S4.3 AND HEREDITARILY EXTREMALLY DISCONNECTED SPACES

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On the occasion of the one hundredth anniversary of the birth of George Chogoshvili.¹

Abstract. The modal logic $S4.3$ defines the class of hereditarily extremally disconnected spaces (HED-spaces). We construct a countable HED-subspace $X$ of the Gleason cover of the real closed unit interval $[0, 1]$ such that $S4.3$ is the logic of $X$.

1. Introduction

It is well known that if we interpret modal diamond as topological closure (and hence modal box as topological interior), then the modal logic $S4$ defines the class of all topological spaces. The celebrated McKinsey-Tarski theorem [14] states that $S4$ is the logic of any dense-in-itself (separable) metrizable space. In particular, $S4$ is the logic of the real closed unit interval $I = [0, 1]$.

We recall that a topological space is extremally disconnected (ED-space) provided the closure of each open set is open. Compact Hausdorff ED-spaces are of major importance in the category of compact Hausdorff spaces as they are the projective objects in the category. In fact, each compact Hausdorff space has the projective cover, called the Gleason cover (see [12, 13, 16]). The class of ED-spaces is definable by the modal logic $S4.2 = S4 + \Box p \rightarrow \Box \Diamond p$ (see, e.g., [3, p. 253]). As was shown in [1], ED-spaces play an important role in modeling full belief. It is a consequence of [5, Prop. 4.3] that $S4.2$ is the logic of the Gleason cover of $I$.

Our main result in this paper is the modal logic $S4.3 = S4 + \Box (\Box p \rightarrow q) \lor \Box (\Box q \rightarrow p)$. This system plays an important role in tense logic. It was studied in detail by Bull [7], Fine [11], and others. In particular, it is known that Kripke frames of $S4.3$ are those $S4.2$-frames whose subframes are also $S4.2$-frames. Similarly, we will see that topological spaces satisfying $S4.3$ are those ED-spaces whose subspaces are also ED-spaces. Because of this, $S4.3$ was recently proposed as the logic of updatable full belief [2].

ED-spaces whose subspaces are also ED-spaces are called hereditarily extremally disconnected spaces (HED-spaces). Unlike compact Hausdorff ED-spaces, which are in abundance, the only compact Hausdorff HED-spaces are finite (see, e.g., [6, p. 82]). On the other hand, there are plenty of non-compact HED-spaces. In fact, as follows from [6, Prop. 2.3], every dense-in-itself topology is contained in a dense-in-itself HED-topology. As we already pointed out above, $S4.2$ is the logic of the Gleason cover of $I$. Our main result yields a countable Hausdorff HED-subspace of the Gleason cover of $I$ whose logic is $S4.3$.

¹George Chogoshvili (1914–1998) was the founder of the Georgian topological school, and one of the most influential Georgian mathematicians of the twentieth century. He has been a source of inspiration for many generations. In particular, Chogoshvili’s ideas influenced Leo Esakia (1934–2010), who was one of the pioneers in developing topological modal logic, an area representing a fruitful cross-fertilization of tools and techniques of topology and logic. Our paper continues this tradition. We are honored to dedicate it to the memory of Professor Chogoshvili.
In proving our main result we utilize two tools, one logical and one topological. On the one hand, we use the fact that \textbf{S4.3} is characterized by finite rooted \textbf{S4.3}-frames (see, e.g., [8, Ch. 5]). On the other hand, we use Efimov’s theorem [9] (see also [15, Thm. 1.4.7]) that each compact Hausdorff ED-space of weight no greater than continuum can be embedded in the Čech-Stone compactification of the natural numbers.

2. Preliminaries

In this section we recall some basic definitions and facts about modal logic and topology. As basic references we use [8] for modal logic and [10] for topology.

2.1. Logical background. The modal logic \textbf{S4} is the least set of formulas containing the classical tautologies, the formulas

\[ \Box(p \to q) \to (\Box p \to \Box q), \]
\[ \Box p \to p, \]
\[ \Box p \to \Box \Box p, \]

and closed under Modus Ponens (MP) \( \phi \wedge \psi \to \phi \), substitution (S) \( \phi^{(p_1, \ldots, p_n)} \), and necessitation (N) \( \Box \phi \). As it is customary, we use \( \Diamond \phi \) to abbreviate \( \neg \Box \neg \phi \). Let

\[ \textbf{S4.2} = \textbf{S4} + \Diamond \Box p \to \Box \Diamond p, \]
\[ \textbf{S4.3} = \textbf{S4} + (\Box(p \to q) \lor \Box(q \to p)). \]

