EXERCISE CLASS 22-11-2017: GENERAL FRAMES AND PDL

1. General Frames

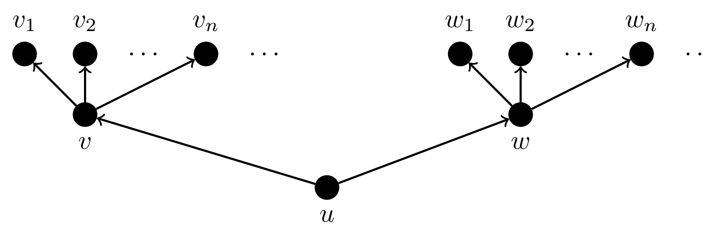
(1) Let $\mathbb{M} := (\mathbb{F}, V)$ be a model with $\mathbb{F} := (W, R)$ and let $\mathcal{A}_V := \{V(\varphi) \colon \varphi \in \text{Form}\}$, where $V(\varphi) = \{w \in W \colon \mathbb{M}, w \Vdash \varphi\}$. Show that $\mathfrak{f}_{\mathbb{M}} := (\mathbb{F}, \mathcal{A}_V)$ is a general frame.

(2) Consider the Kripke frame (W, R) depicted in Figure. Let A be the collection of all finite and co-finite subsets of W. Show that

(a) (W, R, A) is a general frame,

(b) $(W, R), u \nVdash \Diamond\Box p \to \Box\Diamond p$,

(c) $(W, R, A), u \Vdash \Diamond\Box p \to \Box\Diamond p$.



(3) Let $\mathcal{C}$ be a class of general frames. Show that $\text{Log}(\mathcal{C})$ is a normal modal logic. Are all normal modal logics of this form?

2. PDL

(4) Let $\mathfrak{F} = (W, \{R_\pi\}_{\pi\in\Pi})$ be a frame. Show that for each $\pi, \pi_1, \pi_2 \in \Pi$ we have

(a) $\mathfrak{F} \Vdash \langle\pi_1; \pi_2\rangle p \leftrightarrow \langle\pi_1\rangle\langle\pi_2\rangle p$ iff $R_{\pi_1;\pi_2} = R_{\pi_1} \circ R_{\pi_2}$.[1]

(b) $\mathfrak{F} \Vdash \langle\pi_1 \cup \pi_2\rangle p \leftrightarrow \langle\pi_1\rangle p \vee \langle\pi_2\rangle p$ iff $R_{\pi_1\cup\pi_2} = R_{\pi_1} \cup R_{\pi_2}$.

(c) If $(R_\pi)^* = R_{\pi^*}$ then

$$\mathfrak{F} \Vdash \langle\pi^*\rangle p \leftrightarrow p \vee \langle\pi\rangle\langle\pi^*\rangle p$$
$$\mathfrak{F} \Vdash [\pi^*](p \to [\pi]p) \to (p \to [\pi^*]p)$$

(d) If $\mathfrak{F} \Vdash p \vee \langle\pi\rangle\langle\pi^*\rangle p \to \langle\pi^*\rangle p$, then $(R_\pi)^* \subseteq R_{\pi^*}$.

(5) Let $\mathcal{M}$ be the set of all **PDL**-MCSs, and Σ a set of formulas. Show that:

(a) $At(\Sigma) = \{\Gamma \cap \neg\text{FL}(\Sigma) \mid \Gamma \in \mathcal{M}\}$;

(b) If $X \subseteq \neg\text{FL}(\Sigma)$ and X is **PDL**-consistent, then there exists $A \in At(\Sigma)$ such that $X \subseteq A$.

(6) Consider a set of **PDL** formulas Σ and let $A \in At(\Sigma)$ be an atom over Σ. Show that:

(a) For every $\varphi \in \neg\text{FL}(\Sigma)$, exactly one of φ and $\sim\varphi$ is in A;

(b) For every $\varphi \vee \psi \in \neg\text{FL}(\Sigma)$, $\varphi \vee \psi \in A$ iff $\varphi \in A$ or $\psi \in A$;

[1] In BdRV $R_{\pi_1} \circ R_{\pi_2}$ is also denoted $R_{\pi_1}; R_{\pi_2}$.

(c) For every $\langle \pi_1; \pi_2 \rangle \varphi \in \neg\mathrm{FL}(\Sigma)$, $\langle \pi_1; \pi_2 \rangle \varphi \in A$ iff $\langle \pi_1 \rangle \langle \pi_2 \rangle \varphi \in A$;

(d) For every $\langle \pi_1 \cup \pi_2 \rangle \varphi \in \neg\mathrm{FL}(\Sigma)$, $\langle \pi_1 \cup \pi_2 \rangle \varphi \in A$ iff $\langle \pi_1 \rangle \varphi$ or $\langle \pi_2 \rangle \varphi \in A$.

3. Additional exercices

(7) $(*)$ Let $\mathcal{K}$ be the class of Kripke frames satisfying the first-order condition $\forall x \exists y (xRy \mathbin{\&} yRy)$ and let $\mathbf{KMT} \coloneqq \mathrm{Log}(\mathcal{K})$.

(a) Can you find a Kripke frame $\mathbb{F}$ such that $\mathbb{F} \notin \mathcal{K}$ but $\mathbb{F} \Vdash \mathbf{KMT}$? (Hint: think about ultrafilter extensions).

(b) Can you find a finite Kripke frame $\mathbb{F}$ such that $\mathbb{F} \notin \mathcal{K}$ but $\mathbb{F} \Vdash \mathbf{KMT}$? (Hint: Show that the formula $\Diamond((\Box p_1 \to p_1) \wedge \ldots \wedge (\Box p_n \to p_n))$ belongs to $\mathbf{KMT}$ for all $n \in \omega$.)

(c) Show that $\mathbf{KMT} = \mathbf{K} + \{\Diamond((\Box p_1 \to p_1) \wedge \ldots \wedge (\Box p_n \to p_n))\}_{n \in \omega}$.

(d) Is $\mathbf{KMT}$ finitely axiomatisable?