Exercise 1. (30 points) Show that the following frame properties are not modally definable by a single formula in the basic modal language (i.e., that there is no basic modal formula $\phi$ such that a frame $F$ has the given property iff $F \models \phi$):

(a) every point has at least two successors;
(b) every point lies on a cycle (i.e., from every node $s$ there is a path of non-zero length from $s$ to itself).

Exercise 2. (30 points) Let $M$ and $M'$ be image-finite Kripke models, and consider states $s$ and $s'$ in $M$ and $M'$, respectively. Assume that $M, s \equiv_n M', s'$, for all natural numbers $n$.

(a) Prove that $M, s \equiv M', s'$.
(b) Prove that $M, s \equiv M', s'$ without using results from [BdRV].

Exercise 3. (40 points) In this exercise we consider a bimodal language with two diamonds, $\diamond$ and $\langle \ast \rangle$. We call a frame $F = (W, R_\diamond, R_{\langle \ast \rangle})$ for this language regular if $R_{\langle \ast \rangle}$ is the reflexive-transitive closure of $R_\diamond$: $R_{\langle \ast \rangle} = (R_\diamond)^\ast$.

Verify that the formula $\langle \ast \rangle \phi \leftrightarrow (\phi \lor \diamond \langle \ast \rangle \phi)$ is valid in every regular frame (but you do not need to hand in your proof).

Now let $\Sigma$ be a set of formulas which is closed under taking subformulas, and in addition satisfies $\langle \ast \rangle \phi \in \Sigma \Rightarrow \diamond \langle \ast \rangle \phi \in \Sigma$.

(a) Fix a regular frame $F = (W, R_\diamond, R_{\langle \ast \rangle})$ and assume that the relation $R^* \subseteq W_\Sigma \times W_\Sigma$ is the smallest filtration of $R_\diamond$, that is:

$$R^*uv \text{ iff there are } u', v' \in \overline{\sigma} \text{ with } R_\diamond u'v'.$$

Prove that the relation $(R^*)^\ast$ satisfies the filtration conditions for $R^*$ and $\langle \ast \rangle$:

(S$^*$) if $R^*uv$ then $(R^*)^\ast uv$;
(L$^*$) if $(R^*)^\ast uv$ then, for all $\langle \ast \rangle \phi \in \Sigma$: $M, v \models \phi \Rightarrow M, u \models \langle \ast \rangle \phi$.

(b) Show that a formula in this language is satisfiable in a regular model if and only if is satisfiable in a finite regular model.