A \textit{Kripke frame} is a pair \( \mathcal{F} = (W, R) \), where \( W \) is a nonempty set and \( R \) is a binary relation on \( W \). A \textit{valuation} in \( \mathcal{F} \) is a function \( \nu \) assigning subsets of \( \mathcal{F} \) to propositional letters. This assignment extends recursively to all formulas, where Boolean connectives \( \land, \neg \) are interpreted as set-theoretic intersection and complement, and we set

\[ w \vDash \Box \phi \iff (\forall v)(wRv \to v \vDash \phi), \]
\[ w \vDash \Diamond \phi \iff (\exists v)(wRv \land v \vDash \phi). \]

A \textit{model} on \( \mathcal{F} \) is a pair \( \mathcal{M} = (\mathcal{F}, \nu) \), where \( \nu \) is a valuation in \( \mathcal{F} \). A formula \( \phi \) is \textit{true} in a model \( \mathcal{M} = (\mathcal{F}, \nu) \) provided \( w \vDash \phi \) for each \( w \in W \); and \( \phi \) is \textit{valid} in a frame \( \mathcal{F} \) provided \( \phi \) is true in every model on \( \mathcal{F} \). If \( \phi \) is valid in \( \mathcal{F} \), we write \( \mathcal{F} \vDash \phi \). If \( \phi \) is not valid in \( \mathcal{F} \), then we say that \( \mathcal{F} \) \textit{refutes} \( \phi \) and write \( \mathcal{F} \nvdash \phi \).

Let \( \mathcal{F} = (W, R) \) be a Kripke frame. We call \( \mathcal{F} \) a \textit{quasi-order} provided \( R \) is reflexive and transitive. The quasi-order \( \mathcal{F} \) is \textit{directed} provided \( (\forall u, v, v')(uRv \land uRv') \to (\exists w)(vRw \land v'Rw) \), and it is \textit{connected} provided \( (\forall u, v, w)(uRv \land uRw) \to (vRw \lor wRv) \). It is easy to see that \( \mathcal{F} \) is connected iff each subframe of \( \mathcal{F} \) is directed. It is well-known (see, e.g., [8, Ch. 3]) that \( \mathcal{F} \vDash \textbf{S4} \) if \( \mathcal{F} \) is a quasi-order, \( \mathcal{F} \vDash \textbf{S4.2} \) if \( \mathcal{F} \) is a directed quasi-order, and \( \mathcal{F} \vDash \textbf{S4.3} \) if \( \mathcal{F} \) is a connected quasi-order.

For a quasi-order \( \mathcal{F} = (W, R) \), define an equivalence relation \( \sim \) on \( W \) by setting \( w \sim v \) iff \( wRv \) and \( vRw \). The equivalence classes of \( \sim \) are called \textit{clusters} of \( \mathcal{F} \). One can partially order the clusters by setting \( C \preceq C' \) iff there exist \( w \in C \) and \( w' \in C' \) such that \( wRw' \). The resulting partial order is known as the \textit{skeleton} of \( \mathcal{F} \). We say that a cluster \( C \) of \( \mathcal{F} \) is \textit{maximal} provided \( C \) is a maximal element of the skeleton, and we call \( \mathcal{F} \) a \textit{quasi-chain} provided the skeleton of \( \mathcal{F} \) is a chain.

Let \( \mathcal{F} = (W, R) \) be a quasi-order. Then \( w \in W \) is a \textit{root} of \( \mathcal{F} \) provided \( wRv \) for each \( v \in W \), and \( \mathcal{F} \) is \textit{rooted} provided \( \mathcal{F} \) has a root. It is easy to see that a finite rooted quasi-order is directed iff it has a unique maximal cluster, and it is connected iff it is a quasi-chain. It is well known (see, e.g., [8, Ch. 5]) that \textbf{S4} is characterized by finite rooted quasi-orders, \textbf{S4.2}
is characterized by finite rooted quasi-orders having a unique maximal cluster, and \( S4.3 \) is characterized by finite quasi-chains.

2.2. Topological background. Topological semantics generalizes Kripke semantics for \( S4 \). Indeed, we can view quasi-orders as special topological spaces, in which each point has a least neighborhood, namely \( R[w] := \{ v \mid wRv \} \). Such spaces are often referred to as Alexandrov spaces and can equivalently be characterized as those topological spaces in which the intersection of an arbitrary family of opens is open.

