STABLE FORMULAS IN INTUITIONISTIC LOGIC

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Abstract. NNIL-formulas were introduced in [16], where it was shown that these are (up to provable equivalence) exactly the formulas that are preserved under taking submodels of Kripke models. In this paper we show that NNIL-formulas are up to frame equivalence the formulas that are preserved under taking subframes of (descriptive and Kripke) frames. As a result we obtain that NNIL-formulas are subframe formulas and that all subframe logics can be axiomatized by NNIL-formulas.

We also define a new syntactic class of ONNILLI-formulas and show that these are (up to frame equivalence) the formulas that are preserved in monotonic images of (descriptive and Kripke) frames. As a result, we obtain that ONNILLI-formulas are stable formulas as introduced in [1] and that ONNILLI is a syntactically defined set of formulas that axiomatize all stable logics. This resolves an open problem of [1].

1. Introduction

Intermediate logics are logics situated between intuitionistic propositional calculus IPC and classical propositional calculus CPC. One of the central topics in the study of intermediate logics is their axiomatization. Jankov [15], by means of Heyting algebras, and de Jongh [13], via Kripke frames, developed an axiomatization method for intermediate logics using the so-called splitting formulas. These formulas are also referred to as *Jankov-de Jongh formulas*. In algebraic terminology, for each finite subdirectly irreducible Heyting algebra A, its Jankov formula is refuted in an algebra B, if there is a one-one Heyting homomorphism from A into a homomorphic image of B. In other words, the Jankov formula of A axiomatizes the greatest variety of Heyting algebras that does not contain A. In terms of Kripke frames, for each finite rooted frame $\mathfrak{F}$, the Jankov-de Jongh formula of $\mathfrak{F}$ is refuted in a frame $\mathfrak{G}$ iff $\mathfrak{F}$ is a p-morphic image of a generated subframe of $\mathfrak{G}$. In fact, the Jankov-de Jongh formula of $\mathfrak{F}$ axiomatizes the least intermediate logic that does not have $\mathfrak{F}$ as its frame. Large classes of intermediate logics (splitting and join-splitting logics) are axiomatizable by Jankov-de Jongh formulas. However, not every intermediate logic is axiomatizable by such formulas, see e.g., [11, Sec 9.4].

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Zakharyaschev [18, 19] introduced new classes of formulas called *subframe* and *cofinal subframe formulas* that axiomatize large classes of intermediate logics not axiomatizable by Jankov-de Jongh formulas. For each finite rooted frame $\mathfrak{F}$ the (cofinal) subframe formula of $\mathfrak{F}$ is refuted in a frame $\mathfrak{G}$ iff $\mathfrak{F}$ is a p-morphic image of a (cofinal) subframe of $\mathfrak{G}$. Logics axiomatizable by subframe and cofinal subframe formulas are called *subframe* and *cofinal subframe logics*, respectively. There is a continuum of such logics and each of them enjoys the finite model property. Moreover, Zakharyaschev showed that subframe logics are exactly those logics whose frames are closed under taking subframes. He also showed that an intermediate logic L is a subframe logic iff it is axiomatizable by $(\wedge, \to)$-formulas, and L is a cofinal subframe logic iff it is axiomatizable by $(\wedge, \to, \bot)$-formulas. However, there exist intermediate logics that are not axiomatizable by subframe and cofinal subframe formulas, see e.g., [11, Sec 9.4]. Finally, Zakharyaschev [18] introduced *canonical formulas* that generalize these three types of formulas and showed that every intermediate logic is axiomatizable by these formulas.

Zakharyaschev's method was model-theoretic. In [6] an algebraic approach to subframe and cofinal subframe logics was developed and in [2] extended to a full algebraic treatment of canonical formulas. This approach is based on identifying locally finite reducts of Heyting algebras. Recall that a variety V of algebras is called *locally finite* if the finitely generated V-algebras are finite. In logical terminology the corresponding notion is called local tabularity. A logic L is called *locally tabular* if there exist only finitely many non-L-equivalent formulas in finitely many variables. Note that $\vee$-free reducts of Heyting algebras are locally finite.

Based on the above observation, for a finite subdirectly irreducible Heyting algebra A, [2] defined a formula that encodes fully the structure of the $\vee$-free reduct of A, and only partially the behavior of $\vee$. This results in a formula that has properties similar to the Jankov formula of A, but captures the behavior of A not with respect to Heyting homomorphisms, but rather morphisms that preserve the $\vee$-free reduct of A. This formula is called the $(\wedge, \to)$-*canonical formula of* A, and such $(\wedge, \to)$-canonical formulas axiomatize all intermediate logics. In [2], it was shown, via the Esakia duality for Heyting algebras, that $(\wedge, \to)$-canonical formulas are frame-equivalent to Zakharyaschev's canonical formulas, and that so defined subframe and cofinal subframe formulas are frame-equivalent to Zakharyaschev's subframe and cofinal subframe formulas.

However, Heyting algebras also have other locally finite reducts, namely $\to$-free reducts. Recently, [1] developed a theory of canonical formulas for intermediate logics based on these reducts of Heyting algebras. For a finite subdirectly irreducible Heyting algebra A, [1] defined the $(\wedge, \vee)$-*canonical formula of* A that encodes fully the structure of the $\to$-free reduct of A, and only partially the behavior of $\to$. One of the main results of [1] is that each intermediate logic is axiomatizable by $(\wedge, \vee)$-canonical formulas.

This parallels the result on the axiomatization of intermediate logics via $(\wedge, \to)$-canonical formulas.

The $(\wedge, \vee)$-canonical formulas produce a new class of formulas called *stable formulas*. It was shown in [1], via the Esakia duality, that for each finite rooted frame $\mathfrak{F}$ the stable formula of $\mathfrak{F}$ is refuted in a frame $\mathfrak{G}$ iff $\mathfrak{F}$ is an monotonic image of $\mathfrak{G}$. *Stable logics* are intermediate logics axiomatizable by stable formulas. There is a continuum of stable logics and all stable logics have the finite model property. Also, an intermediate logic is stable iff the class of its rooted frames is preserved under monotonic images [1].

Thus, stable formulas play the same role for $(\wedge, \vee)$-canonical formulas that subframe formulas play for $(\wedge, \to)$-canonical formulas. Also the role that subframes play for subframe formulas are played by monotonic images for stable formulas. A syntactic characterization of stable formulas was left in [1] as an open problem. The goal of this paper is to resolve this problem. This is done via the NNIL-formulas of [16].

NNIL-formulas were introduced in [16]. NNIL stands for *no nested implication to the left*. It was shown in [16] that these formulas are exactly the formulas that are closed under taking submodels of Kripke models. This implies that these formulas are also preserved under taking subframes. Moreover, for each finite rooted frame $\mathfrak{F}$, [7] constructs its subframe formula as a NNIL-formula. In Section 3 of this paper we recall this characterization and use it to show that the class of NNIL-formulas is (up to frame equivalence) the same as the class of subframe formulas. Hence, an intermediate logic is a subframe logic iff it is axiomatized by NNIL-formulas. This also implies that each NNIL-formula is frame-equivalent to a $(\wedge, \to)$-formula. We refer to [17] for more details on this.

In this paper we define a new class of ONNILLI-formulas. ONNILLI stands for *only* NNIL *to the left of implications*. We show that each ONNILLI-formula is closed under monotonic images of rooted frames. For each finite rooted frame $\mathfrak{F}$ we also construct an ONNILLI-formula as its stable formula. This shows that the class of stable formulas (up to frame equivalence) is the same as the class of ONNILLI-formulas. We deduce from this that an intermediate logic is stable iff it is axiomatizable by ONNILLI-formulas.

The paper is organized as follows. In Section 2 we recall Kripke and descriptive frames and models of intuitionistic logic and basic operations on them. In Section 3 we discuss in detail the connection between NNIL-formulas and subframe logics. In Section 4 we introduce ONNILLI formulas and prove that they axiomatize stable logics.

2. Preliminaries

For the definition and basic facts about intuitionistic propositional calculus IPC we refer to [11], [12] or [7]. Here we briefly recall the Kripke semantics of intuitionistic logic.

