Definition 1. Let $S$ be a non-empty set. A filter over $S$ is a set $F \subseteq \mathcal{P}(S)$ such that (1) $S \in F$, (2) if $X \in F$ and $X \subseteq Y \subseteq S$ then $Y \in F$, and (3) if $X, Y \in F$ then $X \cap Y \in F$.

A filter $F$ is proper if $F \neq \mathcal{P}(S)$, or equivalently, $\emptyset \notin F$. An ultrafilter is a proper filter $u$ such that for all $X \in \mathcal{P}(S)$: $X \in u$ if and only if $S \setminus X \notin u$. The collection of ultrafilters over a set $S$ is denoted as $\mathcal{Uf}(S)$.

Given an element $s \in S$, we define the principal ultrafilter $\pi_s := \{X \in S \mid s \in X\}$.

Exercise 1. Let $S$ be a non-empty set.

(a) Verify that $\mathcal{P}(S)$ and $\{S\}$ are filters.

(b) Given a subset $X \subseteq S$, verify that $\uparrow X := \{Y \in \mathcal{P}(S) \mid X \subseteq Y\}$ is a filter.

(c) Verify that $\{X \in \mathcal{P}(S) \mid S \setminus X \text{ is finite}\}$ is a filter, if $S$ is infinite.

(d) Given an element $s \in S$, verify that $\pi_s$ is indeed an ultrafilter.

Exercise 2. Let $S$ be some non-empty set, and let $u$ be an ultrafilter over $S$.

(a) Show that for every pair $X, Y \in \mathcal{P}(S)$: $X \cup Y \in u$ if and only if $X \in u$ or $Y \in u$.

(b) Show that $u$ is principal if $S$ is finite.

Definition 2. Let $S$ be some non-empty set. A collection $E \subseteq \mathcal{P}(S)$ has the finite intersection property if the intersection $\bigcap E'$ of any finite subcollection $E' \subseteq E$ is nonempty.

Theorem 1 (Ultrafilter Theorem). Let $S$ be some non-empty set. If the collection $E \subseteq \mathcal{P}(S)$ has the finite intersection property then there is an ultrafilter $u \in \mathcal{Uf}(S)$ such that $E \subseteq u$.

Exercise 3. Show that every infinite set has a non-principal ultrafilter.

Definition 3. Given a binary relation $R \subseteq S \times S$ we define the following operations\(^1\) $\langle R \rangle, [R]$ on the power set of $S$:

$\langle R \rangle(X) := \{s \in S \mid Rx \text{ for some } x \in X\}$,

$[R](X) := \{s \in S \mid Rx \text{ implies } x \in X, \text{ for all } x \in S\}$.

Clearly the operations $\langle R \rangle$ and $[R]$ encode the semantics of the $\Diamond$ and $\Box$ modality (see for instance Exercise 5).

Definition 4. Given a Kripke model $M = (S, R, V)$, we define its ultrafilter extension\(^2\) as the Kripke model $M^* = (S^*, R^*, V^*)$, where

$W^* := \mathcal{Uf}(S)$,

$R^* := \{(u, v) \in \mathcal{Uf}(S) \times \mathcal{Uf}(S) \mid \langle R \rangle X \in u \text{ for all } X \in v\}$

$V^*(p) := \{u \in \mathcal{Uf}(S) \mid V(p) \in u\}$

\(^1\)In [BdRV] these operations are denoted as $m_R$ and $l_R$, respectively.

\(^2\)In [BdRV] the ultrafilter extension of a model $M$ is denoted as $\text{ue}M$. 

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Exercise 4. Given a Kripke model $M = (S, R, V)$, show that $R^*uv$ iff for all $X \in \mathcal{P}(S)$: $[R]X \in u$ implies $X \in v$.

Definition 5. Given a Kripke model $M = (S, R, V)$, we define the extension of a formula $\phi$ as the set $\lbrack\phi\rbrack^M := \{ s \in S \mid M, s \models \phi \}$. 

We may write $\lbrack\phi\rbrack$ instead of $\lbrack\phi\rbrack^M$ if $M$ is clear from context.

Lemma 1 (Key Lemma). Let $M = (S, R, V)$ be a Kripke model. Then for every modal formula $\phi$, and for every ultrafilter $u \in \mathcal{Uf}(S)$ we have $M^*, u \models \phi$ iff $\lbrack\phi\rbrack^M \in u$. 

Exercise 5. The key lemma is proved by induction on $\phi$. In this exercise we focus on the hard part of the inductive case for the formula $\Box \phi$. That is, assume as inductive hypothesis that (1) holds for the formula $\phi$.

(a) Show that $\lbrack\Box \phi\rbrack^M = \langle R \rangle [\phi]^M$.

(b) Suppose that $\langle R \rangle X \in u$, for some set $X \in \mathcal{P}(S)$. Consider the set $E := \{ X \} \cup \{ Y \in \mathcal{P}(S) \mid [R]Y \in u \}$.

(b1) Show that $E$ has the finite intersection property.

(b2) Let $v \in \mathcal{Uf}(S)$ be such that $E \subseteq v$. Show that $R^*uv$ and $X \in v$.

(c) Suppose that $\lbrack\Box \phi\rbrack^M \in u$, and prove that $M^*, u \models \Box \phi$.

Exercise 6. Prove the key lemma.

Exercise 7. Let $M = (S, R, V)$ be a Kripke model, let $u \in \mathcal{Uf}(S)$ be an ultrafilter and let $\Sigma$ be a set of modal formulas. Assume that $\Sigma$ is finitely satisfiable in the set $R^*(u)$ of successors of $u$, in the model $M^*$. Define $H := \{ [\phi]^M \mid \phi \in \Sigma \} \cup \{ Y \in \mathcal{P}(S) \mid [R]Y \in u \}$.

(a) Show that $H$ has the finite intersection property.

(b) Let $v \in \mathcal{Uf}(S)$ be such that $H \subseteq v$. Show that $R^*v$ and that $M^*, v \models \phi$, for all $\phi \in \Sigma$.

(c) Show that $M^*$ is m-saturated.

Exercise 8. The ultrafilter extension of a Kripke frame $F = (S, R)$ is defined as the structure $F^* := (\mathcal{Uf}(S), R^*)$.

(a) Show that if $F^* \models \phi$ then $F \models \phi$.

(b) Show that the frame property $\forall x \exists y(xRy \& yRy)$ is preserved under taking disjoint unions, generated subframes and p-morphic imagesootnote{That is images under p-morphisms also known as bounded morphisms.}, but is nevertheless not modally definable.

(c)* Give a counterexample showing that the converse implication of (a) does not hold. (Hint: you need to find a formula which expresses a property which is not first-order definable.)