Given a topological space \( X \), we interpret formulas as subsets of \( X \), Boolean connectives as the corresponding set-theoretic operations, \( \square \) as interior, and \( \Diamond \) as closure. Consequently, for \( x \in X \), we have

\[
\begin{align*}
\text{if} \quad & x \models \square \varphi \quad \text{iff} \quad \text{there is an open neighborhood } U \text{ of } x \text{ such that } y \models \varphi \text{ for all } y \in U, \\
\text{if} \quad & x \models \Diamond \varphi \quad \text{iff} \quad \text{for each open neighborhood } U \text{ of } x \text{ there is } y \in U \text{ such that } y \models \varphi.
\end{align*}
\]

Since \( S4 \)-axioms correspond to Kuratowski’s axioms, we see that \( S4 \) defines the class of all topological spaces. In addition, since \( S4 \) is Kripke complete, we see that \( S4 \) is the logic of all topological spaces. In fact, by the McKinsey-Tarski theorem [14], \( S4 \) is the logic of an arbitrary dense-in-itself separable metric space. Rasiowa and Sikorski proved in [17] that separability can be dropped from the assumptions, and hence \( S4 \) is the logic of an arbitrary dense-in-itself metric space.

A topological space \( X \) is extremally disconnected (ED-space) if the closure of an open subset of \( X \) is open, and it is hereditarily extremally disconnected (HED-space) if every subspace of \( X \) is an ED-space. Let \( i \) and \( c \) denote the interior and closure. Since \( X \) is an ED-space iff \( ci(A) \subseteq ic(A) \) for each \( A \subseteq X \), we see that \( S4.2 \) defines the class of all ED-spaces. In addition, since \( S4.2 \) is Kripke complete, we see that \( S4.2 \) is the logic of all ED-spaces. It is a corollary of the McKinsey-Tarski theorem that \( S4 \) is the logic of the real closed unit interval \( \mathbb{I} = [0, 1] \). It follows from [5] that \( S4.2 \) is the logic of the Gleason cover of \( \mathbb{I} \).

The Gleason cover of a compact Hausdorff space \( X \) is a pair \( (Y, \pi) \), where \( Y \) is a compact Hausdorff ED-space and \( \pi : Y \rightarrow X \) is an irreducible map (an onto continuous map such that the image of a proper closed subset of the domain is proper). The Gleason cover of \( X \) is unique up to homeomorphism, and can be constructed as follows. A subset \( U \) of \( X \) is regular open if \( U = ic(U) \). Let \( RO(X) \) be the collection of regular open subsets of \( X \). Ordered by inclusion, \( RO(X) \) is a complete Boolean algebra, where \( \bigvee U_i = ic(\bigcup U_i) \) and \( \neg U = i(X \setminus U) \). Let \( Y \) be the Stone space of \( RO(X) \) (the space of ultrafilters of \( RO(X) \)). By Stone duality, since \( RO(X) \) is complete, \( Y \) is a compact Hausdorff ED-space. Define \( \pi : Y \rightarrow X \) by setting \( \pi(\nabla) = \cap \{ c_X(U) \mid U \in \nabla \} \). Then \( (Y, \pi) \) is the Gleason cover of \( X \) [12].

3. Main results

Our goal is to obtain results about \( S4.3 \) and HED-spaces that are similar to the ones about \( S4.2 \) and ED-spaces. We start by showing that \( S4.3 \) defines the class of all HED-spaces (see also [2]). We recall that \( A, B \subseteq X \) are separated provided \( c(A) \cap B = \emptyset = A \cap c(B) \). By [6, Prop. 2.1], \( X \) is HED iff any two separated subsets of \( X \) have disjoint closures.

