INTRODUCTION TO MODAL LOGIC 2018

HOMEWORK 2

- Deadline: October 2 — at the **beginning** of class.
- Please staple and hand in your homework. Submit electronically only (!) in case of emergency.
- Grading is from 0 to 100 points.
- * means that the exercise is a bit tricky.
- Good luck!

Exercise 1. (30 points) Consider the modality $\langle 2 \rangle$ with the following semantics

$$\mathbb{M}, s \Vdash \langle 2 \rangle \phi \text{ iff } \exists t_0, t_1 \text{ with } sRt_0, sRt_1, t_0 \neq t_1, \mathbb{M}, t_0 \Vdash \phi \text{ and } \mathbb{M}, t_1 \Vdash \phi.$$

(a) Is this modality expressible in the language of basic modal logic?
(b) Is this modality expressible in the language of basic modal logic, if we restrict attention to the flows of time of (Homework 1, Exercise 2)?

Exercise 2. (30 points) Show that the following frame properties are not modally definable by a single formula in the basic modal language (i.e., that there is no basic modal formula ϕ such that a frame $\mathbb{F}$ has the given property iff $\mathbb{F} \Vdash \phi$):

(a) $\forall x \forall y \exists z (Rxz \wedge Ryz)$;
(b) $\forall x \exists y (Rxy \wedge \exists z (Rzy \wedge Rzx))$.

Exercise 3. (40 points)

(a) Let $\mathbb{M} = (W, R, V)$ be a model with R a transitive relation and Σ be any set of formulas closed under subformulas. Show that (W_Σ, R^t, V^f)[1], is a filtration and that the relation R^t is transitive, where R^t is the relation defined in Lemma 2.42 of the Blackburn, de Rijke, Venema book.
(b)* Consider a frame $(\mathbb{Q}, <)$, where $\mathbb{Q}$ is the set of rational numbers. Let Σ be a finite set of modal formulas closed under subformulas. Show that any transitive filtration (i.e., a filtration with a transitive relation), through Σ, of a model based on the frame $(\mathbb{Q}, <)$, is a finite linear sequence of clusters, perhaps interspersed with singleton irreflexive points, no two of which can be adjacent.

[1] At the lecture we denoted W_Σ by W^f.

Here a *cluster* on a transitive frame (W, R) is a subset $C \subseteq W$ that is a maximal equivalence relation under R. That is, the restriction of R to C is an equivalence relation, and this is not the case for any other $D \subseteq W$ such that $C \subsetneq D$.

Additional comments. We assume that the order on clusters is defined as follows: a cluster C is related to cluster D if there are $x \in C$ and $y \in D$ such that x and y are related. We have a *linear sequence of clusters* means that this order on clusters is such that for each clusters C and D with $C \neq D$ we have that C is related to D or that D is related to C. (See also Figure 2.7 of the Blackburn, de Rijke, Venema book.)