Introduction to Modal Logic. Exercise class 2

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Exercise 1. Show that the roots of the two models[1] in Figure 2.5 of [BdRV] are not bisimilar.

Exercise 2. Consider the binary modality U ('until') with the following semantics

$$\mathbb{M}, s \Vdash \phi \,\mathrm{U}\, \psi \text{ iff } \begin{cases} \text{there is a } t \text{ such that } Rst \;\&\; \mathbb{M}, t \Vdash \phi, \text{ and} \\ \text{for every } u \text{ such that } Rsu \;\&\; Rut \text{ it holds } \mathbb{M}, u \Vdash \psi. \end{cases}$$

Is U expressible in the language of basic modal logic? And in the language of basic temporal logic?
Hint: consider the models in [BdRV, Exercise 2.2.4].

Exercise 3. Consider the modality $\circ$ with the following semantics

$$\mathbb{M}, s \Vdash \circ\phi \iff \exists t \in W \; (sRt \;\&\; \neg(tRt) \;\&\; \mathbb{M}, t \Vdash \phi).$$

Is $\circ$ expressible in the language of basic modal logic?

Exercise 4. Let $\mathbb{M} = (W, R, V)$ be a Kripke model, and let X be a subset of W. We define $\mathbb{M}_X$ as the restricted model (X, R_X, V_X), where $R_X := R \cap (X \times X)$ and $V_X(p) := V(p) \cap X$. We call $X \subseteq W$ *hereditary* if $s \in X$ and Rst imply $t \in X$; in this case we say that $\mathbb{M}_X$ is a *generated submodel* of $\mathbb{M}$.

(1) Show that $\Delta_X := \{(x, x) \mid x \in X\}$ is a bisimulation between $\mathbb{M}_X$ and $\mathbb{M}$ iff X is hereditary.

(2) Show that if f is a bounded morphism from $\mathbb{M}$ to $\mathbb{M}'$, then the set $f[W] := \{f(s) \mid s \in W\}$ is a hereditary subset of W'.

Exercise 5. A *bounded morphism* between two frames $\mathbb{F} = (W, R)$ and $\mathbb{F}' = (W', R')$ is a map $f : W \to W'$ such that, for all $s, t \in W$ and $t' \in W'$:
(forth) Rst implies $R'f(s)f(t)$;
(back) $R'f(s)t'$ implies the existence of a $t \in W$ with Rst and $f(t) = t'$.
Now let f be such a bounded morphism.

(1) Show that for any valuation V' on $\mathbb{F}'$ one can find a valuation V on $\mathbb{F}$ such that f (or rather, its graph $\{(s, f(s)) \mid s \in W\}$) is a bisimulation between the models $(\mathbb{F}, V)$ and $(\mathbb{F}', V')$.

(2) Show that if f is surjective, then $\mathbb{F} \Vdash \phi$ implies $\mathbb{F}' \Vdash \phi$, for any modal formula ϕ.

[1] We take the set of proposition letters to be *empty* here.

(3) Prove that irreflexivity is not modally definable. That is, show that there is no modal formula ϕ such that ϕ is valid on exactly the frames with an irreflexive accessibility relation.

Exercise 6. Which of the following frame properties are preserved (reflected) by the operations of forming generated subframes, p-morphic images, disjoint unions?

(1) reflexivity;

(2) transitivity;

(3) irreflexivity;

(4) converse seriality ($\forall x \exists y\, Ryx$);

(5) having cardinality at least n, for some natural number n;

(6) having cardinality at most n, for some natural number n.

Exercise 7. Show that the following frame properties cannot be defined in the basic modal language:

(1) converse seriality;

(2) having cardinality at least n, for some natural number n;

(3) having cardinality at most n, for some natural number n;

(4) acyclicity: 'there is no finite path (of non-zero length) from any point to itself'.

Exercise 8 (BdRV, Ex. 2.2.8)**.** Consider a non-empty family $\{Z_i | i \in I\}$ of bisimulations between two models $\mathbb{M}$ and $\mathbb{M}'$.

(1) Show that the union $\bigcup\{Z_i | i \in I\}$ is again a bisimulation;

(2) Use the previous fact to show that there exists a *greatest bisimulation* between $\mathbb{M}$ and $\mathbb{M}'$.

(3) Show that, in the case $\mathbb{M} = \mathbb{M}'$, this greatest bisimulation is an equivalence relation.

(4) Can you always find a smallest bisimulation between $\mathbb{M}$ and $\mathbb{M}'$?

Exercise 9 (*)**.** Let $\mathbb{M} = (W, R, V)$ be a Kripke model; we denote the greatest bisimulation relation on $\mathbb{M}$ (see Exercise 8(3)) simply as $\leftrightarroweq$.

(1) Show that there is a model $\mathbb{M}^*$ such that the greatest bisimulation between $\mathbb{M}$ and $\mathbb{M}^*$ is in fact (the graph of) a surjective bounded morphism π. Hint: take a (suitably defined) *quotient* of $\mathbb{M}$ under $\leftrightarroweq$.

(2) Show that $\mathbb{M}^*$ is uniquely determined modulo isomorphism.

(3) Prove that $\mathbb{M}, s \leftrightarroweq \mathbb{M}', s'$ if and only if there is an isomorphism from $\mathbb{M}^*$ to $(\mathbb{M}')^*$ mapping $\pi(s)$ to $\pi'(s')$.