Let $\mathcal{L}$ denote a *propositional language* consisting of

- infinitely many propositional variables (letters) $p_0, p_1, \dots$,
- propositional connectives $\wedge$, $\vee$, $\rightarrow$,
- a propositional constant $\bot$.

We denote by Prop the set of all propositional variables. Formulas in $\mathcal{L}$ are defined as usual. Denote by Form($\mathcal{L}$) (or simply by Form) the set of all well-formed formulas in the language $\mathcal{L}$. We assume that $p, q, r, \dots$ range over propositional variables and $\varphi, \psi, \chi, \dots$ range over arbitrary formulas. For every formula φ and ψ we let $\neg\varphi$ abbreviate $\varphi \rightarrow \bot$ and $\varphi \leftrightarrow \psi$ abbreviate $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$. We also let $\top$ abbreviate $\neg\bot$.

We now quickly recall the Kripke semantics for intuitionistic logic. Let R be a binary relation on a set W. For every $w, v \in W$ we write wRv if $(w, v) \in R$ and we write $\neg(wRv)$ if $(w, v) \notin R$.

Definition 2.1.

(1) *An* intuitionistic Kripke frame *is a pair* $\mathfrak{F} = (W, R)$*, where* $W \neq \emptyset$ *and* R *is a partial order; that is, a reflexive, transitive and anti-symmetric relation on* W*.*

(2) *An* intuitionistic Kripke model *is a pair* $\mathfrak{M} = (\mathfrak{F}, V)$ *such that* $\mathfrak{F}$ *is an intuitionistic Kripke frame and* V *is an* intuitionistic valuation*, i.e., a map* V *from* Prop *to the powerset* $\mathscr{P}(W)$ *of* W *satisfying the condition:*

$$w \in V(p) \text{ and } wRv \text{ implies } v \in V(p).$$

The definition of the satisfaction relation $\mathfrak{M}, w \models \varphi$ where $\mathfrak{M} = (W, R, V)$ is an intuitionistic Kripke model, $w \in W$ and $\varphi \in$ Form is given in the usual manner (see e.g. [11]). We will write $V(\varphi)$ for $\{w \in W | w \models \varphi\}$. The notions $\mathfrak{M} \models \varphi$ and $\mathfrak{F} \models \varphi$ (where $\mathfrak{F}$ is a Kripke frame) are also introduced as usual.

Let $\mathfrak{F} = (W, R)$ be a Kripke frame. $\mathfrak{F}$ is called *rooted* if there exists $w \in W$ such that for every $v \in W$ we have wRv. It is well known that IPC is complete with respect to finite rooted frames; see, e.g., [11, Thm. 5.12].

Theorem 2.2. *For every formula* φ *we have*

$$\mathsf{IPC} \vdash \varphi \text{ iff } \varphi \text{ is valid in every finite rooted Kripke frame.}$$

Next we recall the main operations on Kripke frames and models. Let $\mathfrak{F} = (W, R)$ be a Kripke frame. For every $w \in W$ and $U \subseteq W$ let $R(w) = \{v \in W : wRv\}$, $R^{-1}(w) = \{v \in W : vRw\}$, $R^{-1}(w) = \{v \in W : vRw\}$, $R(U) = \bigcup_{w \in U} R(w)$, and $R^{-1}(U) = \bigcup_{w \in U} R^{-1}(w)$.

A subset $U \subseteq W$ is called an *upset of* $\mathfrak{F}$ if for every $w, v \in W$ we have that $w \in U$ and wRv imply $v \in U$. A frame $\mathfrak{F}' = (U, R')$ is called a *generated subframe of* $\mathfrak{F}$ if $U \subseteq W$, U is an upset of $\mathfrak{F}$ and R' is the restriction of R to U, i.e., $R' = R \cap U^2$. Let $\mathfrak{M} = (\mathfrak{F}, V)$ be a Kripke model. A model $\mathfrak{M}' = (\mathfrak{F}', V')$ is called a *generated submodel of* $\mathfrak{M}$ if $\mathfrak{F}'$ is a generated subframe of $\mathfrak{F}$ and V' is the restriction of V to U, i.e., $V'(p) = V(p) \cap U$. We write $\mathfrak{M}_w$ for the *submodel of* $\mathfrak{M}$ *generated by* w, i.e. with the domain $R(w)$.

Let $\mathfrak{F} = (W, R)$ and $\mathfrak{F}' = (W', R')$ be Kripke frames. A map $f : W \to W'$ is called a *p-morphism*[1] *between* $\mathfrak{F}$ *and* $\mathfrak{F}'$ if for every $w, v \in W$ and $w' \in W'$:

(1) wRv implies $f(w)R'f(v)$,
(2) $f(w)R'w'$ implies that there exists $u \in W$ such that wRu and $f(u) = w'$.

We call the conditions (1) and (2) the "forth" and "back" conditions, respectively. We say that f is *monotonic* if it satisfies the forth condition. If f is a surjective p-morphism from $\mathfrak{F}$ onto $\mathfrak{F}'$, then $\mathfrak{F}'$ is called a *p-morphic image of* $\mathfrak{F}$. Let $\mathfrak{M} = (\mathfrak{F}, V)$ and $\mathfrak{M}' = (\mathfrak{F}', V')$ be Kripke models. A map $f : W \to W'$ is called a *p-morphism between* $\mathfrak{M}$ *and* $\mathfrak{M}'$ if f is a p-morphism between $\mathfrak{F}$ and $\mathfrak{F}'$ and for every $w \in W$ and $p \in \text{Prop}$:

$$\mathfrak{M}, w \models p \ \text{ iff } \ \mathfrak{M}', f(w) \models p.$$

If a map between models satisfies the above condition, then we call it *valuation preserving*. If f is surjective, then $\mathfrak{M}$ is called a *p-morphic image of* $\mathfrak{M}'$; surjective p-morphisms are also called *reductions*; see, e.g., [11]. A *monotonic map* between two models is a monotonic map between the underlying frames which, in addition, is valuation preserving.

Next we recall the definition of general frames; see, e.g., [11, §8.1 and 8.4].

Definition 2.3. *An* intuitionistic general frame *or simply a* general frame *is a triple* $\mathfrak{F} = (W, R, \mathcal{P})$*, where* (W, R) *is an intuitionistic Kripke frame and* $\mathcal{P}$ *is a set of upsets such that* $\emptyset$ *and* W *belong to* $\mathcal{P}$*, and* $\mathcal{P}$ *is closed under* $\cup$*,* $\cap$ *and* $\Rightarrow$ *defined by*

$$U_1 \Rightarrow U_2 := \{w \in W : \forall v(wRv \wedge v \in U_1 \to v \in U_2)\} = W \setminus R^{-1}(U_1 \setminus U_2).$$

Note that every Kripke frame can be seen as a general frame where $\mathcal{P}$ is the set of all upsets of $\mathfrak{F} = (W, R, \mathcal{P})$. A *valuation* on a general frame is a map $V : \text{Prop} \to \mathcal{P}$. The pair $(\mathfrak{F}, V)$ is called a *general model*. The validity of formulas in general models is defined exactly the same way as for Kripke models.

Definition 2.4. *Let* $\mathfrak{F} = (W, R, \mathcal{P})$ *be a general frame.*

(1) *We call* $\mathfrak{F}$ *refined if for every* $w, v \in W$*:* $\neg(wRv)$ *implies that there is* $U \in \mathcal{P}$ *such that* $w \in U$ *and* $v \notin U$*.*
(2) *We call* $\mathfrak{F}$ *compact if for every* $\mathcal{X} \subseteq \mathcal{P} \cup \{W \setminus U : U \in \mathcal{P}\}$*, if* $\mathcal{X}$ *has the finite intersection property (that is, every intersection of finitely many elements of* $\mathcal{X}$ *is nonempty), then* $\bigcap \mathcal{X} \neq \emptyset$*.*
(3) *We call* $\mathfrak{F}$ *descriptive if it is refined and compact.*

We call the elements of $\mathcal{P}$ *admissible sets.*

Definition 2.5. *Let* $\mathfrak{F} = (W, R, \mathcal{P})$ *be a descriptive frame. A* descriptive valuation *is a map* $V : \text{Prop} \to \mathcal{P}$*. A pair* $(\mathfrak{F}, V)$ *where* V *is a descriptive valuation is called a descriptive model.*

[1] Some authors call such maps *bounded morphisms*; see, e.g., [10].