**Proposition 3.1.** For a topological space \( X \), the following are equivalent:

1. \( X \) is an HED-space.
2. \( X \models \Box(\Box p \rightarrow q) \lor \Box(\Box q \rightarrow p) \).
3. \( c(A \setminus cB) \cap c(B \setminus cA) = \emptyset \) for any \( A, B \subseteq X \).
Proof. It is straightforward to verify that when interpreting □ as i, then \( X \models □□(□p \rightarrow q) \lor □(□q \rightarrow p) \) iff \( c(A \setminus cB) \cap c(B \setminus cA) = \emptyset \) for any \( A, B \subseteq X \). Thus, (2) is equivalent to (3). To see that (1) implies (3), suppose that \( X \) is an HED-space. Since \( A \setminus cB \) and \( B \setminus cA \) are separated, [6, Prop. 2.1] yields \( c(A \setminus cB) \cap c(B \setminus cA) = \emptyset \). Finally, to see that (3) implies (1), suppose that \( A, B \subseteq X \) are separated. Then \( A \setminus cB = A \) and \( B \setminus cA = B \). Therefore, \( c(A) \cap c(B) = c(A \setminus cB) \cap c(B \setminus cA) = \emptyset \). Thus, \( X \) is an HED-space by [6, Prop. 2.1]. □

As a corollary, we obtain that S4.3 defines the class of all HED-spaces. Since S4.3 is Kripke complete, it follows that S4.3 is the logic of all HED-spaces. As we pointed out in the introduction, S4.2 is the logic of the Gleason cover of \( I \). We will construct a countable HED-subspace \( X \) of the Gleason cover of \( I \) whose logic is S4.3.

To see that the logic of an HED-space \( X \) is S4.3, in view of Proposition 3.1, it is sufficient to show that each non-theorem of S4.3 is refuted on \( X \). But since S4.3 is the logic of finite quasi-chains, each non-theorem of S4.3 is refuted on a finite quasi-chain. We call a topological space \( Y \) an interior image of \( X \) provided there is a continuous open surjection \( f : X \rightarrow Y \). Note that \( f \) is continuous and open iff \( c_x f^{-1}(B) = f^{-1}c_Y(B) \) for each \( B \subseteq Y \) (see, e.g., [17, pp. 99–100]); and \( B \) can be replaced by singletons provided \( Y \) is finite. It is well known (see, e.g., [4, Prop. 2.9]) that interior images reflect refutation. Therefore, to conclude that S4.3 is the logic of \( X \), it is sufficient to show that each finite quasi-chain, viewed as a topological space, is an interior image of \( X \).

Let \( (Y, \pi) \) be the Gleason cover of \( I \). As \( I \) is a dense-in-itself separable space, so is \( Y \). Moreover, since \( Y \) is an infinite compact Hausdorff ED-space, it contains a copy of the Čech-Stone compactification of the natural numbers \( β\omega \) (see, e.g., [18, p. 37] or [10, Exercise 6.2.G(b)]). Therefore, the weight of \( Y \) is at least that of continuum. But since \( Y \) is separable, its weight is at most that of continuum. Thus, the weight of \( Y \) is that of continuum. Furthermore, \( \pi^{-1}(x) \) is infinite for each \( x \in I \). To see this, take a pairwise disjoint family \( \{U_n \in RO(I) \mid n \in \omega\} \) such that \( x \in c(U_n) \) for each \( n \in \omega \). The filter in \( RO(I) \) generated by the regular open neighborhoods of \( x \) together with \( U_n \) is proper, hence extends to an ultrafilter in \( RO(I) \). Each such ultrafilter \( \nabla \) contains all regular open neighborhoods of \( x \), so \( \pi(\nabla) = x \). Since the \( U_n \)'s are disjoint, these ultrafilters are distinct, producing infinitely many points in \( \pi^{-1}(x) \). In fact, each \( (\pi_n)^{-1}(x) \) has a large cardinality because as an infinite closed set of a compact Hausdorff ED-space, it contains a copy of \( β\omega \) (see, e.g., [18, p. 37] or [10, Exercise 6.2.G(b)]).

**Lemma 3.2.** There is a pairwise disjoint family \( \{E_n \subseteq Y \mid n \in \omega\} \) such that each \( E_n \) is countably infinite and dense in \( Y \).

Proof. Let \( D \) be a countably infinite dense subset of \( I \) (for example, take \( D = Q \cap I \)). For each \( x \in D \), since \( \pi^{-1}(x) \) is infinite, there is a countably infinite subset \( D_x = \{x_n \mid n \in \omega\} \) of \( \pi^{-1}(x) \). For each \( n \in \omega \), define \( E_n = \{x_n \mid x \in D\} \) (see Figure 1). Clearly each \( E_n \) is countably infinite and \( \{E_n \mid n \in \omega\} \) is pairwise disjoint. It remains to be shown that \( E_n \) is dense in \( Y \) for each \( n \in \omega \). By construction, \( \pi(E_n) = \{\pi(x_n) \mid x \in D\} = D \). Since \( \pi \) is a closed map, \( \pi(cE_n) \) is a closed set in \( I \) containing \( D \). As \( D \) is dense in \( I \), we see that \( \pi(cE_n) = I \). Thus, since \( \pi \) is irreducible, \( cE_n = Y \). □

