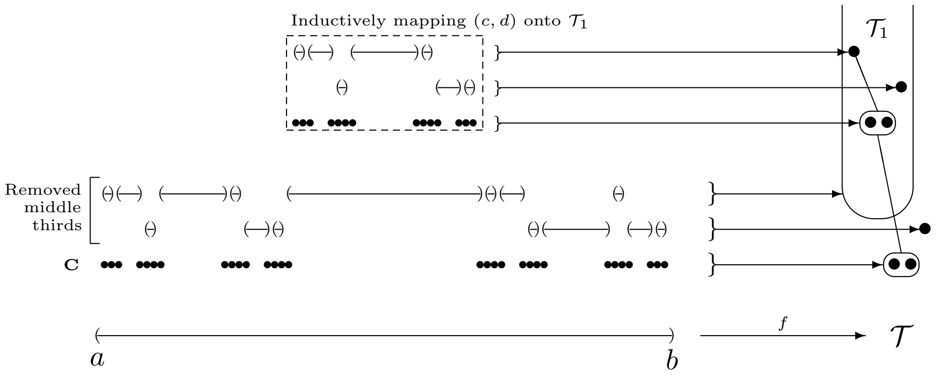

Figure 4. Depiction of $f : (a, b) \to \mathcal{T}$.

4. The new proof

In this section we present a new proof of the McKinsey-Tarski Theorem, in which the Embedding Lemma is replaced by the Mapping Lemma, and the key Dissection Lemma by the simpler Partition Lemma. To prove these two lemmas, we require some preparation. The section is divided into four subsections. The first subsection presents the auxiliary lemmas, culminating in Lemma 4.5; the second subsection proves the Partition Lemma (Lemma 4.13); the third subsection the Mapping Lemma (Lemma 4.22); and the fourth subsection shows that the McKinsey-Tarski Theorem does not generalize to the hereditarily paracompact setting.

4.1. Auxiliary lemmas

We start by recalling some basic definitions; see, e.g., [10]. For a topological space X, we recall that $\mathbf{i}$ and $\mathbf{c}$ stand for the interior and closure operators of X. As usual, we call $U \subseteq X$ *regular open* if $U = \mathbf{ic}U$.

Definition 4.1. Let X be a space and $\mathscr{A}$ a family of subsets of X.

1. Call $\mathscr{A}$ *discrete* if each $x \in X$ has an open neighborhood U such that $\{A \in \mathscr{A} \mid A \cap U \neq \varnothing\}$ consists of at most one element.
2. Call $\mathscr{A}$ *σ-discrete* if $\mathscr{A} = \bigcup_{n\in\omega} \mathscr{A}_n$ and each $\mathscr{A}_n$ is discrete.
3. Call $\mathscr{A}$ *closure preserving* if $\mathbf{c}\bigcup\mathscr{B} = \bigcup\{\mathbf{c}B \mid B \in \mathscr{B}\}$ for each $\mathscr{B} \subseteq \mathscr{A}$.

Remark 4.2. It is easy to see that if $\mathscr{A}$ is discrete, then $\mathscr{A}$ is pairwise disjoint. Moreover, if $\mathscr{A}$ is discrete and $\mathscr{B} \subseteq \mathscr{A}$, then $\mathscr{B}$ is discrete. Furthermore, if $\mathscr{A}$ is finite, then $\mathscr{A}$ is closure preserving.

LEMMA 4.3. *Let X be a space, B an open subset of X, and $\mathscr{U}$ a closure preserving family of nonempty regular open subsets of X such that $\{\mathbf{c}U \mid U \in \mathscr{U}\}$ is pairwise disjoint. Then $B \subseteq \bigcup \mathscr{U}$ iff $B \subseteq \mathbf{c}\bigcup \mathscr{U}$.*

PROOF. We only need to prove the right to left implication. Suppose $B \subseteq \mathbf{c}\bigcup \mathscr{U}$. Let $U \in \mathscr{U}$ and set $\mathscr{V} = \mathscr{U} \setminus \{U\}$. Then $\mathbf{c}(U) \cap \bigcup\{\mathbf{c}V \mid V \in \mathscr{V}\} = \varnothing$ and $B \subseteq \mathbf{c}\bigcup \mathscr{U} = \mathbf{c}(U) \cup \bigcup\{\mathbf{c}V \mid V \in \mathscr{V}\}$. Therefore, $B \cap \mathbf{c}U = B \setminus \bigcup\{\mathbf{c}V \mid V \in \mathscr{V}\} = B \setminus \mathbf{c}\bigcup \mathscr{V}$ is open in X, so $B \cap \mathbf{c}U \subseteq \mathbf{ic}U = U$. Thus, $B = \bigcup\{B \cap \mathbf{c}U \mid U \in \mathscr{U}\} \subseteq \bigcup \mathscr{U}$. ■

LEMMA 4.4. *Let X be a nonempty dense-in-itself regular space and Y a nonempty open subspace of X.*

1. *There is a nonempty regular open subset U of X such that $\mathbf{c}U \subset Y$.*
2. *For each $n \geq 1$, there is a family $\mathscr{U}$ consisting of n nonempty regular open subsets of X such that $\{\mathbf{c}U \mid U \in \mathscr{U}\}$ is pairwise disjoint and $\mathbf{c}\bigcup \mathscr{U} \subset Y$.*

PROOF. (1) Let $x \in Y$. Since X is a dense-in-itself T_1-space, $Y \setminus \{x\}$ is a nonempty open subset of X. As X is regular, there is a nonempty open subset V of X such that $\mathbf{c}V \subseteq Y \setminus \{x\}$. Thus, $U := \mathbf{ic}V$ is as required.