Validity of formulas in a descriptive frame (model) is defined similarly to the Kripke case except that it ranges over all descriptive valuations. It is well-known that every intermediate logic L is complete with respect to a class of descriptive frames, see e.g., [11, Thm. 8.36].

Next we recall the definitions of generated subframes and p-morphisms of descriptive frames.

Definition 2.6.

(1) *A descriptive frame* $\mathfrak{F}' = (W', R', \mathcal{P}')$ *is called a* generated subframe *of a descriptive frame* $\mathfrak{F} = (W, R, \mathcal{P})$ *if* (W', R') *is a generated subframe of* (W, R) *and* $\mathcal{P}' = \{U \cap W' : U \in \mathcal{P}\}$.

(2) *A map* $f : W \to W'$ *is called a* p-morphism *between* $\mathfrak{F} = (W, R, \mathcal{P})$ *and* $\mathfrak{F}' = (W', R', \mathcal{P}')$ *if* f *is a p-morphism between* (W, R) *and* (W', R') *and for every* $U' \in \mathcal{P}'$ *we have* $f^{-1}(U') \in \mathcal{P}$. *If a map between descriptive models satisfies the latter condition, it is called* admissible.

Generated submodels and p-morphisms between descriptive models are defined as in the case of Kripke semantics. For convenience, we will sometimes denote a descriptive frame, just as a pair (W, R), dropping the set $\mathcal{P}$ of admissible sets from the signature.

3. Subframe logics and NNIL-formulas

Subframe formulas for modal logic were first introduced by Fine [14]. Subframe formulas for intuitionistic logic were defined by Zakharyaschev [18]. For an overview of these results see [11, §9.4]. For an algebraic approach to subframe formulas we refer to [6] and [2]. We will define subframe formulas differently and connect them to the NNIL-formulas of [16]. Most of the results in this section have appeared in the PhD thesis [7].

We first recall from [16] and [17] some facts about NNIL-formulas. NNIL-formulas are known to have the following normal form:

Definition 3.1. NNIL-*formulas* (no nested implication to the left) *in* normal form *are defined by:*

$$\varphi := \bot \mid p \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid p \to \varphi.$$

Definition 3.2.

(1) *Let* $\mathfrak{F} = (W, R)$ *be a Kripke frame. A frame* $\mathfrak{F}' = (W', R')$ *is called a* subframe *of* $\mathfrak{F}$ *if* $W' \subseteq W$ *and* R' *is the restriction of* R *to* W'.

(2) *Let* $\mathfrak{F} = (W, R, \mathcal{P})$ *be a descriptive frame. A descriptive frame* $\mathfrak{F}' = (W', R', \mathcal{P}')$ *is called a subframe of* $\mathfrak{F}$ *if* (W', R') *is a subframe of* (W, R), $\mathcal{P}' = \{U \cap W' : U \in \mathcal{P}\}$ *and the following condition, which we call the* topo-subframe condition, *is satisfied:*

For every $U \subseteq W'$ *such that* $W' \setminus U \in \mathcal{P}'$ *we have* $W \setminus R^{-1}(U) \in \mathcal{P}$.

For a detailed discussion about the topological motivation behind the notion of subframes and its connection to nuclei of Heyting algebras we refer to [6] (see also [7]). Here we just note how we are going to use this condition.

Remark 3.3. The reason for adding the topo-subframe condition to the definition of subframes of descriptive frames is explained by the next proposition. The topo-subframe condition allows us to extend a descriptive valuation V' defined on a subframe $\mathfrak{F}'$ of a descriptive frame $\mathfrak{F}$ to a descriptive valuation V of $\mathfrak{F}$ such that the restriction of V to $\mathfrak{F}'$ is equal to V'.

Proposition 3.4. *Let $\mathfrak{F} = (W, R, \mathcal{P})$ and $\mathfrak{F}' = (W', R', \mathcal{P}')$ be descriptive frames. If $\mathfrak{F}'$ is a subframe of $\mathfrak{F}$, then for every descriptive valuation V' on $\mathfrak{F}'$ there exists a descriptive valuation V on $\mathfrak{F}$ such that the restriction of V to W' is V'.*

Proof. For every $p \in \textsc{Prop}$ let $V(p) = W \setminus R^{-1}(W' \setminus V'(p))$. By the topo-subframe condition, $V(p) \in \mathcal{P}$. Now suppose $x \in W'$. Then $x \notin V(p)$ iff $x \in R^{-1}(W' \setminus V'(p))$ iff (there is $y \in W'$ such that $y \notin V'(p)$ and xRy) iff $x \notin V'(p)$, since $V'(p)$ is an upset of $\mathfrak{F}'$. Therefore, $V(p) \cap W' = V'(p)$. □

We say that a formula φ is *preserved under submodels*, if for all models $\mathfrak{M} = (W, R, V)$ and $\mathfrak{N} = (W', R', V')$, if w is in the domain of $\mathfrak{N}$, $\mathfrak{N}$ is a submodel of $\mathfrak{M}$, and $\mathfrak{M}, w \models \varphi$, then $\mathfrak{N}, w \models \varphi$. We say that a formula φ is *preserved under subframes*, if for all (descriptive or Kripke) frames $\mathfrak{F} = (W, R)$ and $\mathfrak{G} = (W', R')$, if $\mathfrak{G}$ is a subframe of $\mathfrak{F}$, and $\mathfrak{F} \models \varphi$, then $\mathfrak{G} \models \varphi$.

We have the following characterization theorem showing that NNIL-formulas are exactly the ones that are preserved under submodels [16].

Theorem 3.5.

(1) *Every $\varphi \in \mathsf{NNIL}$ is preserved under submodels.*
(2) *If φ is preserved under submodels, then there exists $\psi \in \mathsf{NNIL}$ such that $\mathsf{IPC} \vdash \psi \leftrightarrow \varphi$.*

Corollary 3.6. NNIL*-formulas are preserved under subframes.*

Proof. Assume that a NNIL-formula is not preserved under subframes. Then there exists a NNIL-formula φ, (descriptive or Kripke) frames $\mathfrak{G}$ and $\mathfrak{F}$ such that $\mathfrak{F}$ is a subframe of $\mathfrak{G}$, $\mathfrak{G} \models \varphi$ and $\mathfrak{F} \not\models \varphi$. So there exists a valuation V on $\mathfrak{F}$ such that $(\mathfrak{F}, V) \not\models \varphi$. Let V' be a valuation on $\mathfrak{G}$ such that $(\mathfrak{F}, V)$ is a submodel of $(\mathfrak{G}, V')$. If $\mathfrak{F}$ and $\mathfrak{G}$ are descriptive frames, then such a V' exists by Proposition 3.4. If $\mathfrak{F}$ and $\mathfrak{G}$ are Kripke frames, then we again use the valuation V' defined in Proposition 3.4. Thus, we obtain that φ is not preserved under submodels, which contradicts Theorem 3.5. □

A formula is called a *subframe formula* if it is preserved under subframes. An intermediate logic is called a *subframe logic* if it is axiomatizable by subframe formulas. It is proved by Zakharyaschev (see e.g., [11, Thm. 11.25]) that an intermediate logic L is a subframe logic iff L is axiomatizable by

$(\wedge, \to)$-formulas iff descriptive frames of L are closed under subframes. Also, every subframe logic has the finite model property [11, Thm. 11.20].

Definition 3.7. *Let $\mathfrak{M} = (\mathfrak{F}, V)$ be a descriptive model. We fix n propositional variables $p_1, \dots, p_n$. With every point w of $\mathfrak{M}$, we associate a sequence $i_1 \dots i_n$ such that for $k = 1, \dots, n$:*

$$i_k = \begin{cases} 1, & if\ w \models p_k, \\ 0, & if\ w \not\models p_k. \end{cases}$$

We call the sequence $i_1 \dots i_n$ associated with w the color of w (or more specifically the n-color of w) and denote it by $col(w)$.