We are ready to construct an HED-subspace \( X \) of \( Y \) such that each finite quasi-chain is an interior image of \( X \). The space \( X \) will be the union of the spaces \( \{X_n \mid n \in \omega\} \), defined recursively. In defining \( X_n \) we will also define a decreasing sequence \( Y_n \) of subspaces of \( Y \) such that each \( Y_n \) is homeomorphic to \( Y \). Let \( D \) be a fixed countable dense subset of \( I \), and fix \( x \in I \setminus D \). 


Lemma 3.3.
(1) If \( n \geq m \), then \( X_n \subseteq Y_m \); and if \( n < m \), then \( X_n \cap Y_m = \emptyset \).
(2) \( X \) is countable.
(3) \( X \) is a dense subspace of \( Y \).
(4) \( X \) is an HED-space.

Proof. (1) By definition, \( Y_{m+1} \subseteq \beta_m \subseteq (\pi_m)^{-1}(x) \subseteq Y_m \). Therefore, \( n \geq m \) implies \( Y_n \subseteq Y_m \).
Since \( X_n \subseteq Y_n \), we conclude that \( n \geq m \) implies \( X_n \subseteq Y_m \).

By definition, \( X_n \subseteq Y_n \setminus (\pi_n)^{-1}(x) \). Since \( Y_{n+1} \subseteq (\pi_n)^{-1}(x) \), we see that \( X_n \cap Y_{n+1} = \emptyset \).
Therefore, if \( n < m \), then \( n + 1 \leq m \). Thus, \( Y_m \subseteq Y_{n+1} \), yielding \( X_n \cap Y_m \subseteq X_n \cap Y_{n+1} = \emptyset \).

(2) By definition, each \( X_n \) is a countable union of countable sets, hence is countable.
Therefore, \( X \) is a countable union of countable sets, and so is countable.

(3) It is clear by the definition of \( X_0 \) and Lemma 3.2 that \( X_0 \) is dense in \( Y \). Since \( X_0 \subseteq X \), we conclude that \( X \) is dense in \( Y \).
The idea is to map follows

Proof of Claim:
Let Suppose Lemma 3.4. Every finite quasi-chain is an interior image of $X$ a Hausdorff space, $X$ is also clearly Hausdorff. But every countable Hausdorff ED-space is an HED-space (see, e.g., [6, p. 86]). Thus, $X$ is an HED-space.

Lemma 3.4. Every finite quasi-chain is an interior image of $X$.

Proof. Suppose $\mathcal{F} = (W, R)$ is a finite quasi-chain, and suppose its skeleton is ordered as follows

$$C_{k-1} \leq C_{k-2} \leq \cdots \leq C_1 \leq C_0.$$  

The idea is to map $X_i$ to $C_i$ for $i < k$, and to $C_{k-1}$ for $i \geq k$. Recall that $X_i$ is the countably infinite disjoint union of countable infinite dense subsets $\{E_{ij} | j \in \omega\}$ of $Y_i$.

Claim 3.5. For any nonempty $\alpha \subseteq \omega$, we have $c_X(\bigcup \{E_{ij} | j \in \alpha\}) = \bigcup \{X_n | n \geq i\}$.

Proof of Claim: Let $\alpha \subseteq \omega$ be nonempty. Since each $E_{ij}$ is dense in $Y_i$, the set $\bigcup \{E_{ij} | j \in \alpha\}$ is dense in $Y_i$. Therefore,

$$Y_i = c_{Y_i} \left( \bigcup \{E_{ij} | j \in \alpha\} \right) = c_Y \left( \bigcup \{E_{ij} | j \in \alpha\} \right) \cap Y_i \subseteq c_Y \left( \bigcup \{E_{ij} | j \in \alpha\} \right).$$

Conversely, from $E_{ij} \subseteq Y_i$ it follows that $\bigcup \{E_{ij} | j \in \alpha\} \subseteq Y_i$. Therefore, since $Y_i$ is closed in $Y$, we have $c_Y \left( \bigcup \{E_{ij} | j \in \alpha\} \right) \subseteq Y_i$, hence the equality. Thus, by Lemma 3.3(1),

$$c_X \left( \bigcup \{E_{ij} | j \in \alpha\} \right) = c_Y \left( \bigcup \{E_{ij} | j \in \alpha\} \right) \cap X = Y_i \cap X = \bigcup \{X_n | n \geq i\}.$$  

