(1) (30pt) 
(a) Show that if a frame $\mathfrak{F}$ is a bounded morphic image of a frame $\mathfrak{G}$, then $\text{Log}(\mathfrak{G}) \subseteq \text{Log}(\mathfrak{F})$.

(b) Show that if a frame $\mathfrak{F}$ is a generated subframe of a frame $\mathfrak{G}$, then $\text{Log}(\mathfrak{G}) \subseteq \text{Log}(\mathfrak{F})$.

(c) Let $\mathcal{C}$ be a non-empty class of frames. Use (a) and (b) to show that $\text{Log}(\mathcal{C})$ is contained in the logic of a single reflexive point or $\text{Log}(\mathcal{C})$ is contained in the logic of a single irreflexive point.

(2) (30pt) Recall that $\mathbf{S5} = \mathbf{K} + (\Box p \rightarrow p) + (\Box p \rightarrow \Box \Box p) + (p \rightarrow \Box \Diamond p)$. Show:

(a) $\vdash_{\mathbf{S5}} \Diamond p \rightarrow \Box \Diamond p$

(b) Show that $\mathbf{S5}$ is sound and complete with respect to the class of frames $(W, R)$, where $R$ is an equivalence relation.

(c) Use (b) to show that $\not\vdash_{\mathbf{S5}} \Diamond p \rightarrow \Box p$.

(3) (40pt) Recall that $\mathbf{S4.2} = \mathbf{S4} + (\Diamond p \rightarrow \Box \Diamond p)$. Find a class $\mathcal{C}$ of Kripke frames which $\mathbf{S4.2}$ is sound and complete for.

You may use the fact that $\mathbf{S4}$ is sound and complete with respect to reflexive and transitive frames.

You are not allowed to use Sahlqvist’s completeness theorem in any of these exercises.