A finite model $\mathfrak{M}=(W, R, V)$ is colorful *if the number of propositional variables is $|W|$ and, for each $w \in W$, there is a propositional variable p_w such that $v \models p_w$ iff wRv.*

Definition 3.8. *Let $i_1 \dots i_n$ and $j_1 \dots j_n$ be two colors. We write*

$$i_1 \dots i_n \leq j_1 \dots j_n \quad \textit{iff} \quad i_k \leq j_k \textit{ for each } k = 1, \dots, n.$$

We also write $i_1 \dots i_n < j_1 \dots j_n$ if $i_1 \dots i_n \leq j_1 \dots j_n$ and $i_1 \dots i_n \neq j_1 \dots j_n$.

Let $\mathfrak{F}$ be a finite rooted frame. For every point w of $\mathfrak{F}$ we introduce a propositional letter p_w and let V be such that $V(p_w) = R(w)$. We denote the model $(\mathfrak{F}, V)$ by $\mathfrak{M}$. Then $\mathfrak{M}$ is colorful. Thus, with any finite frame $\mathfrak{F}$ we can associate a colorful model $(\mathfrak{F}, V)$. We call $(\mathfrak{F}, V)$ the *colorful model corresponding to $\mathfrak{F}$*.

Lemma 3.9. *Let $(\mathfrak{F}, V)$ be a colorful model. Then, for every $w, v \in W$, we have:*

1. *$w \neq v$ and wRv iff $col(w) < col(v)$,*
2. *$w = v$ iff $col(w) = col(v)$.*

Proof. The proof is just spelling out the definitions. □

Let $\mathfrak{F} = (W, R)$ be a finite rooted frame and $\mathfrak{M} = (\mathfrak{F}, V)$ the corresponding colorful model. Next we inductively define the subframe formula $\beta(\mathfrak{F})$ in the NNIL form. For every $v \in W$, let

$$prop(v) := \{p_k : v \models p_k, k \leq n\}, notprop(v) := \{p_k : v \not\models p_k, k \leq n\}.$$

Definition 3.10. *We define $\beta(\mathfrak{F})$ by induction. If v is a maximal point of $\mathfrak{M}$ then let*

$$\beta(v) := \bigwedge prop(v) \to \bigvee notprop(v)$$

Let w be a point in $\mathfrak{M}$ and let $w_1, \dots, w_m$ be all the immediate successors of w. We assume that $\beta(w_i)$ is already defined, for every w_i. We define $\beta(w)$ by

$$\beta(w) := \bigwedge prop(w) \to \bigvee notprop(w) \vee \bigvee_{i=1}^{m} \beta(w_i).$$

Let r be the root of $\mathfrak{F}$. We define $\beta(\mathfrak{F})$ by

$$\beta(\mathfrak{F}) := \beta(r).$$

We call $\beta(\mathfrak{F})$ the subframe formula of $\mathfrak{F}$.

Note that $\beta(\mathfrak{F})$ is a NNIL-formula. We will need the next lemma for establishing the crucial property of subframe formulas. We first recall the definition of depth of a frame and of a point.

Definition 3.11. *Let $\mathfrak{F}$ be a (descriptive or Kripke) frame.*

(1) *We say that $\mathfrak{F}$ is of depth $n < \omega$, denoted $d(\mathfrak{F}) = n$, if there is a chain of n points in $\mathfrak{F}$ and no other chain in $\mathfrak{F}$ contains more than n points. The frame $\mathfrak{F}$ is of finite depth if $d(\mathfrak{F}) < \omega$.*
(2) *We say that $\mathfrak{F}$ is of an infinite depth, denoted $d(\mathfrak{F}) = \omega$, if for every $n \in \omega$, $\mathfrak{F}$ contains a chain consisting of n points.*
(3) *The depth of a point $w \in W$ is the depth of $\mathfrak{F}_w$, i.e., the depth of the subframe of $\mathfrak{F}$ generated by w. We denote the depth of w by $d(w)$.*

Lemma 3.12. *Let $\mathfrak{F} = (W, R)$ be a finite rooted frame and let V be defined as above. Let $\mathfrak{M}' = (W', R', V')$ be an arbitrary (descriptive or Kripke) model. For every $w, v \in W$ and $x \in W'$, if wRv, then*

$$\mathfrak{M}', x \not\models \beta(w) \text{ implies } \mathfrak{M}', x \not\models \beta(v).$$

Proof. The proof is a simple induction on the depth of v. If $d(v) = d(w) - 1$ and wRv, then v is an immediate successor of w. Then $\mathfrak{M}', x \not\models \beta(w)$ implies $\mathfrak{M}', x \not\models \beta(v)$, by the definition of $\beta(w)$. Now suppose $d(v) = d(w) - (k+1)$ and the lemma is true for every u such that wRu and $d(u) = d(w) - k$, for every k. Let u' be an immediate predecessor of v such that wRu'. Such a point clearly exists since we have wRv. Then $d(u') = d(w) - k$ and by the induction hypothesis $\mathfrak{M}, x \not\models \beta(u')$. This, by definition of $\beta(u')$, means that $\mathfrak{M}', x \not\models \beta(v)$. ☐

The next theorem states the crucial property of subframe formulas (see also [7, Thm. 3.3.16]).

Theorem 3.13. *Let $\mathfrak{G} = (W', R', \mathcal{P}')$ be a descriptive frame and let $\mathfrak{F} = (W, R)$ be a finite rooted frame. Then*

$$\mathfrak{G} \not\models \beta(\mathfrak{F}) \text{ iff } \mathfrak{F} \text{ is a p-morphic image of a subframe of } \mathfrak{G}.$$

Proof. Suppose $\mathfrak{G} \not\models \beta(\mathfrak{F})$. Then there exists a valuation V' on $\mathfrak{G}$ such that $(\mathfrak{G}, V') \not\models \beta(\mathfrak{F})$. For every $w \in W$, let $\{w_1, \ldots w_m\}$ denote the set of all immediate successors of w. Let $p_1, \ldots, p_n$ be the propositional variables occurring in $\beta(\mathfrak{F})$ (in fact $n = |W|$). Then, V' defines a coloring of $\mathfrak{G}$. Let

$$P_w := \{x \in W' : col(x) = col(w) \text{ and } x \not\models \bigvee_{i=1}^{m} \beta(w_i)\}.$$

Take $Y := \bigcup_{w \in W} P_w$ and $\mathfrak{H} := (Y, S, \mathcal{Q})$, where S is the restriction of R' to Y, and $\mathcal{Q} = \{U' \cap Y : U' \in \mathcal{P}'\}$. We show that $\mathfrak{H}$ is a subframe of $\mathfrak{G}$ and $\mathfrak{F}$ is a p-morphic image of $\mathfrak{H}$.

For the proof that $\mathfrak{H}$ is a subframe of $\mathfrak{G}$ we just check the topo-subframe condition. The other conditions are clear from the definition of $\mathfrak{H}$. So, assume $Y \setminus U' \in \mathcal{Q}$. We have to show that $W' \setminus R'^{-1}(U') \in \mathcal{P}'$.

Note that $x \in W' \setminus R'^{-1}(U')$ iff $x \notin R'^{-1}(U')$ iff $\neg\exists y(xRy \wedge y \in U')$ iff $\forall y(xRy \to y \notin U')$ iff $\forall y(xRy \to y \notin Y \vee y \in Y \setminus U')$ iff $\forall y(xRy \wedge y \in Y \to y \in Y \setminus U')$ iff (for $U'' \in \mathcal{P}'$ such that $Y \setminus U' = U'' \cap Y$) $\forall y(xRy \wedge y \in Y \to y \in U'')$. Since $Y = \bigcup_{w \in W} P_w$, the latter is equivalent to the conjunction of all the $\forall y(xRy \wedge y \in P_w \to y \in U'')$ for $w \in W$. Then $\forall y(xRy \wedge y \in P_w \to y \in U'')$ iff $\forall y(xRy \wedge col(y) = col(w) \wedge y \not\models \bigvee_{i=1}^m \beta(w_i) \to y \in U'')$ iff $\forall y(xRy \wedge y \models \bigwedge prop(w) \to y \models \bigvee notprop(w) \vee y \models \bigvee_{i=1}^m \beta(w_i) \vee y \in U'')$. The sets $\{x \mid \forall y(xRy \wedge y \models \bigwedge prop(w) \to y \models \bigvee notprop(w) \vee y \models \bigvee_{i=1}^m \beta(w_i) \vee y \in U'')\}$ are equal to $V'(\bigwedge prop(w)) \Rightarrow (V'(\bigvee notprop(w)) \cup V'(\bigvee_{i=1}^m \beta(w_i)) \cup U'')$ and therefore are in $\mathcal{P}'$, as $\mathcal{P}'$ is closed under $\Rightarrow$ and union. So their intersection (the conjunction of the corresponding formulas) is also in $\mathcal{P}'$.