Let $\equiv_n$ be the congruence on $\omega$ modulo $n$. For $i < k$, let $C_i = \{w_0, \ldots, w_{n_i - 1}\}$. Partition $X_i$ into

$$\bigcup \{E_{ij} | j \equiv_{n_i} 0\}, \bigcup \{E_{ij} | j \equiv_{n_i} 1\}, \ldots, \bigcup \{E_{ij} | j \equiv_{n_i} n_i - 1\}.$$  

Define $f : X \to W$ as follows. If $x \in X_i$ for $i < k$, then set $f(x) = w_n$ provided $x \in \bigcup \{E_{ij} | j \equiv_{n_i} n\}$. If $x \in X_i$ for $i \geq k$, then set $f(x) = v$ for some $v \in C_{k-1}$.

The map $f : X \to W$ is well defined since $\{X_i | i \in \omega\}$ partitions $X$ and the sets

$$\bigcup \{E_{ij} | j \equiv_{n_i} 0\}, \bigcup \{E_{ij} | j \equiv_{n_i} 1\}, \ldots, \bigcup \{E_{ij} | j \equiv_{n_i} n_i - 1\}$$

Figure 2
partition $X_i$ for $i < k$. Furthermore, for each $i < k$, we have

$$f(X_i) = f \left( \bigcup_{n < n_i, j \equiv n_i} E_{ij} \right) = \bigcup_{n < n_i} f \left( \bigcup_{j \equiv n_i} E_{ij} \right) = \bigcup_{n < n_i} \{w_n\} = C_i.$$ 

Therefore, $f$ is onto.

Viewing $\mathfrak{F}$ as an Alexandroff space, the closure of $w \in W$ is $R^{-1}[w] := \{v \mid vRw\}$. Therefore, to see that $f$ is interior, since $W$ is finite, it is sufficient to show that $c_Xf^{-1}(w) = f^{-1}R^{-1}[w]$ for each $w \in W$. Let $w \in W$. Then $w \in C_i$ for some $i < k$. Therefore, $w = w_m$ for some $m \leq n_i - 1$. First suppose that $x \in c_Xf^{-1}(w)$. Then, by Claim 3.5,

$$x \in c_Xf^{-1}(w_m) = c_X \left( \bigcup_{j \equiv n_i} E_{ij} \right) = \bigcup_{n \geq i} X_n,$$

giving

$$f(x) \in f \left( \bigcup_{n \geq i} X_n \right) = \bigcup_{n \geq i} f(X_n) = \bigcup_{k > n \geq i} C_n = R^{-1}[C_i] = R^{-1}[w].$$

Thus, $x \in f^{-1}R^{-1}[w]$.

Conversely, suppose $x \in f^{-1}R^{-1}[w]$. Then $f(x)Rw$, giving that $f(x) \in C_j$ for $i \leq j < k$. By the definition of $f$, it must be the case that $x \in X_j$ when $j < k - 1$ and $x \in \bigcup_{n \geq k-1} X_n$ when $j = k - 1$. Therefore, by Claim 3.5,

$$x \in \bigcup_{n \geq i} X_n = c_X \left( \bigcup_{j \equiv n_i} E_{ij} \right) = c_Xf^{-1}(w_m) = c_Xf^{-1}(w).$$

Thus, $c_Xf^{-1}(w) = f^{-1}R^{-1}[w]$, completing the proof.

**Theorem 3.6.** $S4.3$ is the logic of a countable HED-subspace of the Gleason cover of $\mathbb{I}$.

**Proof.** Let $X$ be the countable subspace of the Gleason cover of $\mathbb{I}$ constructed above. By Lemma 3.3(4), $X$ is an HED-space. Therefore, by Proposition 3.1, $X \models S4.3$. Suppose $S4.3 \not \vdash \varphi$. Since $S4.3$ is the logic of finite quasi-chains, there is a finite quasi-chain $\mathfrak{F}$ refuting $\varphi$. By Lemma 3.4, $\mathfrak{F}$ is an interior image of $X$. Since interior images reflect refutations, $X$ refutes $\varphi$. Thus, $S4.3$ is the logic of $X$. 

**References**


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