EXERCISE CLASS 08-11-2017:
CANONICAL MODELS AND COMPLETENESS-VIA-CANONICITY

(1) Let L be a normal modal logic and let Γ be an L-MCS. Show that
  (i) If $\varphi \in \Gamma$ and $\varphi \to \psi \in \Gamma$ then $\psi \in \Gamma$;
  (ii) $L \subseteq \Gamma$;
  (iii) For every formula φ either $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$;
  (iv) For every pair of formulas φ and ψ we have that $\varphi \vee \psi \in \Gamma$ iff $\varphi \in \Gamma$ or $\psi \in \Gamma$.
  (v) For every pair of formulas φ and ψ we have that $\varphi \wedge \psi \in \Gamma$ iff $\varphi \in \Gamma$ and $\psi \in \Gamma$;

(2) Let L be a normal modal logic and define a relation R'' on the canonical model for L by

$$R''(\Gamma, \Delta) \quad \text{iff} \quad \forall\varphi(\varphi \in \Delta \implies \Diamond\varphi \in \Gamma),$$

where Γ and Δ are L-MCSs. Show that $R'' = R'$, where R' is the relation

$$R'(\Gamma, \Delta) \quad \text{iff} \quad \forall\varphi(\Box\varphi \in \Gamma \implies \varphi \in \Delta),$$

where Γ and Δ are L-MCSs. Thus, for any normal modal logic L, we may define the canonical relation R^L as either R' or R''.

(3) (a) Show that the normal modal logic $\mathbf{KD} := \mathbf{K} + (\Diamond\top)$ is sound and complete with respect to the class of serial Kripke frames, i.e., Kripke frames satisfying the first-order condition $\forall x \exists y(xRy)$.
  (b) Show that the normal modal logic $\mathbf{KT} := \mathbf{K} + (\Box p \to p)$ is sound and complete with respect to the class of reflexive Kripke frames.
  (c) Show that the normal modal logic $\mathbf{S4} := \mathbf{K} + (\Box p \to p) + (\Box p \to \Box\Box p)$ is sound a complete with respect to the class of reflexive and transitive Kripke frames.
  (d) Show that the normal modal logic $\mathbf{KB} := \mathbf{K} + (p \to \Box\Diamond p)$ is sound and complete with respect to the class of symmetric Kripke frames.
  (e) Show that the normal modal logic $\mathbf{Den} := \mathbf{K} + (\Diamond p \to \Diamond\Diamond p)$ is sound and complete with respect to the class of dense Kripke frames, i.e., Kripke frames satisfying the first-order condition $\forall x \forall z(xRz \implies \exists y(xRy \wedge yRz))$. *Hint: This is not so easy*[1].

(4) Let Γ be a set of formulas (say, in the language of basic modal logic). Prove that if Γ is satisfiable then it is consistent. Can you generalise this to cover L-consistency for an arbitrary normal modal logic L?

(5) Let L be a normal modal logic. Given a world w in an L-model $\mathbb{M}$, show that the set of formulas $\{\varphi \colon \mathbb{M}, w \Vdash \varphi\}$ is an L-MCS.

(6) Show that in the canonical model for $\mathbf{K}$ (or any other consistent normal modal logic L) there exist (L-)MCSs Γ and Δ that are incomparable (i.e., we have neither $R^L(\Gamma, \Delta)$ nor $R^L(\Delta, \Gamma)$).

(7) (a) Let $\Gamma := \{p, q, p \wedge q, \Box p, \Box q, \Box(p \wedge q)\}$, $\Delta := \{p, \neg q, \Box p\}$, and $\Delta' := \{\Box p, \Box q, \Box(p \wedge q)\}$ be sets of formulas.
  (b) Are these sets maximal consistent (in some language)?
  (c) Let the relation R' on $\{\Gamma, \Delta, \Delta'\}$ and the valuation V' on $\{\Gamma, \Delta, \Delta'\}$ be as defined on the canonical model. Draw the resulting Kripke model.

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[1] Given **Den**-MCSs Γ and Δ such that $\Gamma R^{\mathbf{Den}} \Delta$ You need to show that the set of formulas $\Sigma_0^- \cup \Sigma_1^-$ is **Den**-consistent, where $\Sigma_0^- := \{\varphi \colon \Box\varphi \in \Gamma\}$ and $\Sigma_1^- := \{\Diamond\psi \colon \psi \in \Delta\}$. To that end you might find it helpful to show that $\vdash_{\mathbf{K}} \Diamond(p \wedge q) \to \Diamond p \wedge \Diamond q$ and that $((p \wedge q) \to r) \to (p \to (q \to r))$ is a propositional tautology.

Additional exercises

Here are a few additional exercises for those of you who want to know more about the canonical model. They are not part of the core curiculum.

(8) Let $\mathfrak{M} = (W, R, V)$ be a Kripke model we say that

(i) The Kripke model $\mathfrak{M}$ is *tight* if

$$\forall w, w' \in W((\{\varphi \colon \mathfrak{M}, w \Vdash \Box\varphi\} \subseteq \{\varphi \colon \mathfrak{M}, w' \Vdash \varphi\}) \implies wRw');$$

(ii) The Kripke model $\mathfrak{M}$ is *differentiated* if

$$\forall w, w' \in W((\{\varphi \colon \mathfrak{M}, w \Vdash \varphi\} = \{\varphi \colon \mathfrak{M}, w' \Vdash \varphi\}) \implies w = w');$$

(iii) The Kripke model $\mathfrak{M}$ is *compact* if for every set of formulas Σ have that

$$\exists w(\mathfrak{M}, w \Vdash \Sigma) \quad \text{iff} \quad \forall \Sigma_0 \subseteq_\omega \Sigma \exists w(\mathfrak{M}, w \Vdash \Sigma_0)$$

(iv) The Kripke model $\mathfrak{M}$ is *refined* if it is both tight and differentiated.

Let L be a consistent normal modal logic. Show that the canonical model $\mathfrak{M}^L$ for L is a refined and compact Kripke model.

(9) (*For those that know a bit of topology:*) Let L be a consistent normal modal logic and let $\mathfrak{M}^L$ be the canonical model for L. Show that the collection of sets

$$V^L(\varphi) = \{\Gamma \in W^L \colon \mathfrak{M}^L, \Gamma \Vdash \varphi\},$$

with φ ranging over the set of formulas in the language of basic modal logic, generates a topology on the set W^L which is compact, Hausdorff and zero-dimensional.