INTRODUCTION TO MODAL LOGIC 2017

HOMEWORK 1

- Deadline: September 21 — at the **beginning** of class.
- Grading is from 0 to 100 points.
- Success!

Exercise 1. (30 points) A *flow of time* is a relational structure $\mathbb{T} = (T, <)$ such that $<$ is an unbounded strict linear order[1] on T. A flow of time is called *discrete* if it satisfies the formula $\forall x \exists y\, (x < y \wedge \neg \exists z\, (x < z \wedge z < y))$. In words, every point has an *immediate* successor. Now consider the temporal logic formula

$$\delta := (q \wedge \Box_P q) \to \Diamond_F \Box_P q.$$

Show that for any flow of time $\mathbb{T} = (T, <)$, seen as a bidirectional frame for the temporal language, it holds that $\mathbb{T}$ is discrete iff $\mathbb{T} \Vdash \delta$.

Exercise 2. (30 points) Consider the modality $\langle 2 \rangle$ with the following semantics

$\mathbb{M}, s \Vdash \langle 2 \rangle \phi$ iff there are $t_0, t_1 \in R[s]$ with $t_0 \neq t_1, \mathbb{M}, t_0 \Vdash \phi$ and $\mathbb{M}, t_1 \Vdash \phi$).

(a) Is this modality expressible in the language of basic modal logic?
(b) Is this modality expressible in the language of basic modal logic, if we restrict attention to the flows of time of Exercise 1?

Exercise 3. (40 points) Given a finite set Φ of basic modal logic formulas, we define the formula

$$\nabla \Phi := \bigwedge \Diamond \Phi \wedge \Box \bigvee \Phi,$$

where $\Diamond \Phi$ denotes the set $\{\Diamond \phi \mid \phi \in \Phi\}$, and we understand that $\bigwedge \varnothing = \top$ and $\bigvee \varnothing = \bot$. Then we have, for any Kripke model $\mathbb{M}$ and any state s in $\mathbb{M}$, that $\nabla \Phi$ holds at s iff every formula $\phi \in \Phi$ holds at some successor of s, and, conversely, every successor of s satisfies one of the formulas in Φ.

(a) Show that, for any finite set Φ, we have that $\nabla \Phi$ is satisfiable iff every member of Φ is satisfiable.
(b)* Give an example of two formulas ϕ_0 and ϕ_1 such that
   (1) both ϕ_0 and ϕ_1 are satisfiable in some reflexive frame, while
   (2) $\nabla \{\phi_0, \phi_1\}$ is not satisfiable in any reflexive frame.

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[1]That is, $<$ is irreflexive ($\forall x\, \neg x < x$), transitive ($\forall xyz\, (x < y < z \to x < z)$), total ($\forall xy\, (x < y \vee x = y \vee y < x)$), and unbounded ($\forall x \exists y\, x < y$ and $\forall x \exists y\, y < x$).