Define a map $f : Y \to W$ by

$$f(x) = w \text{ if } x \in P_w.$$

We show that f is a well-defined onto p-morphism. By Lemma 3.9, distinct points of W have distinct colors. Therefore, $P_w \cap P_{w'} = \emptyset$ if $w \neq w'$. This means that f is well-defined.

To prove that f is onto, by the definition of f, it is sufficient to show that $P_w \neq \emptyset$ for every $w \in W$. If r is the root of $\mathfrak{F}$, then since $(\mathfrak{G}, V') \not\models \beta(\mathfrak{F})$, there exists a point $x \in W'$ such that $x \models \bigwedge prop(r)$ and $x \not\models \bigvee notprop(r)$ and $x \not\models \bigvee_{i=1}^m \beta(r_i)$. This means that $x \in P_r$. If w is not the root of $\mathfrak{F}$ then we have rRw. Therefore, by Lemma 3.12, we have $x \not\models \beta(w)$. This means that there is a successor y of x such that $y \models \bigwedge prop(w)$, $y \not\models \bigvee notprop(w)$ and $y \not\models \beta(w_i)$, for every immediate successor w_i of w. Therefore, $y \in P_w$ and f is surjective.

To show that f is admissible we first note that to show an onto p-morphism to a finite frame to be admissible it is sufficient to show that for every upset U of W we have $f^{-1}(U) \in \mathcal{Q}$. It is clear that $U = R(u_1) \cup \cdots \cup R(u_k)$ for some $u_1, \ldots, u_k \in W$. Since $(\mathfrak{F}, V)$ is a colorful model, we have $R(u_i) = V(p_{u_i})$ for each $i = 1, \ldots, k$. Now observe that by the definition of f, for each $x \in Y$ we have $col(f(x)) = col(x)$. So $f(x) \in V(p_{u_i})$ if and only if $x \in V'(p_{u_i})$. This means that $f^{-1}(U) = (V'(p_{u_1}) \cup \cdots \cup V'(p_{u_k})) \cap Y$, which clearly is in $\mathcal{Q}$.

Next assume that $x, y \in Y$ and xSy. Note that by the definition of f, for every $t \in Y$ we have

$$col(t) = col(f(t)).$$

Obviously, xSy implies $col(x) \leq col(y)$. Therefore, $col(f(x)) = col(x) \leq col(y) = col(f(y))$. By Lemma 3.9, this yields $f(x)Rf(y)$. Now suppose $f(x)Rf(y)$. Then by the definition of f we have that $x \not\models \beta(f(x))$ and by Lemma 3.12, $x \not\models \beta(f(y))$. This means that there is $z \in W'$ such that $xR'z$,

$col(z) = col(f(y))$, and $z \not\models \beta(u)$, for every immediate successor u of $f(y)$. Thus, $z \in P_{f(y)}$ and $f(z) = f(y)$. Therefore, $\mathfrak{F}$ is a p-morphic image of $\mathfrak{H}$.

Conversely, suppose $\mathfrak{H}$ is a subframe of a descriptive frame $\mathfrak{G}$ and $f : \mathfrak{H} \to \mathfrak{F}$ is a p-morphism. Clearly, $\mathfrak{F} \not\models \beta(\mathfrak{F})$ and since f is a p-morphism, we have that $\mathfrak{H} \not\models \beta(\mathfrak{F})$. This means that there is a valuation V' on $\mathfrak{H}$ such that $(\mathfrak{H}, V') \not\models \beta(\mathfrak{F})$. By Proposition 3.4, V' can be extended to a valuation V on $\mathfrak{G}$ such that the restriction of V to $\mathfrak{G}'$ is equal to V'. Finally, recall that $\mathfrak{G}$ is a NNIL-formula. This, by Theorem 3.5(1), implies that $\mathfrak{G} \not\models \beta(\mathfrak{F})$. □

Zakharyaschev [18] showed that every subframe logic is axiomatizable by the formulas satisfying the condition of Theorem 3.13. We will now put this result in the context of frame-based formulas of [7] and [8]. We will use the same argument in the next section for stable logics and ONNILI-formulas.

For each intermediate logic L let $\mathbb{DF}(L)$ be the class of rooted descriptive frames of L. Note that [7] and [8] work with finitely generated descriptive frames. But for our purposes this restriction is not essential.

Definition 3.14. *Call a reflexive and transitive relation $\trianglelefteq$ on $\mathbb{DF}(\mathsf{IPC})$ a* frame order *if the following two conditions are satisfied:*

1. *For every $\mathfrak{F}, \mathfrak{G} \in \mathbb{DF}(\mathsf{IPC})$, $\mathfrak{G}$ is finite and $\mathfrak{F} \triangleleft \mathfrak{G}$ imply $|\mathfrak{F}| < |\mathfrak{G}|$.*
2. *For every finite rooted frame $\mathfrak{F}$ there exists a formula $\alpha(\mathfrak{F})$ such that for every $\mathfrak{G} \in \mathbb{DF}(\mathsf{IPC})$*

$$\mathfrak{G} \not\models \alpha(\mathfrak{F}) \quad \textit{iff} \quad \mathfrak{F} \trianglelefteq \mathfrak{G}.$$

The formula $\alpha(\mathfrak{F})$ is called the frame-based formula for $\trianglelefteq$.

Definition 3.15. *Let L be an intermediate logic. We let*

$$\mathbf{M}(L, \trianglelefteq) := min_{\trianglelefteq}(\mathbb{DF}(\mathsf{IPC}) \setminus \mathbb{DF}(L))$$

Theorem 3.16. [7, 8] *Let L be an intermediate logic and let $\trianglelefteq$ be a frame order on $\mathbb{DF}(\mathsf{IPC})$. Then L is axiomatized by frame-based formulas for $\trianglelefteq$ iff the following two conditions are satisfied.*

1. *$\mathbb{DF}(L)$ is a $\trianglelefteq$-downset. That is, for every $\mathfrak{F}, \mathfrak{G} \in \mathbb{DF}(\mathsf{IPC})$, if $\mathfrak{G} \in \mathbb{DF}(L)$ and $\mathfrak{F} \trianglelefteq \mathfrak{G}$, then $\mathfrak{F} \in \mathbb{DF}(L)$.*
2. *For every $\mathfrak{G} \in \mathbb{DF}(\mathsf{IPC}) \setminus \mathbb{DF}(L)$ there exists a finite $\mathfrak{F} \in \mathbf{M}(L, \trianglelefteq)$ such that $\mathfrak{F} \trianglelefteq \mathfrak{G}$.*

The formula $\beta(\mathfrak{F})$ is a particular case of a frame-based formula for a relation $\preccurlyeq$, where $\mathfrak{F} \preccurlyeq \mathfrak{G}$ if $\mathfrak{F}$ is a p-morphic image of a subframe of $\mathfrak{G}$. Condition (2) of Theorem 3.16 is always satisfied by $\preccurlyeq$ [11, Thm. 9.36], for an algebraic proof of this fact see [6] and [2]. So an intermediate logic L is a subframe logic iff L is axiomatizable by these formulas iff $\mathbb{DF}(L)$ is a $\preccurlyeq$-downset. As p-morphic images preserve the validity of formulas we obtain that $\mathbb{DF}(L)$ is a $\preccurlyeq$-downset iff $\mathbb{DF}(L)$ is closed under subframes. Thus, L is a subframe logic iff L is axiomatizable by these formulas iff $\mathbb{DF}(L)$ is closed under subframes.

