INTRODUCTION TO MODAL LOGIC 2017
HOMEWORK 3

• Deadline: October 16 — at the beginning of class.
• Please staple and hand in your homework. Submit electronically only (!) in case of emergency.
• Grading is from 0 to 100 points.
• Good luck!

(1) (30pt) (From 2017 Final Exam) Let $F = (W, R)$ be a Kripke frame. The frame $F^r = (W, R)$ is called the reflexivization of $F$ if $R \cup \{(x, x) \mid x \in W\}$.

For each modal formula $\varphi$ let $\varphi^+$ be the formula obtained from $\varphi$ by replacing each subformula of the form $\Box \psi$ with $\Box (\psi \lor \psi)$ (we assume that $\bot, \lor, \neg$ and $\Box$ are the primitive symbols of the language). More precisely, $\varphi^+$ is defined inductively as follows:

- $(\bot)^+ = \bot$;
- $(p)^+ = p$, where $p$ is a propositional letter;
- $(\varphi \lor \psi)^+ = \varphi^+ \lor \psi^+$;
- $(-\varphi)^+ = \neg (\varphi^+)$;
- $(\Box \varphi)^+ = \Box (\varphi^+) \lor \varphi^+$.

Show that for each frame $F = (W, R)$ and each modal formula $\varphi$:

$$F \vdash \varphi^+ \iff F^r \vdash \varphi.$$

(2) (30pt) Suppose that $\gamma(x), \delta(x)$ are first-order formulas. We say that $\gamma(x)$ semantically entails $\delta(x)$ (Notation: $\gamma(x) \models \delta(x)$) provided that for any model $M = (W, R, V)$ and any $w \in W$,

$$M \models \gamma(x)[w] \text{ implies } M \models \delta(x)[w].$$

We say that $\gamma(x)$ entails $\delta(x)$ along bisimulation provided that for any models $M = (W, R, V)$ and $M' = (W', R', V')$, and any $w \in W$, $w' \in W'$,

$$(M, w \equiv M', w' \text{ and } M \models \gamma(x)[w]) \text{ implies } M' \models \delta(x)[w'].$$

A modal formula $\varphi$ is called a modal interpolant of $(\gamma(x), \delta(x))$ provided that

$$\gamma(x) \models \text{ST}_x(\varphi) \text{ and } \text{ST}_x(\varphi) \models \delta(x).$$

In the following let $\alpha(x)$ and $\beta(x)$ be first-order formulas.

(a) Show that if $\alpha(x)$ entails $\beta(x)$ along bisimulation, then $\alpha(x) \models \beta(x)$. 

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(b) Suppose that the pair \((\alpha(x), \beta(x))\) has a modal interpolant. Show that \(\alpha(x)\) entails \(\beta(x)\) along bisimulation.

(c) In fact, the converse of (b) is true (but non-trivial to show), i.e. if \(\alpha(x)\) entails \(\beta(x)\) along bisimulation then the pair \((\alpha(x), \beta(x))\) has a modal interpolant. Show that this implies (the non-obvious direction of) the van Benthem Characterization Theorem.

(3) (40pt) Show that Grzegorczyk’s formula

\[ \Box(\Box(p \rightarrow \Box p) \rightarrow p) \rightarrow p \]

defines the class of frames \(\mathcal{F} = (W, R)\) satisfying (i) \(R\) is reflexive, (ii) \(R\) is transitive and (iii) there are no infinite paths \(x_0Rx_1Rx_2R...\) such that for all \(i\) we have \(x_i \neq x_{i+1}\).