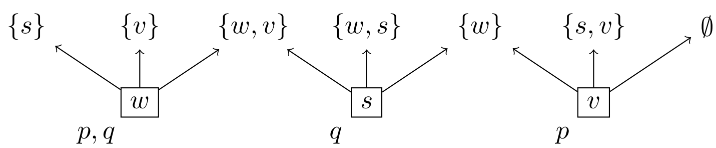
EXERCISE CLASS 6-12-2017:
NEIGHBORHOOD SEMANTICS

(1) ($\Diamond\Box$ modality) Prove that
- (a) $\vdash_{S4} \Diamond\Box(p \wedge q) \to (\Diamond\Box p \wedge \Diamond\Box q)$,
- (b) $\nvdash_{S4} (\Diamond\Box p \wedge \Diamond\Box q) \to \Diamond\Box(p \wedge q)$,
- (c) $\vdash_{S4.2} (\Diamond\Box p \wedge \Diamond\Box q) \to \Diamond\Box(p \wedge q)$,

(2) Consider the NBD-model[1] $\mathbb{M} = (W, N, V)$ here defined.

$$W = \{w, s, v\} \qquad V(p) = \{w, s\} \qquad V(q) = \{s, v\}$$

$$N(w) = \{\,\{s\}, \{v\}, \{w, v\}\,\} \qquad N(s) = \{\,\{w, v\}, \{w, s\}, \{w\}\,\} \qquad N(v) = \{\,\{w\}, \{s, v\}, \emptyset\,\}$$



Compute the set of states that satisfy:
- (a) $\Box\bot$,
- (b) $\Box p$,
- (c) $\Diamond p$,
- (d) $\Box\Diamond p$,
- (e) $\Box\Box p$.

(3) (Logic of an NBD-frame) Given an NBD-frame $\mathbb{F}$, define $\text{Log}(\mathbb{F}) = \{\varphi | \, \mathbb{F} \vDash \varphi\}$. We say that a formula φ is *valid* on $\mathbb{F}$ if $\varphi \in \text{Log}(\mathbb{F})$.
- (a) Show that $\text{Log}(\mathbb{F})$ contains the **Dual** axiom and it is closed under MP, US and RE.

$$\text{RE}\ \frac{p \leftrightarrow q}{\Box p \leftrightarrow \Box q}$$

- (b) Show that the **K** axiom is not valid on every NBD-frame.
- (c) Show that $\text{Log}(\mathbb{F})$ is not closed under generalization in general.
- (d) Show that if $\mathbb{F}$ is monotone[2], then the axiom

$$(\mathbf{M}) \quad \Box(p \wedge q) \to (\Box p \wedge \Box q)$$

is valid on $\mathbb{F}$. Is it true in general?

(4) What class of NBD-frames do the following formulas define?
- (a) $\Box\top$,
- (b) $\Box p \wedge \Box q \to \Box(p \wedge q)$,
- (c) $\Box(p \wedge q) \to \Box p \wedge \Box q$,
- (d) $\Box(p \to q) \to (\Box p \to \Box q)$.

[1] NBD-model and NBD-frame stand for neighborhood model and neighborhood frame respectively.

[2] An NBD-frame is called *monotone* if $N(w)$ is closed under upsets for every $w \in W$.

(5) Call an NBD-model $\mathbb{M} = (W, N, V)$ a *filter NBD-model* if the neighborhood of each point is a filter, i.e. $N(w)$ is closed under upsets and intersections for every $w \in W$.

(a) Let $\mathcal{M} = (W, R, V)$ be a Kripke model. Define an NBD-model $\mathbb{M} = (W, N, V)$ such that for each $w \in W$ and each modal formula φ we have

$$\mathcal{M}, w \Vdash \varphi \iff \mathbb{M}, w \Vdash \varphi \qquad (*)$$

(b) Let $\mathbb{M} = (W, N, V)$ be a filter NBD-model. Define a Kripke model $\mathbb{M} = (W, N, V)$ such that for each $w \in W$ and each modal formula φ, (*) holds.

(c) Is it possible to find $\mathcal{M}$ as in point 5b for a generic $\mathbb{M}$?

(6) Define $\mathbf{E} \oplus \gamma$ as the smallest logic containing $\mathbf{E}$, γ and closed under MP, US and RM. Prove that

(a) $\mathbf{EM} = \mathbf{E} \oplus (\ \Box(p \wedge q) \to \Box p \wedge \Box q\ )$ is the smallest logic containing E and closed under the rule RM.

$$\text{RM}\ \frac{p \to q}{\Box p \to \Box q}$$

(b) $\mathbf{EN} = \mathbf{E} \oplus (\ \Box\top\ )$ is the smallest logic containing E and closed under Generalization.

(7) Define the modality $\langle\ ]$ as follows.

$$\mathbb{M}, w \langle\ ] \varphi \iff \exists X \in N(w),\ X \subseteq [\![\varphi]\!]_{\mathbb{M}}$$

Prove that $\langle\ ]$ and $\Box$ coincide on monotone NBD-frames.