INTRODUCTION TO MODAL LOGIC 2018

HOMEWORK 1

- Deadline: September 18 — at the **beginning** of class.
- Grading is from 0 to 100 points.
- Good luck!

Exercise 1. (30 points) Let Z_i be a bisimulation between models $\mathbb{M}_1$ and $\mathbb{M}_2$, for each $i \in I$. Are the following relations bisimulations between $\mathbb{M}_1$ and $\mathbb{M}_2$:

- the union $\bigcup_{i \in I} Z_i$?
- the intersection $\bigcap_{i \in I} Z_i$?

Give a proof or a counter-example.

Exercise 2. (30 points) A *flow of time* is a relational structure $\mathbb{T} = (T, <)$ such that $<$ is an unbounded strict linear order[1] on T. A flow of time is called *discrete* if it satisfies the formula $\forall x \exists y \, (x < y \wedge \neg \exists z \, (x < z \wedge z < y))$. In words, every point has an *immediate* successor. Now consider the temporal logic formula

$$\delta := (q \wedge \Box_P q) \to \Diamond_F \Box_P q.$$

Show that for any flow of time $\mathbb{T} = (T, <)$, seen as a bidirectional frame for the temporal language, it holds that $\mathbb{T}$ is discrete iff $\mathbb{T} \Vdash \delta$.

Exercise 3. (40 points) Given a finite set Φ of basic modal logic formulas, we define the formula

$$\nabla \Phi := \bigwedge \Diamond \Phi \wedge \Box \bigvee \Phi,$$

where $\Diamond \Phi$ denotes the set $\{\Diamond \phi \mid \phi \in \Phi\}$, and we understand that $\bigwedge \varnothing = \top$ and $\bigvee \varnothing = \bot$. Then we have, for any Kripke model $\mathbb{M}$ and any state s in $\mathbb{M}$, that $\nabla \Phi$ holds at s iff every formula $\phi \in \Phi$ holds at some successor of s, and, conversely, every successor of s satisfies one of the formulas in Φ.

(a) Show that, for any finite set Φ, we have that $\nabla \Phi$ is satisfiable iff every member of Φ is satisfiable.

(b)* Give an example of two formulas ϕ_0 and ϕ_1 such that
(1) both ϕ_0 and ϕ_1 are satisfiable in some reflexive frame, while
(2) $\nabla \{\phi_0, \phi_1\}$ is not satisfiable in any reflexive frame.

[1]That is, $<$ is irreflexive ($\forall x \, \neg x < x$), transitive ($\forall xyz \, (x < y < z \to x < z)$), total ($\forall xy \, (x < y \vee x = y \vee y < x)$), and unbounded ($\forall x \exists y \, x < y$ and $\forall x \exists y \, y < x$).