We say that formulas φ and ψ are *frame-equivalent* if for any (descriptive) frame $\mathfrak{F}$ we have $\mathfrak{F} \models \varphi$ iff $\mathfrak{F} \models \psi$.

Corollary 3.17.

(1) *An intermediate logic L is a subframe logic iff L is axiomatizable by* NNIL*-formulas.*
(2) *The class of* NNIL*-formulas (up to frame equivalence) coincides with the class of subframe formulas.*
(3) *Each* NNIL*-formula is frame-equivalent to a $(\wedge, \to)$-formula.*

Proof. (1) As we showed above L is a subframe logic iff it is axiomatizable by the formulas of type $\beta(\mathfrak{F})$. As each $\beta(\mathfrak{F})$ is NNIL, subframe logics are axiomatizable by NNIL-formulas. Conversely, by Corollary 3.6, every NNIL-formula is preserved under subframes. Therefore, if L is axiomatizable by NNIL-formulas, $\mathbb{DF}(L)$ is closed under subframes. Thus, L is a subframe logic.

(2) By Corollary 3.6, every NNIL-formula is preserved under subframes. So every NNIL-formula is a subframe formula. Now suppose that φ is preserved under subframes. Then $\mathsf{IPC} + \varphi$ (where $\mathsf{IPC} + \varphi$ is the least intermediate logic containing formula φ) is a subframe logic. By (1) subframe logics are axiomatizable by the formulas $\beta(\mathfrak{F})$. Then there exists $\mathfrak{F}_1, \dots, \mathfrak{F}_n$ such that $\mathsf{IPC} + \varphi = \mathsf{IPC} + \bigwedge_{i=1}^{n} \beta(\mathfrak{F}_i)$. Note that $n \in \omega$, otherwise $\mathsf{IPC} + \varphi$ is infinitely axiomatizable, a contradiction. Each $\beta(\mathfrak{F}_i)$ is a NNIL-formula, so $\bigwedge_{i=1}^{n} \beta(\mathfrak{F}_i)$ is also a NNIL-formula. Thus, φ is frame-equivalent to a NNIL-formula and NNIL is (up to frame equivalence) the class of formulas preserved under subframes.

(3) also follows from (1) and the fact that subframe formulas are frame-equivalent to $(\wedge, \to)$-formulas [11, Thm 11.25]. A direct syntactic proof that each NNIL-formula is frame-equivalent to some $(\wedge, \to)$-formula can be found in [17]. □

We do not treat cofinal subframe logics here as they are not axiomatized by NNIL-formulas. We refer to [11, Sec 9.4] for a detailed treatment of these logics, to [6] and [2] for their algebraic analysis and to [7, Sec. 3.3.3] for the details on how to obtain cofinal subframe formulas from the subframe formulas introduced in this paper.

4. Stable logics and ONNILLI-formulas

In this section we construct a new class of formulas, ONNILLI, that turns out to be the class of formulas preserved by onto monotonic maps. This class is defined using the class of NNIL-formulas.

Proposition 4.1. *Let $\mathfrak{M} = (X, R, V)$ and $\mathfrak{N} = (Y, R', V')$ be two intuitionistic (Kripke or descriptive) models and $f \colon X \to Y$ a monotonic map on these models. Then, for each $x \in X$ and each $\varphi \in$ NNIL we have*

$$f(x) \models \varphi \Rightarrow x \models \varphi.$$

Proof. By induction on the normal form of φ as in Definition 3.1. The basic steps are trivial. Assume the induction hypothesis holds for φ and ψ and

$f(x) \models \varphi \wedge \psi$. Then, $f(x) \models \varphi$ and $f(x) \models \psi$. By the induction hypothesis, $x \models \varphi$ and $x \models \psi$. So, $x \models \varphi \wedge \psi$. The case for $\vee$ is similar.

So, finally assume the induction hypothesis holds for φ, and $f(x) \models p \to \varphi$. Now let xRy and $y \models p$. Then $f(x)Rf(y)$ and, as f is valuation preserving, $f(y) \models p$. So, $f(y) \models \varphi$. By the induction hypothesis, $y \models \varphi$. So $x \models p \to \varphi$. □

Corollary 4.2. *For each formula ψ there exists a* NNIL*-formula φ such that* IPC $\vdash \varphi \leftrightarrow \psi$ *iff for any pair of intuitionistic* (*Kripke or descriptive*) *models* $\mathfrak{M} = (X, R, V)$ *and* $\mathfrak{N} = (Y, R', V')$ *with a monotonic map* $f : X \to Y$ *and* $x \in X$*, we have*

$$f(x) \models \psi \Rightarrow x \models \psi. \tag{1}$$

Proof. The left to right direction follows from Proposition 4.1. Conversely, note that the identity function from a submodel into the larger model is always a monotonic map. Thus, if ψ satisfies (1), then ψ is preserved in submodels and, by Theorem 3.5, is equivalent to some NNIL-formula φ. □

Definition 4.3.

(1) BASIC *is the closure of the set of the propositional variables plus* $\top$ *and* $\bot$ *under conjunctions and disjunctions.*
(2) *Formulas of the form* $\varphi \to \psi$ *with* $\varphi \in$ NNIL *and* $\psi \in$ BASIC *are called* simple ONNILLI-formulas*.*
(3) *The class* ONNILLI (only NNIL to the left of implications) *is defined as the closure of the set of simple* ONNILLI*-formulas under conjunctions and disjunctions.*

Note that there are no iterations of implications in ONNILLI-formulas except inside the NNIL-part. Note also that, if $\psi \in$ BASIC and f is valuation-preserving, then by a simple induction argument we obtain that

$$y \models \psi \Leftrightarrow f(y) \models \psi. \tag{2}$$

Example 4.4. $\neg p \vee \neg\neg p$ is ONNILLI. To see this, write it as $(p \to \bot) \vee (\neg p \to \bot)$, and note that $\neg p$ is in NNIL. It is well-known that $\neg p \vee \neg\neg p$ is not preserved under taking subframes. (Note however that $\neg p \vee \neg\neg p$ is preserved under taking cofinal subframes e.g., [11, Sec. 9.4].) So, by Corollary 3.17 it cannot be equivalent to a NNIL-formula. Thus the class NNIL does not contain ONNILLI. We will see later that ONNILLI also does not contain NNIL.

Proposition 4.5.

(1) *Let* $\mathfrak{M} = (X, R, V)$ *and* $\mathfrak{N} = (Y, R', V')$ *be two rooted intuitionistic* (*Kripke or descriptive*) *models,* $f : X \to Y$ *a surjective monotonic map and* $\varphi \in$ ONNILLI *such that* $\mathfrak{M} \models \varphi$*. Then* $\mathfrak{N} \models \varphi$*.*
(2) *Let* $\mathfrak{F} = (X, R)$ *and* $\mathfrak{G} = (Y, R')$ *be two rooted intuitionistic* (*Kripke or descriptive*) *frames and* $f : X \to Y$ *a monotonic map from* $\mathfrak{F}$ *onto* $\mathfrak{G}$*. Then, for each* $\varphi \in$ ONNILLI*, if* $\mathfrak{F} \models \varphi$*, then* $\mathfrak{G} \models \varphi$*.*

Proof. (1) This is proved by induction on the form of φ. For the base case, let us consider a simple ONNILLI-formula $\varphi = \psi \to \chi$ with $\psi \in$ NNIL and $\chi \in$ BASIC, and let $\mathfrak{M} \models \psi \to \chi$, i.e., $x \models \psi \to \chi$ for all $x \in X$. Now assume $yR'y'$ and $y' \models \psi$. Note that because f is surjective, all elements of Y are of the form $f(x)$ for some $x \in X$. So, assume $y' = f(x)$. Then $f(x) \models \psi$. By Proposition 4.1 we know that $x \models \psi$. But then, since $x \models \psi \to \chi$ we have $x \models \chi$. As $\chi \in$ BASIC, we can, as we noted above (see (2)), conclude that $y' \models \chi$. Hence, $y \models \psi \to \chi$. Thus, $\mathfrak{N} \models \psi \to \chi$.

