EXERCISE CLASS 15-12-2017: EVEN MORE NEIGHBORHOOD SEMANTICS

(1) Call an NBD-model $\mathbb{M} = (W, N, V)$ a *filter NBD-model* if the neighborhood of each point is a filter, i.e., for each $w \in W$ the collection $N(w)$ is non-empty, closed under supersets and binary intersections.

(a) Let $\mathcal{M} = (W, R, V)$ be a Kripke model. Define an NBD-model $\mathbb{M} = (W, N_R, V)$ by $N_R(w) \coloneqq \{X \in \wp(W) \colon R[w] \subseteq X\}$. Show that for each $w \in W$ and each modal formula φ we have

$$\mathcal{M}, w \Vdash \varphi \iff \mathbb{M}, w \Vdash \varphi \qquad (*)$$

(b) Let $\mathbb{M} = (W, N, V)$ be a filter NBD-model with the property that for each $w \in W$ the collection $N(w)$ is closed under arbitrary intersections[1]. Define a Kripke model $\mathcal{M} = (W, R_N, V)$ by $wR_N v$ iff $v \in \bigcap_{X \in N(w)} X$. Show that for each $w \in W$ and each modal formula φ, (*) holds.

(2) Define $\mathbf{E} \oplus \gamma$ as the smallest logic containing $\mathbf{E}$, γ and closed under MP, US and RE. Prove that

(a) $\mathbf{EM} = \mathbf{E} \oplus (\ \Box(p \wedge q) \to \Box p \wedge \Box q\ )$ is the smallest logic containing $\mathbf{E}$ which is closed under the rule RM.

$$\frac{p \to q}{\Box p \to \Box q}\ (\text{RM})$$

(b) $\mathbf{EN} = \mathbf{E} \oplus (\ \Box\top\ )$ is the smallest logic containing E which is closed under Generalization.

(c) $\mathbf{EMCN} = \mathbf{E} \oplus (\ \Box(p \wedge q) \to \Box p \wedge \Box q\ ) \oplus (\ \Box p \wedge \Box q \to \Box(p \wedge q)\ ) \oplus (\ \Box\top\ )$ is the in fact the modal logic $\mathbf{K}$.

(3) Define a modality $\langle\ ]$ as follows.

$$\mathbb{M}, w\, \langle\ ]\, \varphi \iff \exists X \in N(w),\ X \subseteq [\![\varphi]\!]_{\mathbb{M}}$$

Prove that $\langle\ ]$ and $\Box$ coincide on monotone NBD-frames.

(4) Let $\mathbf{L}$ be a modal logic. Given a formula in the language of basic modal logic φ define

$$|\varphi|_{\mathbf{L}} \coloneqq \{\Gamma \in M_{\mathbf{L}} \colon \varphi \in \Gamma\},$$

where $M_{\mathbf{L}}$ is the set of maximal $\mathbf{L}$-consistent sets. Show that for formulas φ and ψ,

$$|\varphi|_{\mathbf{L}} \subseteq |\psi|_{\mathbf{L}} \iff \vdash_{\mathbf{L}} \varphi \to \psi.$$

(5) Show that

(a) The logic $\mathbf{EC}$ is sound and complete with respect to the class of neighborhood frames that are closed under intersections;

(b) The logic $\mathbf{EN}$ is sound and complete with respect to the class of neighborhood frames that contains the unit;

(c) $(*)$ The logic $\mathbf{EM}$ is sound and complete with respect to the class of monotone neighborhood frames;

(d) The logic $\mathbf{K}$ is sound and complete with respect to the class of (augmented) filter models.

The exercises here are all taken from (Pacuit 2017).

[1] Such models are called *augmented* in (Pacuit 2017).

Hint: For item (3) consider the minimal canonical model $\mathcal{M}^{\min}_{\mathbf{EM}} = (W_{\mathbf{EM}}, N^{\min}_{\mathbf{EM}}, V_{\mathbf{EM}})$ for **EM** and consider the model $\mathcal{M}^{\mathrm{mm}}_{\mathbf{EM}} \coloneqq (W_{\mathbf{EM}}, N^{\mathrm{mm}}_{\mathbf{EM}}, V_{\mathbf{EM}})$, where for each $w \in W$, we let

$$N^{\mathrm{mm}}_{\mathbf{EM}}(w) \coloneqq \{X \in \wp(W_{\mathbf{EM}}) \colon \exists Y \in N^{\min}_{\mathbf{EM}}(w)\ (Y \subseteq X)\}.$$