(2) Induction on $n \geq 1$. Applying (1) renders the base case $n = 1$. Suppose $n \geq 1$ and there is a family $\mathscr{V}$ consisting of n nonempty regular open subsets of X such that $\{\mathbf{c}V \mid V \in \mathscr{V}\}$ is pairwise disjoint and $\mathbf{c}\bigcup \mathscr{V} \subset Y$. Then $Y \setminus \mathbf{c}\bigcup \mathscr{V}$ is a nonempty open subset of X. Applying (1) yields a nonempty regular open subset W of X such that $\mathbf{c}W \subset Y \setminus \mathbf{c}\bigcup \mathscr{V}$. The family $\mathscr{U} := \mathscr{V} \cup \{W\}$ is as required. ■

LEMMA 4.5. *Let X be a dense-in-itself regular space, F a closed discrete subspace of X, and $\mathscr{U}_1, \ldots, \mathscr{U}_n$ families of subsets of X such that $\mathscr{U} := \bigcup_{i=1}^n \mathscr{U}_i$ is a closure preserving family of nonempty regular open subsets of X satisfying $\{\mathbf{c}U \mid U \in \mathscr{U}\}$ is pairwise disjoint and $\mathbf{c}(U) \cap F = \varnothing$ for each $U \in \mathscr{U}$.*

If $\mathscr{B}$ is a discrete family of open subsets of X, then there are families $\mathscr{V}_1, \ldots, \mathscr{V}_n$ of subsets of X and a closed discrete subspace D of X such that:

1. *$\mathscr{U}_i \subseteq \mathscr{V}_i$ for each $i \in \{1, \ldots, n\}$.*
2. *The family $\mathscr{V} := \bigcup_{i=1}^n \mathscr{V}_i$ is a closure preserving family of nonempty regular open subsets of X such that $\{\mathbf{c}V \mid V \in \mathscr{V}\}$ is pairwise disjoint.*
3. *$\mathbf{c}(V) \cap (F \cup D) = \varnothing$ for each $V \in \mathscr{V}$.*
4. *If $B \in \mathscr{B}$ and $B \not\subseteq \bigcup \mathscr{V}$, then:*

(a) For each $i \in \{1, \dots, n\}$ there is $V_i \in \mathscr{V}_i$ such that $\mathbf{c}V_i \subseteq B$.

(b) The set $B \cap D$ contains at least two elements.

Proof. Let $\mathscr{C} = \{B \in \mathscr{B} \mid B \not\subseteq \bigcup \mathscr{U}\}$. Suppose that $\mathscr{C} = \varnothing$. Set $\mathscr{V}_i = \mathscr{U}_i$ for $i \in \{1, \dots, n\}$ and $D = \varnothing$. Then D is a closed discrete subspace of X and $\mathscr{V} = \mathscr{U}$. Therefore, conditions (1)–(4) are satisfied trivially.

Suppose that $\mathscr{C} \neq \varnothing$. Let $B \in \mathscr{C}$. Then $B \not\subseteq \bigcup \mathscr{U}$. By Lemma 4.3, $B \not\subseteq \mathbf{c}\bigcup \mathscr{U}$, so $B \setminus \mathbf{c}\bigcup \mathscr{U}$ is a nonempty open subset of X. Because F is a closed discrete subspace of X and X is dense-in-itself, $B \setminus (F \cup \mathbf{c}\bigcup \mathscr{U})$ is a nonempty open subset of X. Lemma 4.4(2) delivers a family $\{B_1, \dots, B_n\}$ of nonempty regular open subsets of X such that $\{\mathbf{c}B_1, \dots, \mathbf{c}B_n\}$ is pairwise disjoint and $\bigcup_{i=1}^n \mathbf{c}B_i = \mathbf{c}\bigcup_{i=1}^n B_i \subset B \setminus (F \cup \mathbf{c}\bigcup \mathscr{U})$. Let D_B consist of any two points in the nonempty open subset $(B \setminus (F \cup \mathbf{c}\bigcup \mathscr{U})) \setminus \bigcup_{i=1}^n \mathbf{c}B_i$ of X. Set $\mathscr{V}_i = \mathscr{U}_i \cup \{B_i \mid B \in \mathscr{C}\}$ and $D = \bigcup\{D_B \mid B \in \mathscr{C}\}$.

Claim 4.6. *D is a closed discrete subspace of X.*

Proof. Let $x \in \mathbf{c}D$. Since $\mathscr{B}$ is a discrete family, there is an open neighborhood U of x such that $\{B \in \mathscr{B} \mid U \cap B \neq \varnothing\}$ consists of at most one element. Because $\varnothing \neq U \cap D \subseteq U \cap \bigcup \mathscr{C} \subseteq \bigcup\{U \cap B \mid B \in \mathscr{B}\}$, there is $B' \in \mathscr{C}$ such that $\{B \in \mathscr{B} \mid U \cap B \neq \varnothing\} = \{B'\}$. Note that $D_{B'} \setminus \{x\}$ is finite and hence closed since $D_{B'}$ consists of two points. Therefore, $U \setminus (D_{B'} \setminus \{x\})$ is an open neighborhood of x, and so

$$\begin{aligned}
\varnothing &\neq (U \setminus (D_{B'} \setminus \{x\})) \cap D = (U \setminus (D_{B'} \setminus \{x\})) \cap \bigcup\{D_B \mid B \in \mathscr{C}\} \\
&= \bigcup\{(U \setminus (D_{B'} \setminus \{x\})) \cap D_B \mid B \in \mathscr{C}\} \\
&= (U \setminus (D_{B'} \setminus \{x\})) \cap D_{B'} \subseteq \{x\},
\end{aligned}$$

giving that $(U \setminus (D_{B'} \setminus \{x\})) \cap D = \{x\}$. This shows that D is both closed and discrete. ■

We now verify that conditions (1)–(4) hold. Clearly condition (1) holds by the definition of the $\mathscr{V}_i$. That condition (2) holds follows from Claims 4.7, 4.8, and 4.10 below.

Claim 4.7. *Each $V \in \mathscr{V}$ is a nonempty regular open subset of X.*

Proof. Since for each $B \in \mathscr{C}$ and $i \in \{1, \dots, n\}$, the families $\{B_1, \dots, B_n\}$ and $\mathscr{U}_i$ consist of nonempty regular open subsets of X, each $V \in \mathscr{V}_i = \mathscr{U}_i \cup \{B_i \mid B \in \mathscr{C}\}$ is a nonempty regular open subset of X. The result follows since $\mathscr{V} = \bigcup_{i=1}^n \mathscr{V}_i$. ■

Claim 4.8. *The family* $\{\mathbf{c}V \mid V \in \mathscr{V}\}$ *is pairwise disjoint.*