The induction step for conjunctions is straightforward. For the induction step for disjunctions it is necessary to require that the models are rooted. Indeed, let $\varphi = \psi \vee \chi$, and let the induction step hold for ψ and χ. Furthermore, let r and r' be the roots of $\mathfrak{M}$ and $\mathfrak{N}$ respectively. Then $\mathfrak{M} \models \psi \vee \chi$ implies that $r \models \psi$ or $r \models \chi$. Then, since $f(r) = r'$, by the above, we have $r' \models \psi$ or $r' \models \chi$. So $r' \models \psi \vee \chi$ and therefore $\mathfrak{N} \models \psi \vee \chi$.

(2) The proof is similar to the proof of Corollary 3.6 and follows immediately from part (1). □

In general, this proposition holds definitely only for rooted models, and not for truth in a node. Also surjectivity is an essential feature. Let us give an example to show that rootedness is needed for the result.

Example 4.6. Consider a non-rooted model $\mathfrak{M}$ which is the disjoint union of two linear 2-point models $\mathfrak{M}_1$ and $\mathfrak{M}_2$, $\mathfrak{M}_1$ consisting of two nodes with colors 00 and 10, and $\mathfrak{M}_2$ consisting of two nodes with colors 00 and 01. Furthermore, consider the 3-point model $\mathfrak{N}$ consisting of a root of color 00 and successors with colors 01 and 10 (see Figure 1). We have $\mathfrak{M} \models (p \to q) \vee (q \to p)$ because $\mathfrak{M}_1 \models q \to p$ and $\mathfrak{M}_2 \models p \to q$. But although there is an obvious surjective monotonic map from $\mathfrak{M}$ to $\mathfrak{N}$, clearly $\mathfrak{N} \not\models (p \to q) \vee (q \to p)$ in spite of the fact that $(p \to q) \vee (q \to p)$ is even in the intersection of NNIL and ONNILLI.

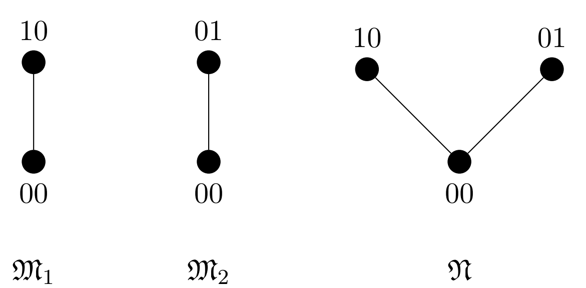

Figure 1

Definition 4.7.

(1) *If c is an n-color we write ψ_c for $p_1 \wedge \cdots \wedge p_k \to q_1 \vee \cdots \vee q_m$ if $p_1 \ldots p_k$ are the propositional variables that are associated with 1 in c and $q_1 \ldots q_m$ the ones that are associated with 0 in c.*

(2) *If* $\mathfrak{M}$ *is colorful and* $w \in W$*, we write* $Colors(\mathfrak{M}_w)$ *for the formula* $prop(w) \wedge \bigwedge\{\psi_c \mid c$ *is not the color of any point in* $\mathfrak{M}_w\}$*.*
(3) *If* $\mathfrak{M} = (W, R, V)$ *is finite and colorful,* $\gamma(\mathfrak{M}) = \bigvee\{Colors(\mathfrak{M}_w) \to p_{w_1} \vee \cdots \vee p_{w_m} \mid w \in W,\ w_1, \dots w_m$ *all immediate successors of* $w\}$*.*
(4) *Let* $\mathfrak{F}$ *be a finite rooted frame and* $\mathfrak{M} = (\mathfrak{F}, V)$ *the colorful model corresponding to* $\mathfrak{F}$*. We define*

$$\gamma(\mathfrak{F}) := \gamma(\mathfrak{M}).$$

We call $\gamma(\mathfrak{F})$ *the stable formula of* $\mathfrak{F}$*.*

Note that $\gamma(\mathfrak{F})$ is an ONNILLI-formula.

Lemma 4.8. *Let* $\mathfrak{M} = (W, R, V)$ *be a colorful model and* $\mathfrak{N} = (W', R', V')$ *be a model.*

(1) *The formula* ψ_c *expresses that the color* c *does not occur, more precisely, for* $u' \in W'$ *we have* $u' \models \psi_c$ *iff, for all* v' *with* $u' R' v'$*,* v' *does not have color* c*.*
(2) *The formula* $Colors(\mathfrak{M}_w)$ *expresses that only the colors in* $\mathfrak{M}_w$ *occur, i.e., if* $w \in W$ *and* $u' \in W'$*, then* $u' \models Colors(\mathfrak{M}_w)$ *iff, for all* v' *with* $u'R'v'$*,* v' *has the color of some node in* $\mathfrak{M}_w$*.*
(3) *Let* $w, w' \in W$ *such that* wRw'*, and let* $u' \in W'$*. Then* $u' \models Colors(\mathfrak{M}_w)$ *implies that for any* $v' \in W'$*,* $u'R'v'$ *and* $v' \models prop(w')$ *iff* $v' \models Colors(\mathfrak{M}_{w'})$*.*
(4) *Let* $w \in W$*,* $u' \in W'$*, and* $u' \not\models Colors(\mathfrak{M}_w) \to p_{w_1} \vee \cdots \vee p_{w_m}$*, where* $w_1, \dots, w_m$ *are the immediate successors of* w *in* $\mathfrak{M}$*. Then there is* $v' \in W'$ *such that* $u'Rv'$*,* $v' \models Colors(\mathfrak{M}_w)$ *and* $col(v') = col(w)$*.*

Proof. (1) is obvious. For (2) just note that the presence of $prop(w)$ means that only colors $\geq col(w)$ occur, and then apply (1).

For (3) apply (2) and note that then $v' \models Colors(\mathfrak{M}_{w'})$ iff $v' \models Colors(\mathfrak{M}_w)$ and $col(v') \geq col(w')$ (using for the $\Leftarrow$-direction the fact that $\mathfrak{M}$ is colorful) iff $v' \models Colors(\mathfrak{M}_w)$ and $v' \models prop(w')$.

(4) Let $u' \not\models Colors(\mathfrak{M}_w) \to p_{w_1} \vee \cdots \vee p_{w_m}$. Then there is $v' \in W'$ with $u' R v'$ such that $v' \models Colors(\mathfrak{M}_w)$ and $v' \not\models p_{w_1} \vee \cdots \vee p_{w_m}$. By (2), $col(v')$ should occur in $\mathfrak{M}_w$. As $v' \not\models p_{w_1}, \dots, p_{w_m}$, this color must be the color of w. ☐

It may be good to stress the fact that truth of $Colors(\mathfrak{M}_w)$ in a node u does not imply that all colors in $\mathfrak{M}_w$ occur above u, but only that different colors do not occur above u.

Lemma 4.9. *Let* $\mathfrak{F}$ *be a finite rooted frame. Then* $\mathfrak{F} \not\models \gamma(\mathfrak{F})$*.*

Proof. It is easy to see that if $\mathfrak{M}$ is a finite rooted colorful model with a root r, then $r \not\models Colors(\mathfrak{M}_w) \to p_{w_1} \vee \cdots \vee p_{w_m}$ for each $w \in W$ and $w_1, \dots w_m$ all immediate successors of w. The result follows from Lemma 4.8(4). ☐

Theorem 4.10. *Let* $\mathfrak{F} = (W, R)$ *be a finite rooted frame and let* $\mathfrak{G} = (W', R')$ *a rooted (Kripke or descriptive) frame. Then*

$\mathfrak{G} \not\models \gamma(\mathfrak{F})$ iff there is a surjective monotonic map from $\mathfrak{G}$ onto $\mathfrak{F}$.

Proof. $\Leftarrow$: Let $\mathfrak{F}$ and $\mathfrak{G}$ be as in the above assumptions of the theorem and assume there is a surjective monotonic map from $\mathfrak{G}$ onto $\mathfrak{F}$. Furthermore, let $\mathfrak{M}$ be a colorful model on $\mathfrak{F}$. By Definition 4.7, $\gamma(\mathfrak{F}) = \gamma(\mathfrak{M})$. By Lemma 4.9, $\mathfrak{F} \not\models \gamma(\mathfrak{F})$. Since $\gamma(\mathfrak{F})$ is an ONNILLI-formula, by Proposition 4.5(2), it is preserved under monotonic images of rooted frames. Thus, $\mathfrak{G} \not\models \gamma(\mathfrak{F})$.

