INTRODUCTION TO MODAL LOGIC 2017
HOMEWORK 3

• Deadline: October 19 — at the beginning of class.
• Grading is from 0 to 100 points.
• Success!

(1) (30pt) Suppose that $\gamma(x), \delta(x)$ are first-order formulas. We say that $\gamma(x)$ semantically entails $\delta(x)$ (Notation: $\gamma(x) \models \delta(x)$) provided that for any model $M = (W, R, V)$ and any $w \in W$,

$$M \models \gamma(x)[w] \text{ implies } M \models \delta(x)[w].$$

We say that $\gamma(x)$ entails $\delta(x)$ along bisimulation provided that for any models $M = (W, R, V)$ and $M' = (W', R', V')$, and any $w \in W, w' \in W'$,

$$(M, w \sim M', w' \text{ and } M \models \gamma(x)[w]) \text{ implies } M' \models \delta(x)[w'].$$

A modal formula $\varphi$ is called a modal interpolant of $(\gamma(x), \delta(x))$ provided that $\gamma(x) \models ST_x(\varphi)$ and $ST_x(\varphi) \models \delta(x)$.

In the following let $\alpha(x)$ and $\beta(x)$ be first-order formulas.

(a) Show that if $\alpha(x)$ entails $\beta(x)$ along bisimulation, then $\alpha(x) \models \beta(x)$.

(b) Suppose that the pair $(\alpha(x), \beta(x))$ has a modal interpolant. Show that $\alpha(x)$ entails $\beta(x)$ along bisimulation.

(c) In fact, the converse of (b) is true (but non-trivial to show), i.e. if $\alpha(x)$ entails $\beta(x)$ along bisimulation then the pair $(\alpha(x), \beta(x))$ has a modal interpolant. Show that this implies (the non-obvious direction of) the van Benthem Characterization Theorem.

(2) (40pt) Show that Grzegorczyk’s formula

$$\Box(\Box(p \to \Box p) \to p) \to p$$

characterizes the class of frames $F = (W, R)$ satisfying (i) $R$ is reflexive, (ii) $R$ is transitive and (iii) there are no infinite paths $x_0Rx_1Rx_2R...$ such that for all $i$ we have $x_i \neq x_{i+1}$.
(3) (30pt) Recall that a Kripke frame $F = (W, R)$ is rooted (or point generated) if there is an element $r \in W$ such that the subframe of $F$ generated by $\{r\}$ is $F$.

(a) Give an example of a rooted Kripke frame $F$ such that its ultrafilter extension $F^*$ is not rooted.
(b) Show that if $F$ is a transitive Kripke frame, then the ultrafilter extension $F^*$ is rooted whenever $F$ is.
(c) Show that any image finite Kripke frame $F$ is (isomorphic to) a generated subframe of its ultrafilter extension $F^*$.

(Hint: Show that any ultrafilter containing a finite set is principal and use Exercise 4 on the fifth tutorial sheet.)