Proof. Suppose $V, W \in \mathscr{V}$ are distinct. If $V, W \in \mathscr{U}$, then $\mathbf{c}V \cap \mathbf{c}W = \varnothing$ since $\{\mathbf{c}U \mid U \in \mathscr{U}\}$ is pairwise disjoint. If $V \in \mathscr{U}$ and $W \notin \mathscr{U}$, then $W \in \{B_1, \dots, B_n\}$ for some $B \in \mathscr{C}$. Therefore, $\mathbf{c}W \subseteq \bigcup_{i=1}^n \mathbf{c}B_i \subset B \setminus (F \cup \mathbf{c}\bigcup \mathscr{U}) \subseteq B \setminus \mathbf{c}V \subseteq X \setminus \mathbf{c}V$, which gives $\mathbf{c}V \cap \mathbf{c}W = \varnothing$. The case $W \in \mathscr{U}$ and $V \notin \mathscr{U}$ is similar. If $V, W \notin \mathscr{U}$, then $V \in \{B_1, \dots, B_n\}$ and $W \in \{B'_1, \dots, B'_n\}$ for some $B, B' \in \mathscr{C}$. If $B = B'$, then $\mathbf{c}V \cap \mathbf{c}W = \varnothing$ since $\{B_1, \dots, B_n\} = \{B'_1, \dots, B'_n\}$ and $\{\mathbf{c}B_1, \dots, \mathbf{c}B_n\}$ is pairwise disjoint. If $B \neq B'$, then $B \cap B' = \varnothing$ since $\mathscr{B}$ is discrete, and hence pairwise disjoint. Thus,

$$
\begin{aligned}
\mathbf{c}V \cap \mathbf{c}W &\subseteq \left(\bigcup_{i=1}^{n} \mathbf{c}B_i\right) \cap \left(\bigcup_{i=1}^{n} \mathbf{c}B'_i\right) \\
&\subseteq \left[B \setminus \left(F \cup \mathbf{c}\bigcup \mathscr{U}\right)\right] \cap \left[B' \setminus \left(F \cup \mathbf{c}\bigcup \mathscr{U}\right)\right] \subseteq B \cap B' = \varnothing,
\end{aligned}
$$

and hence $\{\mathbf{c}V \mid V \in \mathscr{V}\}$ is pairwise disjoint. ■

Claim 4.9. *The family* $\mathscr{D} := \{B_i \mid B \in \mathscr{C}$ *and* $i \in \{1, \dots, n\}\}$ *is discrete.*

Proof. Let $x \in X$. Since $\mathscr{B}$ is discrete, there is an open neighborhood N_x of x such that $\{B \in \mathscr{B} \mid B \cap N_x \neq \varnothing\}$ consists of at most one element. If $\{B \in \mathscr{B} \mid B \cap N_x \neq \varnothing\}$ is empty, then $\{B_i \mid B \in \mathscr{C},\ i \in \{1, \dots, n\},\ B_i \cap N_x \neq \varnothing\}$ is empty because $N_x \cap B_i \subseteq N_x \cap B$ for each $B \in \mathscr{C}$ and $i \in \{1, \dots, n\}$.

Suppose $B' \in \mathscr{B}$ is the unique element of $\{B \in \mathscr{B} \mid B \cap N_x \neq \varnothing\}$. If $B' \notin \mathscr{C}$, then $\{B_i \mid B \in \mathscr{C},\ i \in \{1, \dots, n\},\ N_x \cap B_i \neq \varnothing\}$ is empty. Suppose $B' \in \mathscr{C}$. If $x \notin \mathbf{c}\bigcup_{i=1}^n B'_i$, then $U := N_x \setminus \mathbf{c}\bigcup_{i=1}^n B'_i$ is an open neighborhood of x and $\{B_i \mid B \in \mathscr{C},\ i \in \{1, \dots, n\},\ U \cap B_i \neq \varnothing\}$ is empty.

If $x \in \mathbf{c}\bigcup_{i=1}^n B'_i$, then since $\{\mathbf{c}B'_1, \dots, \mathbf{c}B'_n\}$ is pairwise disjoint, there is a unique $j \in \{1, \dots, n\}$ such that $x \in \mathbf{c}B'_j$. Therefore, $U := N_x \setminus \bigcup\{\mathbf{c}B'_i \mid i \neq j\}$ is an open neighborhood of x and $\{B_i \mid B \in \mathscr{C},\ i \in \{1, \dots, n\},\ U \cap B_i \neq \varnothing\} = \{B'_j\}$. Since $x \in \mathbf{c}B'_j$ and U is an open neighborhood of x, we have $U \cap B'_j \neq \varnothing$. If $U \cap B_i \neq \varnothing$ for some $B \in \mathscr{C}$ and $i \in \{1, \dots, n\}$, then $\varnothing \neq U \cap B_i \subseteq N_x \cap B$ gives $B = B'$ and $\varnothing \neq U \cap B'_i = (N_x \setminus \bigcup\{\mathbf{c}B'_i \mid i \neq j\}) \cap B'_i = \varnothing$ for $i \neq j$. Thus, $\mathscr{D}$ is discrete. ■

Claim 4.10. *The family* $\mathscr{V}$ *is closure preserving.*

Proof. Let $\mathscr{W} \subseteq \mathscr{V}$. Using $\mathscr{D}$ as defined in Claim 4.9, we have that $\mathscr{V} = \mathscr{U} \cup \mathscr{D}$ and $\mathscr{W} = (\mathscr{W} \cap \mathscr{U}) \cup (\mathscr{W} \cap \mathscr{D})$. Since $\mathscr{D}$ is discrete, so is $\mathscr{W} \cap \mathscr{D}$. Therefore, $\mathscr{W} \cap \mathscr{D}$ is closure preserving (which follows from [10, Thm. 1.1.11] since a discrete family is locally finite). Because $\mathscr{U}$ is closure preserving, so is $\mathscr{W}\cap\mathscr{U}$. Therefore,

$$\begin{aligned}
\mathbf{c}\bigcup\mathscr{W} &= \mathbf{c}\bigcup((\mathscr{W}\cap\mathscr{U})\cup(\mathscr{W}\cap\mathscr{D})) = \mathbf{c}\left(\bigcup(\mathscr{W}\cap\mathscr{U})\cup\bigcup(\mathscr{W}\cap\mathscr{D})\right)\\
&= \mathbf{c}\bigcup(\mathscr{W}\cap\mathscr{U})\cup\mathbf{c}\bigcup(\mathscr{W}\cap\mathscr{D})\\
&= \bigcup\{\mathbf{c}V \mid V\in\mathscr{W}\cap\mathscr{U}\}\cup\bigcup\{\mathbf{c}V \mid V\in\mathscr{W}\cap\mathscr{D}\}\\
&= \bigcup\{\mathbf{c}V \mid V\in(\mathscr{W}\cap\mathscr{U})\cup(\mathscr{W}\cap\mathscr{D})\} = \bigcup\{\mathbf{c}V \mid V\in\mathscr{W}\}.
\end{aligned}$$

Thus, $\mathscr{V}$ is closure preserving. ■

Claim 4.11. *Condition* (3) *holds.*

Proof. Let $V\in\mathscr{V}$. If $V\in\mathscr{U}$, then $\mathbf{c}V\cap F=\varnothing$ by assumption. Moreover, $\mathbf{c}V\cap D=\varnothing$ because for each $B\in\mathscr{C}$, from

$$\begin{aligned}
D_B &\subseteq \left(B\setminus(F\cup\mathbf{c}\bigcup\mathscr{U})\right)\setminus\bigcup\nolimits_{i=1}^{n}\mathbf{c}B_i\\
&\subseteq B\setminus(F\cup\mathbf{c}\bigcup\mathscr{U})\subseteq X\setminus\mathbf{c}\bigcup\mathscr{U}\subseteq X\setminus\mathbf{c}V
\end{aligned}$$

it follows that $D=\bigcup\{D_B \mid B\in\mathscr{C}\}\subseteq X\setminus\mathbf{c}V$. Thus, $\mathbf{c}V\cap(F\cup D)=\varnothing$.

