EXERCISE CLASS 15-11-2017: CANONICAL MODELS AND THE FINITE MODEL PROPERTY

1. More on completeness and the canonical model

(1) Let Γ be a set of formulas (say, in the language of basic modal logic). Prove that if Γ is satisfiable then it is consistent. Can you generalise this to cover L-consistency for an arbitrary normal modal logic L?

(2) Let L be a consistent normal modal logic. Given a world w in a model $\mathbb{M}$ based on an L-frame, show that the set of formulas $\{\varphi\colon \mathbb{M}, w \Vdash \varphi\}$ is an L-MCS.

(3) Show that in the canonical model for $\mathbf{K}$ (or any other consistent normal modal logic L) there exist (L-)MCSs Γ and Δ that are incomparable (i.e., we have neither $R^L(\Gamma, \Delta)$ nor $R^L(\Delta, \Gamma)$).

(4) Show that if $L = \mathrm{Log}(\mathcal{K})$, for some class of (finite) Kripke frames $\mathcal{K}$, then $L = \mathrm{Log}(\mathcal{K}')$ for some class of (finite) rooted Kripke frames $\mathcal{K}'$.

(5) ($**$) show that the set of formulas $\mathbf{KL} \cup \{\Box\varphi \to \varphi\colon \varphi \in \mathrm{Form}\}$ is $\mathbf{KL}$-consistent. Conclude that $\mathbf{KL}$ is not canonical[1].

2. Finite model property

(1) Show that the following normal modal logics have the finite model property

- (a) The normal modal logic $\mathbf{K}$;
- (b) The normal modal logic $\mathbf{KD} \coloneqq \mathbf{K} + \Diamond\top$;
- (c) The normal modal logic $\mathbf{KT} \coloneqq \mathbf{K} + p \to \Diamond p$;
- (d) The normal modal logic $\mathbf{K4} \coloneqq \mathbf{K} + \Diamond\Diamond p \to \Diamond p$;
- (e) The normal modal logic $\mathbf{S4} \coloneqq \mathbf{KT} + \Diamond\Diamond p \to \Diamond p$;
- (f) The normal modal logic $\mathbf{S5} \coloneqq \mathbf{S4} + p \to \Box\Diamond p$;
- (g) The normal modal logic $\mathbf{S4.2} \coloneqq \mathbf{S4} + \Diamond\Box p \to \Box\Diamond p$.

(2) Which of the normal modal logics above are decidable?

(3) Let $\mathcal{K}$ be the class of Kripke frames satisfying the first-order condition $\forall x \exists y (xRy \ \&\ yRy)$. Does the normal modal logic $\mathrm{Log}(\mathcal{K})$ enjoy the finite model property?

(4) Let $\mathcal{R}$ be the class of frames regular frames, viz., frames $(W, R_\Diamond, R_{\langle * \rangle})$ such that $R^*_\Diamond = R_{\langle * \rangle}$. Show that the bimodal logic $\mathrm{Log}(\mathcal{R})$ enjoys the finite model property.

(5) ($*$) Let $\mathcal{K}$ be the class of Kripke frames satisfying the first-order condition $\forall x \exists y (xRy \ \&\ yRy)$ and let $\mathbf{KMT} \coloneqq \mathrm{Log}(\mathcal{K})$.
- (a) Can you find a Kripke frame $\mathbb{F}$ such that $\mathbb{F} \notin \mathcal{K}$ but $\mathbb{F} \Vdash \mathbf{KMT}$? (Hint: think about ultrafilter extensions).
- (b) Can you find a finite Kripke frame $\mathbb{F}$ such that $\mathbb{F} \notin \mathcal{K}$ but $\mathbb{F} \Vdash \mathbf{KMT}$? (Hint: Show that the formula $\Diamond((\Box p_1 \to p_1) \wedge \ldots \wedge (\Box p_n \to p_n))$ belongs to $\mathbf{KMT}$ for all $n \in \omega$.)

[1] Hint: Consider the general frame $(\mathbb{N} \cup \{\infty\}, R, \mathcal{A})$ where $R \coloneqq \{(\infty, n)\colon n \in \mathbb{N}\} \cup \{(n, m)\colon m < n\}$ and $\mathcal{A}$ is the set of finite subsets of $\mathbb{N}$ and the co-finite subsets of $\mathbb{N} \cup \{\infty\}$ which contains ∞. Of course, you may also try a more syntactic approach.