(1) (30pt) (From the 2014 Exam) In the following exercise you can use that the canonical model for $\textbf{S4.3}$ is reflexive and transitive.

(a) Show that the canonical model for the modal logic

$$\textbf{S4.3} = \textbf{S4} + \Box(\Box p \rightarrow q) \lor \Box(\Box q \rightarrow p)$$

has no branching to the right. Recall that a reflexive Kripke frame has no branching to the right if

$$\forall x\forall y\forall z((Rxy \land Rxz) \rightarrow (Ryz \lor Rzy)).$$

You are not allowed to use Sahlqvist completeness theorem.

(b) Deduce that $\textbf{S4.3}$ is sound and complete with respect to reflexive transitive frames with no branching to the right.

(2) (30pt)

(a) Let $\Sigma$ be a finite subformula closed set. Let $M = (W, R, V)$ be a model such that $(W, R)$ is a rooted transitive reflexive frame with no branching to the right. Show that a transitive filtration of $M$ through $\Sigma$ is a rooted reflexive transitive frame with no branching to the right. (Hint: start by showing that if $r$ is a root of $M$, then $[r]$ is a root of the filtrated model $M_{\Sigma}$.)

Recall that a reflexive and transitive frame $(W, R)$ is \textit{rooted} if there is $x \in W$ such that for each $y \in W$ we have $Rxy$.

(b) Deduce that $\textbf{S4.3}$ has the finite model property.

(c) Deduce that $\textbf{S4.3}$ is decidable.

(3) (40pt) (Item (a) is from the 2014 Exam)

(a) Show that for any modal formulas $\varphi$ and $\psi$ we have

$$\vdash_{\textbf{K}} \Box \varphi \lor \Box \psi \text{ implies } \vdash_{\textbf{K}} \varphi \text{ or } \vdash_{\textbf{K}} \psi.$$
(b) Show that for any modal formulas $\varphi$ and $\psi$ we have:

$$\vdash_{S5} \varphi \rightarrow \Box \psi \text{ iff } \vdash_{S5} \Diamond \varphi \rightarrow \psi.$$ 

(c) Show that the above properties do not hold for all normal modal logics. That is, give an example of normal modal logics $L_1$ and $L_2$ which do not satisfy (a) and (b), respectively.

(Hint: use soundness and completeness of $K$ and $S5$ with respect to Kripke frames.)