$\Rightarrow$: Let $\mathfrak{N} = (W', R', V')$ be a model on $\mathfrak{G}$ such that $\mathfrak{N}, u \not\models \gamma(\mathfrak{F})$ for some $u \in W'$. Then u has, for each element $w \in W$, a successor u' that makes $Colors(\mathfrak{M}_w)$ true and $p_{w_1}, \dots, p_{w_m}$ false if $p_{w_1}, \dots, p_{w_m}$ are the immediate successors of w. This means, by Lemma 4.8(4) and 4.8(2), that u' has the color of w and its successors have colors of successors of w. Let U be the set of all such u's, i.e. $U = \{u' \mid \exists w \in W(u' \models Colors(\mathfrak{M}_w) \text{ and } col(u') = col(w))\}$. By Lemma 4.8(2) and 4.8(3), U is an upset of W'.

Let r be the root of $\mathfrak{F}$. Define a map $f : W' \to W$ by

$$f(u) = \begin{cases} w, & \text{if } u \in U,\ u \models Colors(\mathfrak{M}_w)\ \&\ col(u) = col(w), \\ r, & \text{otherwise.} \end{cases}$$

Because each point of W has a distinct color, f is well-defined.

If $u', v' \in U$ are such that $u'Rv'$, then there are $u, v \in W$ such that $col(u') = col(u)$ and $col(v') = col(v)$. By Lemma 3.9, we have uRv. So $f(u')Rf(v')$ and f is monotonic on U. Mapping the other nodes to the root of $\mathfrak{F}$ preserves this monotonicity.

We already saw that, for each $w \in W$, there exists $u \in U$ such that $u \models Colors(\mathfrak{M}_w)$ and $col(u) = col(w)$. Thus, $f(u) = w$ and f is also surjective. So, f is monotonic and surjective.

If $\mathfrak{N}$ is a descriptive model it remains to prove that f is admissible. For that it is sufficient to prove that, for each $w \in W$, $f^{-1}(R(w))$ is definable, i.e. $V'(\varphi)$ for some φ. But that is straightforward. If $f(r') = w$ for the root r' of $\mathfrak{G}$ it is trivial: $f^{-1}(R(w)) = W'$. Otherwise, $f^{-1}(R(w)) = V'(Colors(\mathfrak{M}_w))$. Namely, if $f(u) = w$, then $u \models Colors(\mathfrak{M}_w)$, and, if $f(u) = w'$ for some w' with wRw', then $u \models Colors(\mathfrak{M}_{w'})$, so $u \models Colors(\mathfrak{M}_w)$ as well. On the other hand, if $u \models Colors(\mathfrak{M}_w)$, then, by Lemma 4.8(2) and 4.8(3), for some w' with wRw', $u \models Colors(\mathfrak{M}_{w'})$ and $col(u) = col(w')$, so that $f(u) = w'$. □

If we define an order $\leq$ on (Kripke or descriptive) frames by putting $\mathfrak{F} \leq \mathfrak{G}$ if $\mathfrak{F}$ is a monotonic image of $\mathfrak{G}$. Then the formula $\gamma(\mathfrak{F})$ becomes a frame-based formula for $\leq$. Note that similarly to subframe formulas Condition (2) of Theorem 3.16 is always satisfied by $\leq$ [1] . Thus, an intermediate logic L is axiomatizable by these formulas iff $\mathbb{DF}(L)$ is a $\leq$-downset. Intermediate logics axiomatizable by these formulas are called *stable logics*. Therefore, a logic L is stable iff $\mathbb{DF}(L)$ is closed under monotonic images. Formulas closed under monotonic images are called *stable formulas*. There are continuum many stable logics and all of them enjoy the finite model property [1]. Now we are ready to prove our main theorem resolving an open problem of [1] on syntactically characterizing formulas that axiomatize stable logics.

Theorem 4.11.
(1) *An intermediate logic* L *is stable iff* L *is axiomatized by* $\mathsf{ONNILLI}$*-formulas.*
(2) *The class of* $\mathsf{ONNILLI}$*-formulas is up to frame equivalence the class all stable formulas.*

Proof. (1) As each $\gamma(\mathfrak{F})$ is $\mathsf{ONNILLI}$, all stable logics are axiomatized by $\mathsf{ONNILLI}$-formulas. By Proposition 4.5(2), every $\mathsf{ONNILLI}$-formula is preserved under monotonic images. Therefore, if L is axiomatized by $\mathsf{ONNILLI}$-formulas, $\mathbb{DF}(L)$ is closed under monotonic images. So L is stable.

(2) By Proposition 4.5(2), every $\mathsf{ONNILLI}$-formula is preserved under monotonic images. So $\mathsf{ONNILLI}$-formulas are stable. Now suppose that φ is preserved under monotonic images. Then $\mathsf{IPC} + \varphi$ is a stable logic. Stable logics are axiomatized by the formulas $\gamma(\mathfrak{F})$. So there exist $\mathfrak{F}_1, \ldots, \mathfrak{F}_n$ such that $\mathsf{IPC}+\varphi = \mathsf{IPC}+\bigwedge_{i=1}^{n} \gamma(\mathfrak{F}_i)$. Note that $n \in \omega$, otherwise $\mathsf{IPC}+\varphi$ is infinitely axiomatizable, a contradiction. Each $\gamma(\mathfrak{F}_i)$ is $\mathsf{ONNILLI}$, so $\bigwedge_{i=1}^{n} \gamma(\mathfrak{F}_i)$ is also $\mathsf{ONNILLI}$. Thus, φ is frame-equivalent to an $\mathsf{ONNILLI}$ formula and $\mathsf{ONNILLI}$ is (up to frame equivalence) the class of formulas closed under monotonic images. □

Example 4.12. It is now easy to construct NNIL-formulas that are not equivalent to an $\mathsf{ONNILLI}$-formula. Note that the logic BD_n of all frames of depth n for each $n \in \omega$ is closed under taking subframes. Thus, it is a subframe logic and hence by Corollary 3.17 is axiomatizable by NNIL-formulas. On the other hand it is easy to see that there are frames of depth n having frames of depth $m > n$ as monotonic images. So BD_n is not a stable logic. Therefore, it cannot be axiomatized by $\mathsf{ONNILLI}$-formulas. Thus, the class of $\mathsf{ONNILLI}$-formulas does not contain the class of NNIL-formulas (up to frame equivalence).

Example 4.13. We list some more examples of stable logics. Let LC_n be the logic of all linear rooted frames of depth $\leq n$, BW_n be the logic of all rooted frames of width $\leq n$ and BTW_n be the logic of all rooted descriptive frames of cofinal width $\leq n$. For the definition of width and cofinal width we refer to [11]. Then, for each $n \in \omega$, the logics LC_n, BW_n and BTW_n are stable. For the proofs we refer to [1].

It remains an open problem whether $\mathsf{ONNILLI}$-formulas are exactly the ones that are preserved under monotonic maps of models in the sense of Proposition 4.5(1).

We finish the paper by mentioning the connection to modal logic. Modal analogues of subframe formulas were defined by Fine [14]. Analogues of $(\wedge, \to)$-canonical formulas for transitive modal logics were investigated by

Zakharyaschev, see [11, Ch. 9] for an overview. An algebraic approach to these formulas was developed in [3] and generalized to weak transitive logics in [4]. Modal analogues of $(\wedge, \vee)$-canonical formulas are studied in [5], where modal analogues of stable logics are also defined. In [9] it is shown that modal stable logics have nice proof-theoretic properties. In particular, they have the bounded proof property.

It still remains open how to define modal analogues of NNIL and $\mathsf{ONNILLI}$ formulas for transitive modal logics and whether these formulas axiomatize all subframe and stable transitive modal logics, respectively. We note that a syntactic characterization of formulas axiomatizing subframe transitive modal logics is a long standing open problem [11].

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