If $V\notin\mathscr{U}$, then $V=B'_j$ for some $B'\in\mathscr{C}$ and $j\in\{1,\dots,n\}$. From

$$\mathbf{c}B'_j\subseteq\bigcup\nolimits_{i=1}^{n}\mathbf{c}B'_i\subseteq B'\setminus(F\cup\mathbf{c}\bigcup\mathscr{U})\subseteq B'\setminus F\subseteq X\setminus F$$

it follows that $\mathbf{c}(B'_j)\cap F=\varnothing$. Also, from

$$D_{B'}\subseteq\left(B'\setminus(F\cup\mathbf{c}\bigcup\mathscr{U})\right)\setminus\bigcup\nolimits_{i=1}^{n}\mathbf{c}B'_i\subseteq X\setminus\bigcup\nolimits_{i=1}^{n}\mathbf{c}B'_i\subseteq X\setminus\mathbf{c}B'_j$$

it follows that $\mathbf{c}(B'_j)\cap D_{B'}=\varnothing$. Since $\mathscr{B}$ is pairwise disjoint, we have

$$\begin{aligned}
\mathbf{c}(B'_j)\cap D &= \mathbf{c}(B'_j)\cap\bigcup\{D_B \mid B\in\mathscr{C}\} = \bigcup\left\{\mathbf{c}(B'_j)\cap D_B \mid B\in\mathscr{C}\right\}\\
&= \left(\mathbf{c}(B'_j)\cap D_{B'}\right)\cup\bigcup\left\{\mathbf{c}(B'_j)\cap D_B \mid B\in\mathscr{C}\setminus\{B'\}\right\}\\
&\subseteq \varnothing\cup\bigcup\left\{\mathbf{c}(B'_j)\cap B \mid B\in\mathscr{C}\setminus\{B'\}\right\}\\
&\subseteq \bigcup\left\{B'\cap B \mid B\in\mathscr{C}\setminus\{B'\}\right\}=\varnothing.
\end{aligned}$$

Therefore, $\mathbf{c}V\cap(F\cup D)=\varnothing$, and hence condition (3) holds. ■

Claim 4.12. *Condition* (4) *holds.*

PROOF. Suppose $B \in \mathscr{B}$ and $B \not\subseteq \bigcup \mathscr{V}$. Then $B \not\subseteq \bigcup \mathscr{U}$ since $\mathscr{U} \subseteq \mathscr{V}$. Therefore, $B \in \mathscr{C}$. Let $i \in \{1, \dots, n\}$. Then $\mathbf{c}B_i \subseteq \bigcup_{j=1}^{n} \mathbf{c}B_j \subseteq B \setminus (F \cup \mathbf{c} \bigcup \mathscr{U}) \subseteq B$. Since $B_i \in \mathscr{V}_i$, condition (4a) holds. Because $\mathscr{B}$ is pairwise disjoint and $D_{B'} \subseteq B'$ for each $B' \in \mathscr{C}$, we have

$$B \cap D = B \cap \bigcup \{D_{B'} \mid B' \in \mathscr{C}\} = \bigcup \{B \cap D_{B'} \mid B' \in \mathscr{C}\} = B \cap D_B = D_B.$$

Thus, $B \cap D_B$ consists of two elements, and hence condition (4b) holds. ■

This completes the proof of Lemma 4.5. ■

With these preliminary results established we are ready to prove the Partition Lemma.

4.2. The Partition Lemma

This subsection is dedicated to proving the Partition Lemma, and it is exactly here where Bing's Metrization Theorem will be utilized.

LEMMA 4.13 (**Partition Lemma**). *Let X be a dense-in-itself metrizable space, F a nonempty closed discrete subspace of X, and $n \geq 1$. Then there is a partition $\{G, U_1, \dots, U_n\}$ of X such that*

1. *G is a dense-in-itself closed nowhere dense subspace of X containing F.*
2. *Each U_i is an open subspace of X such that there is a discrete subspace F_i of U_i with $\mathbf{c}F_i = F_i \cup G$.*

PROOF. By Bing's Metrization Theorem (see, e.g., [10, Thm. 4.4.8]), X has a σ-discrete basis $\mathscr{B} = \bigcup\{\mathscr{B}_m \mid m \geq 1\}$, where each $\mathscr{B}_m$ is a discrete family of open subsets of X. By Lemma 4.4(2), there is a family $\mathscr{V}^0 = \{W_1, \dots, W_n\}$ of nonempty regular open subsets of X such that $\{\mathbf{c}W_1, \dots, \mathbf{c}W_n\}$ is pairwise disjoint and $\mathbf{c} \bigcup \mathscr{V}^0 \subset X \setminus F$. Put $\mathscr{V}_i^0 = \{W_i\}$ for each $i \in \{1, \dots, n\}$ and $D_0 = F$. Then $\mathscr{V}_1^0, \dots, \mathscr{V}_n^0$, $\mathscr{V}^0 = \bigcup_{i=1}^{n} \mathscr{V}_i^0$, and D_0 satisfy the conditions of Lemma 4.5.

