EXERCISE CLASS 15-12-2017:
EVEN MORE NEIGHBORHOOD SEMANTICS

(1) Call an NBD-model $\mathcal{M} = (W, N, V)$ a \textit{filter NBD-model} if the neighborhood of each point is a filter, i.e., for each $w \in W$ the collection $N(w)$ is non-empty, closed under supersets and binary intersections.

(a) Let $\mathcal{M} = (W, R, V)$ be a Kripke model. Define an NBD-model $\mathcal{M} = (W, N, R, V)$ by $N_R(w) := \{X \in \wp(W): R[w] \subseteq X\}$. Show that for each $w \in W$ and each modal formula $\varphi$ we have $\mathcal{M}, w \models \varphi \iff \mathcal{M}, w \models \varphi^*$.

(b) Let $\mathcal{M} = (W, N, V)$ be a filter NBD-model with the property that for each $w \in W$ the collection $N(w)$ is closed under arbitrary intersections. Define a Kripke model $\mathcal{M} = (W, R, N, V)$ by $wR_N v$ if $v \in \bigcap_{X \in N(w)} X$. Show that for each $w \in W$ and each modal formula $\varphi$, (*) holds.

(2) Define $E \oplus \gamma$ as the smallest logic containing $E, \gamma$ and closed under MP, US and RE. Prove that

(a) $EM = E(\Box(p \land q) \rightarrow \Box p \land \Box q)$ is the smallest logic containing $E$ which is closed under the rule RM.

(b) $EN = E \oplus (\Box \top)$ is the smallest logic containing $E$ which is closed under Generalization.

(c) $EMCN = E \oplus (\Box(p \land q) \rightarrow \Box p \land \Box q) \oplus (\Box p \land \Box q \rightarrow \Box(p \land q)) \oplus (\Box \top)$ is the in fact the modal logic $K$.

(3) Define a modality $[]$ as follows.

$\mathcal{M}, w [\Box \varphi] \iff \exists X \in N(w), X \subseteq [\varphi]_M$

Prove that $[]$ and $\Box$ coincide on monotone NBD-frames.

(4) Let $L$ be a modal logic. Given a formula in the language of basic modal logic $\varphi$ define

$|\varphi|_L := \{\Gamma \in M_L: \varphi \in \Gamma\}$,

where $M_L$ is the set of maximal $L$-consistent sets. Show that for formulas $\varphi$ and $\psi$,

$|\varphi|_L \subseteq |\psi|_L \iff \vdash_L \varphi \rightarrow \psi$.

(5) Show that

(a) The logic $EC$ is sound and complete with respect to the class of neighborhood frames that are closed under intersections;

(b) The logic $EN$ is sound and complete with respect to the class of neighborhood frames that contains the unit;

(c) (*) The logic $EM$ is sound and complete with respect to the class of monotone neighborhood frames;

(d) The logic $K$ is sound and complete with respect to the class of (augmented) filter models.

The exercises here are all taken from (Pacuit 2017).

1Such models are called \textit{augmented} in (Pacuit 2017).
Hint: For item (3) consider the minimal canonical model $\mathcal{M}_{\text{EM}}^{\text{min}} = (W_{\text{EM}}, N_{\text{EM}}^{\text{min}}, V_{\text{EM}})$ for $\text{EM}$ and consider the model $\mathcal{M}_{\text{EM}}^{\text{mm}} := (W_{\text{EM}}, N_{\text{EM}}^{\text{mm}}, V_{\text{EM}})$, where for each $w \in W$, we let

$$N_{\text{EM}}^{\text{mm}}(w) := \{X \in \wp(W_{\text{EM}}) : \exists Y \in N_{\text{EM}}^{\text{min}}(w) (Y \subseteq X)\}.$$