EXERCISE CLASS 29-11-2017: PDL EXTRAVAGANZA

(1) Let $\mathfrak{F} = (W, \{R_\pi\}_{\pi\in\Pi})$ be a frame. Show that for each $\pi, \pi_1, \pi_2 \in \Pi$ we have

(a) $\mathfrak{F} \Vdash \langle \pi_1; \pi_2\rangle p \leftrightarrow \langle \pi_1\rangle\langle \pi_2\rangle p$ iff $R_{\pi_1;\pi_2} = R_{\pi_1} \circ R_{\pi_2}$.[1]

(b) $\mathfrak{F} \Vdash \langle \pi_1 \cup \pi_2\rangle p \leftrightarrow \langle \pi_1\rangle p \vee \langle \pi_2\rangle p$ iff $R_{\pi_1\cup\pi_2} = R_{\pi_1} \cup R_{\pi_2}$.

(c) If $(R_\pi)^* = R_{\pi^*}$ then

$$\mathfrak{F} \Vdash \langle \pi^*\rangle p \leftrightarrow p \vee \langle \pi\rangle\langle \pi^*\rangle p \text{ and}$$
$$\mathfrak{F} \Vdash [\pi^*](p \to [\pi]p) \to (p \to [\pi^*]p)$$

(d) If $\mathfrak{F} \Vdash p \vee \langle \pi\rangle\langle \pi^*\rangle p \to \langle \pi^*\rangle p$, then $(R_\pi)^* \subseteq R_{\pi^*}$.

(2) Let $\mathcal{M}$ be the set of all **PDL**-MCSs, and Σ a set of formulas in the language of **PDL**. Show that:

(a) $At(\Sigma) = \{\Gamma \cap \neg\mathrm{FL}(\Sigma) \mid \Gamma \in \mathcal{M}\}$;

(b) If $X \subseteq \neg\mathrm{FL}(\Sigma)$ and X is **PDL**-consistent, then there exists $A \in At(\Sigma)$ such that $X \subseteq A$.

(3) Show that the finite models from [Def. 4.84, BdRV] used in the **PDL** completeness proof can be obtain (up to isomorphism) via certain filtrations.

(4) Consider a set of **PDL** formulas Σ and let $A \in At(\Sigma)$ be an atom over Σ. Show that:

(a) For every $\varphi \in \neg\mathrm{FL}(\Sigma)$, exactly one of φ and $\sim\varphi$ is in A;

(b) For every $\varphi \vee \psi \in \neg\mathrm{FL}(\Sigma)$, $\varphi \vee \psi \in A$ iff $\varphi \in A$ or $\psi \in A$;

(c) For every $\langle \pi_1; \pi_2\rangle\varphi \in \neg\mathrm{FL}(\Sigma)$, $\langle \pi_1; \pi_2\rangle\varphi \in A$ iff $\langle \pi_1\rangle\langle \pi_2\rangle\varphi \in A$;

(d) For every $\langle \pi_1 \cup \pi_2\rangle\varphi \in \neg\mathrm{FL}(\Sigma)$, $\langle \pi_1 \cup \pi_2\rangle\varphi \in A$ iff $\langle \pi_1\rangle\varphi$ or $\langle \pi_2\rangle\varphi \in A$.

(5) A logic L is *compact for the class $\mathcal{C}$ of Kripke frames* if the following condition is met: for every set of formulas Σ, if every finite subset of Σ is satisfiable in a model based on a frame in $\mathcal{C}$, then Σ itself can be satisfied in a model based on a frame in $\mathcal{C}$. Show that **PDL** is not compact for the class of regular frames. Conclude that **PDL** is not strongly complete with respect to the class of regular frames.

(6) Explain why all the axioms of **PDL**, in the standard axiomatisation, with the exception of Segerberg's induction axiom

$$[\pi^*](p \to [\pi]p) \to (p \to [\pi^*]p)$$

are canonical.

1. Additional exercises

(7) Is **PDL** a decidable logic?

(8) $(*)$ Let Σ be a non-empty finite set of **PDL**-formulas. Show that $\vdash_{\mathbf{PDL}} \bigvee_{A\in At(\Sigma)} \widehat{A}$.

(9) $(**)$ Let φ be a formula in the language of **PDL** and let $\Sigma = \{\varphi\}$. Show that $\neg\mathrm{FL}(\Sigma)$ is finite. *Hint: This is not so easy.*

[1] In BdRV $R_{\pi_1} \circ R_{\pi_2}$ is also denoted $R_{\pi_1}; R_{\pi_2}$.