For each $m \geq 1$, define recursively families $\mathscr{V}_1^m, \dots, \mathscr{V}_n^m$ of subsets of X and a closed discrete subspace D_m of X as follows. Suppose for some $m \geq 1$ the families $\mathscr{V}_1^{m-1}, \dots, \mathscr{V}_n^{m-1}$ and the closed discrete subspace D_{m-1} are already defined so that $\mathscr{V}^{m-1} := \bigcup_{i=1}^{n} \mathscr{V}_i^{m-1}$ is a closure preserving family of nonempty regular open subsets of X satisfying $\{\mathbf{c}V \mid V \in \mathscr{V}^{m-1}\}$ is pairwise disjoint and $\mathbf{c}V \cap D_{m-1} = \varnothing$ for each $V \in \mathscr{V}^{m-1}$. Then Lemma 4.5 applied to $\mathscr{U}_i = \mathscr{V}_i^{m-1}$, $F = D_{m-1}$, and $\mathscr{B} = \mathscr{B}_m$ yields families $\mathscr{V}_1^m, \dots \mathscr{V}_n^m$ and a closed discrete subspace D'_m such that:

1. $\mathscr{V}_i^{m-1} \subseteq \mathscr{V}_i^m$ for each $i \in \{1,\dots,n\}$.
2. The family $\mathscr{V}^m := \bigcup_{i=1}^n \mathscr{V}_i^m$ is a closure preserving family of nonempty regular open subsets of X such that $\{\mathbf{c}V \mid V \in \mathscr{V}^m\}$ is pairwise disjoint.
3. $\mathbf{c}V \cap (D_{m-1} \cup D'_m) = \varnothing$ for each $V \in \mathscr{V}^m$.
4. If $B \in \mathscr{B}_m$ and $B \not\subseteq \bigcup \mathscr{V}^m$, then:
   (a) For each $i \in \{1,\dots,n\}$ there is $V_i \in \mathscr{V}_i^m$ such that $\mathbf{c}V_i \subseteq B$.
   (b) The set $B \cap D'_m$ contains at least two elements.

Set $D_m = D_{m-1} \cup D'_m$. Then D_m is a closed discrete subset of X since a finite union of closed discrete subsets of any space is closed and discrete. Therefore, we have:

1. $\mathscr{V}_i^m \subseteq \mathscr{V}_i^{m+1}$ and $D_m \subseteq D_{m+1}$ for each $i \in \{1,\dots,n\}$.
2. The family $\mathscr{V}^m := \bigcup_{i=1}^n \mathscr{V}_i^m$ is a closure preserving family of nonempty regular open subsets of X such that $\{\mathbf{c}V \mid V \in \mathscr{V}^m\}$ is pairwise disjoint.
3. $\mathbf{c}V \cap D_m = \varnothing$ for each $V \in \mathscr{V}^m$.
4. If $B \in \mathscr{B}_m$ and $B \not\subseteq \bigcup \mathscr{V}^m$, then:
   (a) For each $i \in \{1,\dots,n\}$ there is $V_i \in \mathscr{V}_i^m$ such that $\mathbf{c}V_i \subseteq B$.
   (b) The set $B \cap D_m$ contains at least two elements.

For each $i \in \{1,\dots,n\}$, set $\mathscr{V}_i = \bigcup_{m\in\omega} \mathscr{V}_i^m$, $U_i = \bigcup \mathscr{V}_i$, and $G = X \setminus \bigcup_{i=1}^n U_i$. It remains to prove that $\{G, U_1, \dots, U_n\}$ is as desired.

Claim 4.14. *$\{\mathscr{V}_i^m \mid i \in \{1,\dots,n\}\}$ is pairwise disjoint for all $m \in \omega$.*

Proof. By induction on $m \in \omega$. For $m = 0$, the family $\{W_1,\dots,W_n\}$ is chosen so that $\{\mathbf{c}W_1,\dots,\mathbf{c}W_n\}$ is pairwise disjoint. Therefore, $\{\mathscr{V}_i^0 \mid i \in \{1,\dots,n\}\} = \{\{W_1\},\dots,\{W_n\}\}$ is pairwise disjoint, and hence the base case holds.

Let $m \geq 1$ and $\{\mathscr{V}_i^{m-1} \mid i \in \{1,\dots,n\}\}$ be pairwise disjoint. Observe that for each $i \in \{1,\dots,n\}$ we have

$$\mathscr{V}_i^m = \mathscr{V}_i^{m-1} \cup \left\{ B_i \mid B \in \mathscr{B}_{m-1} \text{ and } B \not\subseteq \bigcup \mathscr{V}^{m-1} \right\}.$$

Also, for any $i, k \in \{1,\dots,n\}$, $V \in \mathscr{V}_i^{m-1}$, and $B \in \mathscr{B}_{m-1}$ such that $B \not\subseteq \bigcup \mathscr{V}^{m-1}$, we have that $V \cap B_k = \varnothing$ because

$$B_k \subseteq \mathbf{c}\bigcup_{j=1}^n B_j \subset B \setminus (D_{m-1} \cup \mathbf{c}\bigcup \mathscr{V}^{m-1}) \subseteq X \setminus \bigcup \mathscr{V}^{m-1} \subseteq X \setminus V.$$

Let $V \in \mathscr{V}_i^m \cap \mathscr{V}_j^m$ for some $i, j \in \{1, \ldots, n\}$. Then $V \in \mathscr{V}_i^m$. If $V \in \mathscr{V}_i^{m-1}$, then the above observations yield that $V \in \mathscr{V}_j^{m-1}$. So $i = j$ by the inductive hypothesis. Suppose $V \notin \mathscr{V}_i^{m-1}$. Then $V \notin \mathscr{V}_j^{m-1}$. Therefore, $V = B_i$ and $V = B'_j$ for some $B, B' \in \mathscr{B}_{m-1}$ such that $B, B' \not\subseteq \bigcup \mathscr{V}^{m-1}$. Thus, $\varnothing \neq V = B_i \cap B'_j \subseteq B \cap B'$. Consequently, $B = B'$, so $B_i = B'_j$, and hence $i = j$, yielding that $\{\mathscr{V}_i^m \mid i \in \{1, \ldots, n\}\}$ is pairwise disjoint. ■

Claim 4.15. *$\mathscr{V}_i$ is pairwise disjoint for each $i \in \{1, \ldots, n\}$.*

Proof. Let $V, W \in \mathscr{V}_i = \bigcup_{m\in\omega} \mathscr{V}_i^m$ be such that $V \cap W \neq \varnothing$. Then there are $m', m'' \in \omega$ such that $V \in \mathscr{V}_i^{m'}$ and $W \in \mathscr{V}_i^{m''}$. Let $m = \max\{m', m''\}$. Then $V, W \in \mathscr{V}_i^m \subseteq \mathscr{V}^m$. Since $\mathscr{V}^m$ is pairwise disjoint, $V = W$. Thus, $\mathscr{V}_i$ is pairwise disjoint. ■

Claim 4.16. *For any $m \geq 1$ and $B \in \mathscr{B}_m$, if $B \cap G \neq \varnothing$, then $B \not\subseteq \bigcup \mathscr{V}^m$.*

