INTRODUCTION TO MODAL LOGIC 2017
HOMEWORK 3

- Deadline: October 19 — at the **beginning** of class.
- Grading is from 0 to 100 points.
- Success!

(1) (30pt) Suppose that $\gamma(x), \delta(x)$ are first–order formulas. We say that $\gamma(x)$ *semantically entails* $\delta(x)$ (Notation: $\gamma(x) \models \delta(x)$) provided that for any model $\mathbb{M} = (W, R, V)$ and any $w \in W$,

$$\mathbb{M} \models \gamma(x)[w] \text{ implies } \mathbb{M} \models \delta(x)[w].$$

We say that $\gamma(x)$ *entails* $\delta(x)$ *along bisimulation* provided that for any models $\mathbb{M} = (W, R, V)$ and $\mathbb{M}' = (W', R', V')$, and any $w \in W$, $w' \in W'$,

$$(\mathbb{M}, w \leftrightarrows \mathbb{M}', w' \text{ and } \mathbb{M} \models \gamma(x)[w]) \text{ implies } \mathbb{M}' \models \delta(x)[w'].$$

A modal formula φ is called a *modal interpolant* of $(\gamma(x), \delta(x))$ provided that

$$\gamma(x) \models \mathsf{ST}_x(\varphi) \text{ and } \mathsf{ST}_x(\varphi) \models \delta(x).$$

In the following let $\alpha(x)$ and $\beta(x)$ be first–order formulas.

(a) Show that if $\alpha(x)$ entails $\beta(x)$ along bisimulation, then $\alpha(x) \models \beta(x)$.

(b) Suppose that the pair $(\alpha(x), \beta(x))$ has a modal interpolant. Show that $\alpha(x)$ entails $\beta(x)$ along bisimulation.

(c) In fact, the converse of (b) is true (but non-trivial to show), i.e. if $\alpha(x)$ entails $\beta(x)$ along bisimulation then the pair $(\alpha(x), \beta(x))$ has a modal interpolant. Show that this implies (the non-obvious direction of) the van Benthem Characterization Theorem.

(2) (40pt) Show that Grzegorczyk's formula

$$\Box(\Box(p \to \Box p) \to p) \to p$$

characterizes the class of frames $\mathbb{F} = (W, R)$ satisfying (i) R is reflexive, (ii) R is transitive and (iii) there are no infinite paths $x_0 R x_1 R x_2 R \ldots$ such that for all i we have $x_i \neq x_{i+1}$.

(3) (30pt) Recall that a Kripke frame $\mathbb{F} = (W, R)$ is *rooted* (or *point generated*) if there is an element $r \in W$ such that the subframe of $\mathbb{F}$ generated by $\{r\}$ is $\mathbb{F}$.

(a) Give an example of a rooted Kripke frame $\mathbb{F}$ such that its ultrafilter extension $\mathbb{F}^*$ is not rooted.
(b) Show that if $\mathbb{F}$ is a *transitive* Kripke frame, then the ultrafilter extension $\mathbb{F}^*$ is rooted whenever $\mathbb{F}$ is.
(c) Show that any image finite Kripke frame $\mathbb{F}$ is (isomorphic to) a *generated* subframe of its ultrafilter extension $\mathbb{F}^*$.

(Hint: Show that any ultrafilter containing a finite set is principal and use Exercise 4 on the fifth tutorial sheet.)