Proof. Let $m \geq 1$, $B \in \mathscr{B}_m$, and $x \in B \cap G$. Then $x \in G$, giving that $x \notin \bigcup_{i=1}^n U_i$. Since $\mathscr{V}_i^m \subseteq \bigcup_{m'\in\omega} \mathscr{V}_i^{m'} = \mathscr{V}_i$ for each $i \in \{1, \ldots, n\}$, we have $\bigcup \mathscr{V}_i^m \subseteq \bigcup \mathscr{V}_i = U_i$ for each $i \in \{1, \ldots, n\}$, and hence $\bigcup \mathscr{V}^m = \bigcup \bigcup_{i=1}^n \mathscr{V}_i^m = \bigcup_{i=1}^n \bigcup \mathscr{V}_i^m \subseteq \bigcup_{i=1}^n U_i$. Therefore, $x \notin \bigcup \mathscr{V}^m$, and hence $B \not\subseteq \bigcup \mathscr{V}^m$. ■

Claim 4.17. *Let $D = \bigcup\{D_m \mid m \in \omega\}$. Then $D \cap \bigcup_{i=1}^n U_i = \varnothing$.*

Proof. Since $D \cap \bigcup_{i=1}^n U_i = \bigcup_{i=1}^n (D \cap U_i)$, it is sufficient to show $D \cap U_i = \varnothing$ for all $i \in \{1, \ldots, n\}$. Since $D \cap U_i = \left(\bigcup_{m\in\omega} D_m\right) \cap U_i = \bigcup_{m\in\omega} (D_m \cap U_i)$, it is sufficient to show that $D_m \cap U_i = \varnothing$ for each $m \in \omega$. Since

$$\begin{aligned} D_m \cap U_i &= D_m \cap \bigcup \mathscr{V}_i \\ &= D_m \cap \bigcup \left\{ V \in \mathscr{V}_i^{m'} \mid m' \in \omega \right\} \\ &= \bigcup \left\{ D_m \cap V \mid m' \in \omega,\ V \in \mathscr{V}_i^{m'} \right\}, \end{aligned}$$

we only need to show that $D_m \cap V = \varnothing$ for each $m' \in \omega$ and $V \in \mathscr{V}_i^{m'}$. But it follows from conditions (1) and (3) that $D_m \cap V \subseteq D_{\max\{m,m'\}} \cap V \subseteq D_{\max\{m,m'\}} \cap \mathbf{c}V = \varnothing$ since $V \in \mathscr{V}_i^{m'} \subseteq \mathscr{V}_i^{\max\{m,m'\}}$, completing the proof. ■

Claim 4.18. *The family $\{G, U_1, \ldots, U_n\}$ is a partition of X such that U_i is an open subset of X for each $i \in \{1, \ldots, n\}$ and G is a closed subset of X containing F.*

Proof. Let $i \in \{1, \ldots, n\}$. Because $U_i = \bigcup \mathscr{V}_i$ and each $V \in \mathscr{V}_i$ is a (regular) open subset of X, U_i is an open subset of X. Also $U_i \supseteq W_i \neq \varnothing$ since $\mathscr{V}_i \supseteq \mathscr{V}_i^m \supseteq \mathscr{V}_i^0 = \{W_i\}$.

To see that $\{U_1, \ldots, U_n\}$ is pairwise disjoint, let $x \in U_i \cap U_j$. Then there are $m_i, m_j \in \omega$, $V \in \mathscr{V}_i^{m_i}$, and $W \in \mathscr{V}_j^{m_j}$ such that $x \in V$ and $x \in W$. Let $m = \max\{m_i, m_j\}$. Then $V \in \mathscr{V}_i^m \cap \mathscr{V}_j^m$, giving $\mathscr{V}_i^m \cap \mathscr{V}_j^m \neq \varnothing$. Claim 4.14 then yields $i = j$, and so $\{U_1, \ldots, U_n\}$ is pairwise disjoint.

Clearly $G = X \setminus \bigcup_{i=1}^n U_i$ is a closed subset of X. Because $\{U_1, \ldots, U_n\}$ is a pairwise disjoint family of nonempty sets and $G = X \setminus \bigcup_{i=1}^n U_i$, we only need to verify that G is nonempty to conclude that $\{G, U_1, \ldots, U_n\}$ is a partition. But, by Claim 4.17, $\varnothing \neq F = D_0 \subseteq \bigcup\{D_m \mid m \in \omega\} = D \subseteq X \setminus \bigcup_{i=1}^n U_i = G$, completing the proof. ■

Claim 4.19. *G is a nowhere dense and dense-in-itself subspace of X.*

Proof. Since G is closed, to see that G is nowhere dense, let $\mathbf{i}G \neq \varnothing$. Then there are $m \geq 1$ and a nonempty $B \in \mathscr{B}_m$ such that $B \subseteq G$. By Claim 4.16, $B \not\subseteq \bigcup \mathscr{V}^m$. By condition (4a), there is (a nonempty) $V_1 \in \mathscr{V}_1^m$ such that $\mathbf{c}V_1 \subseteq B$. But then

$$\begin{array}{rcl} \varnothing & \neq & V_1 = B \cap V_1 \subseteq G \cap V_1 \\ & \subseteq & G \cap \bigcup \mathscr{V}_1^m \subseteq G \cap \bigcup \mathscr{V}_1 \\ & = & G \cap U_1 = \varnothing, \end{array}$$

which is a contradiction.

To see that G is dense-in-itself, let $m \geq 1$ and $B \in \mathscr{B}_m$ be such that $B \cap G \neq \varnothing$. By Claim 4.16, $B \not\subseteq \bigcup \mathscr{V}^m$. By condition (4b), $B \cap G \supseteq B \cap D \supseteq B \cap D_m$ contains at least two points. ■

Claim 4.20. *For each $i \in \{1, \ldots, n\}$, there is $F_i \subseteq U_i$ that is discrete and $\mathbf{c}F_i = F_i \cup G$.*

Proof. Let $i \in \{1, \ldots, n\}$. Each $V \in \mathscr{V}_i$ is nonempty, and hence we may choose $x_V \in V$. Set $F_i = \{x_V \mid V \in \mathscr{V}_i\}$. Since $U_i = \bigcup \mathscr{V}_i$, we clearly have that $F_i \subseteq U_i$. By Claim 4.15, $\mathscr{V}_i$ is pairwise disjoint, and so $\{x_V\} = V \cap F_i$ for each $V \in \mathscr{V}_i$. As each $V \in \mathscr{V}_i$ is an open subset of X, we have that F_i is discrete.

Let $x \in G$, $m \geq 1$, and $B \in \mathscr{B}_m$ be arbitrary with $x \in B$. Then $x \in B \cap G$, and so $B \not\subseteq \bigcup \mathscr{V}^m$ by Claim 4.16. By condition (4a), there is $V \in \mathscr{V}_i^m$ such that $\mathbf{c}V \subseteq B$. Therefore, $V \in \mathscr{V}_i$, and so $B \cap F_i \supseteq V \cap F_i = \{x_V\} \neq \varnothing$, giving $x \in \mathbf{c}F_i$. Thus, $G \subseteq \mathbf{c}F_i$, and hence $F_i \cup G \subseteq \mathbf{c}F_i$.

For the reverse inclusion, suppose $x \notin F_i \cup G$. Then $x \notin G$, so $x \in \bigcup_{j=1}^{n} U_j$. If $x \notin U_i$, then $\bigcup \{U_j \mid j \neq i\}$ is an open neighborhood of x that is disjoint from F_i. If $x \in U_i$, then $x \in V$ for some $V \in \mathscr{V}_i$ (such V is unique by Claim 4.15). Noting that $x \neq x_V$ (since $x \notin F_i$) gives that $V \setminus \{x_V\}$ is an open neighborhood of x that is disjoint from F_i (since $V \cap F_i = \{x_V\}$). In both cases, $x \notin \mathbf{c}F_i$. Thus, $\mathbf{c}F_i \subseteq F_i \cup G$, and the equality follows. ■

This completes the proof of the Partition Lemma. ■

4.3. The Mapping Lemma

In this subsection we prove the Mapping Lemma, which yields a new proof of the McKinsey-Tarski Theorem. This requires the following lemma.

LEMMA 4.21. *A dense-in-itself metrizable space X is m-resolvable for every $m \geq 1$.*

PROOF. For each $m \geq 1$, we recursively construct a pairwise disjoint family $\{A_1, \dots, A_m\}$ of dense subsets of X. Put $X_0 = X$. Suppose X_n is a dense subset of X for some $n \in \omega$. Then X_n is a dense-in-itself metrizable space. By [11, Thm. 41], X_n is resolvable. So there is $A_{n+1} \subseteq X_n$ such that A_{n+1} and $X_{n+1} := X_n \setminus A_{n+1}$ are both dense in X_n. Therefore, $X_n = \mathbf{c}_{X_n} A_{n+1} = \mathbf{c}(A_{n+1}) \cap X_n \subseteq \mathbf{c}A_{n+1}$, and similarly $X_n \subseteq \mathbf{c}X_{n+1}$. Thus, both A_{n+1} and X_{n+1} are dense in X since X_n is dense in X. An easy inductive argument gives that $X_m \subseteq X_n$ whenever $m \geq n$ since by definition $X_{n+1} \subseteq X_n$. To see that $\{A_1, \dots, A_m\}$ is pairwise disjoint, without loss of generality let $i > j \geq 1$. Then

$$A_i \cap A_j \subseteq X_{i-1} \cap A_j \subseteq X_i \cap A_j = (X_{i-1} \setminus A_j) \cap A_j = \varnothing.$$

Clearly $\left\{A_1, \dots, A_{m-1}, X \setminus \bigcup_{i=1}^{m-1} A_i\right\}$ is a dense partition of X of cardinality $m \geq 1$. ■

LEMMA 4.22 (**Mapping Lemma**). *Let X be a dense-in-itself metrizable space and F a nonempty closed discrete subspace of X. Then for any finite quasi-tree $\mathcal{T}$, there is an interior mapping of X onto $\mathcal{T}$ such that the image of F is contained in the root cluster of $\mathcal{T}$.*

PROOF. Let the root cluster C of $\mathcal{T} = (W, R)$ consist of m elements, say $C = \{r_1, \dots, r_m\}$. The proof is by induction on depth$(\mathcal{T})$.

First suppose depth$(\mathcal{T}) = 1$. Then $W = C$. By Lemma 4.21, X is m-resolvable. Let $\{A_1, \dots, A_m\}$ be a dense partition of X. Define $f : X \to W$

by $f(x) = r_i$ when $x \in A_i$. By [5, Lem. 5.9], f is a well-defined onto interior map.

Next suppose $\text{depth}(\mathcal{T}) \geq 2$. By the inductive hypothesis, for every dense-in-itself metrizable space Y, a nonempty closed discrete subspace Z of Y, and a finite quasi-tree $\mathcal{S}$ of depth $< \text{depth}(\mathcal{T})$, there is an interior map g of Y onto $\mathcal{S}$ such that $g(Z)$ is contained in the root cluster of $\mathcal{S}$. Let $w_1, \dots, w_n \in W$ be such that $\{C, R[w_1], \dots, R[w_n]\}$ is a partition of W as depicted in Figure 3. For $i \in \{1, \dots, n\}$, let $\mathcal{T}_i = (W_i, R_i)$ be the generated subframe of $\mathcal{T}$ such that $W_i = R[w_i]$. Then $\mathcal{T}_i$ is a finite quasi-tree such that $\text{depth}(\mathcal{T}_i) < \text{depth}(\mathcal{T})$ and $C_i := R_i^{-1}[w_i]$ is the root cluster of $\mathcal{T}_i$.

By Lemma 4.13, there is a partition $\{G, U_1, \dots, U_n\}$ of X such that G is a dense-in-itself closed nowhere dense subspace of X containing F and each U_i is an open subspace of X containing a nonempty discrete subspace F_i such that $\mathbf{c}F_i = F_i \cup G$. Since G is a dense-in-itself metrizable space, Lemma 4.21 yields a dense partition $\{A_1, \dots, A_m\}$ of G. Also, each U_i is a dense-in-itself metrizable space and F_i is closed relative to U_i because

$$\mathbf{c}_{U_i} F_i = \mathbf{c}(F_i) \cap U_i = (F_i \cup G) \cap U_i = F_i.$$

By the inductive hypothesis, there is an interior map f_i of U_i onto $\mathcal{T}_i$ such that $f_i(F_i) \subseteq C_i$. Define $f : X \to W$ by

$$f(x) = \begin{cases} r_i & \text{if } x \in A_i \text{ for } i \in \{1, \dots, m\}, \\ f_j(x) & \text{if } x \in U_j \text{ for } j \in \{1, \dots, n\}. \end{cases}$$

Then f is well defined since $\{A_1, \dots, A_m, U_1, \dots, U_m\}$ is a partition of X. It is onto because $W = C \cup \bigcup_{i=1}^n W_i$, $f(G) = \bigcup_{i=1}^m f(A_i) = \bigcup_{i=1}^m \{r_i\} = C$, and f_i maps U_i onto W_i. It is also clear that $f(F) \subseteq f(G) = C$ (see Figure 5).

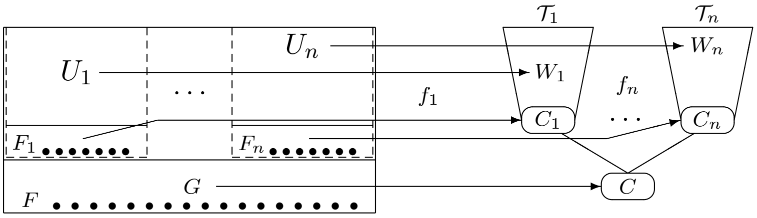

Figure 5. Depiction of $f : X \to \mathcal{T}$.

To see that f is continuous, it is sufficient to show that $f^{-1}(R[w])$ is open in X for each $w \in W$. If $w \in C$, then $R[w] = W$, and so $f^{-1}(R[w]) = f^{-1}(W) = X$. If $w \notin C$, then $w \in W_i$ for a unique $i \in \{1, \dots, n\}$, and hence

$R[w] = R_i[w] \subseteq W_i$. Because f_i is an interior mapping, $f_i^{-1}(R_i[w])$ is an open subset of U_i. Since U_i is open in X, we conclude that $f^{-1}(R[w]) = f_i^{-1}(R_i[w])$ is an open subset of X. Thus, f is continuous.

To see that f is open, let U be an open subset of X. Recalling that $\{G, U_1, \ldots, U_n\}$ is a partition of X, we have

$$\begin{aligned} f(U) &= f\left(U \cap \left(G \cup \bigcup_{i=1}^{n} U_i\right)\right) = f\left((U \cap G) \cup \bigcup_{i=1}^{n}(U \cap U_i)\right) \\ &= f(U \cap G) \cup \bigcup_{i=1}^{n} f(U \cap U_i) = f(U \cap G) \cup \bigcup_{i=1}^{n} f_i(U \cap U_i). \end{aligned}$$

Each $U \cap U_i$ is an open subset of U_i. Since f_i is interior, $f_i(U \cap U_i)$ is an R_i-cone of $\mathcal{T}_i$, and hence an R-cone of $\mathcal{T}$. If $U \cap G = \varnothing$, then $f(U) = \bigcup_{i=1}^{n} f_i(U \cap U_i)$ is a union of R-cones of $\mathcal{T}$, so is an R-cone of $\mathcal{T}$.

Suppose $U \cap G \neq \varnothing$. We show that $f(U) = W$. Since $\{A_1, \ldots, A_m\}$ is a dense partition of G, it follows that $(U \cap G) \cap A_i \neq \varnothing$ for each $i \in \{1, \ldots, m\}$. Therefore, $\{r_i\} = f(U \cap G \cap A_i) \subseteq f(U \cap G)$ for each $i \in \{1, \ldots, m\}$, yielding that $f(U \cap G) = C$. Since $U \cap G \neq \varnothing$, we have $U \cap \mathbf{c}F_j \neq \varnothing$, so $U \cap F_j \neq \varnothing$ for each $j \in \{1, \ldots, n\}$. Let $x_j \in U \cap F_j$. Then $f(x_j) = f_j(x_j) \in C_j$, which is the root cluster of $\mathcal{T}_j$. But $f(x_j) \in f_j(U \cap U_j)$, which is an R_j-cone of $\mathcal{T}_j$ since f_j is interior. Thus, $f_j(U \cap U_j) = W_j$. Consequently, $f(U) = C \cup \bigcup_{i=1}^{n} W_i = W$, and hence f is open, completing the proof. ■

We conclude the section by reiterating how the above delivers a modern proof of the McKinsey-Tarski Theorem that $\mathsf{S4}$ is the logic of any dense-in-itself metrizable space. Let X be a dense-in-itself metrizable space. Then $X \models \mathsf{S4}$. Suppose that $\mathsf{S4} \nvdash \varphi$. Then there is a finite quasi-tree $\mathcal{T}$ refuting φ. By the Mapping Lemma, $\mathcal{T}$ is an interior image of X. Thus, $X \not\models \varphi$.

5. Conclusions

In this final section we indicate how the new proof of the McKinsey-Tarski Theorem could be utilized for better understanding of the modal logics arising from topological spaces.

For a topological space X, let $\mathsf{Log}(X)$ be the set of modal formulas valid in X. Then $\mathsf{Log}(X)$ is a normal extension of $\mathsf{S4}$, and one of the key problems in topological semantics of modal logic is to axiomatize $\mathsf{Log}(X)$. By the McKinsey-Tarski Theorem, if X is a dense-in-itself metrizable space, then $\mathsf{Log}(X) = \mathsf{S4}$. This result was utilized in [6] to axiomatize $\mathsf{Log}(X)$ for an arbitrary metrizable space X. It is desirable to generalize this result to cover larger classes of well-studied topological spaces.

Tychonoff spaces are some of the most studied topological spaces. The class of Tychonoff spaces includes metrizable spaces, compact Hausdorff spaces, paracompact spaces, normal spaces etc. It is a challenging open problem to determine $\mathsf{Log}(X)$ for an arbitrary Tychonoff space.

The new proof technique developed in this paper does not generalize to the general setting of Tychonoff spaces. In fact, the McKinsey-Tarski Theorem is no longer true already for hereditarily paracompact spaces. In [4, Sec. 3] a countable extremally disconnected dense subspace X of the Gleason cover of the closed real unit interval $[0, 1]$ is exhibited whose logic is $\mathsf{S4.3} := \mathsf{S4} + \Box(\Box p \to q) \vee \Box(\Box q \to p)$. Clearly X is dense-in-itself. As a countable space, X is hereditarily Lindelöf, and hence hereditarily paracompact (e.g., by [10, Thm. 5.1.2]). Therefore, there are dense-in-itself hereditarily paracompact spaces for which the McKinsey-Tarski Theorem is no longer true.

However, we are currently working on the possibility of generalizing the new proof technique developed in this paper to determine $\mathsf{Log}(X)$, where X is either a first-countable Tychonoff space or a monotonically normal space. In this direction, we conclude the paper with the following two open problems:

- Is $\mathsf{Log}(X) = \mathsf{S4}$ for an arbitrary dense-in-itself first countable Tychonoff space? If not, then characterize the dense-in-itself first countable Tychonoff spaces for which $\mathsf{Log}(X) = \mathsf{S4}$.
- Is $\mathsf{Log}(X) = \mathsf{S4}$ for an arbitrary dense-in-itself monotonically normal space? If not, then characterize the dense-in-itself monotonically normal spaces for which $\mathsf{Log}(X) = \mathsf{S4}$.

Acknowledgements. We are thankful to the referee who suggested to add Section 5 to the paper. The first two authors were partially supported by Shota Rustaveli National Science Foundation grant # DI-2016-25.

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