INTRODUCTION TO MODAL LOGIC 2017
HOMEWORK 3

- Deadline: October 16 — at the **beginning** of class.
- Please staple and hand in your homework. Submit electronically only (!) in case of emergency.
- Grading is from 0 to 100 points.
- Good luck!

(1) (30pt) (From 2017 Final Exam) Let $\mathbb{F} = (W, R)$ be a Kripke frame. The frame $\mathbb{F}^r = (W, \overline{R})$ is called the *reflexivization of* $\mathbb{F}$ if $\overline{R} = R \cup \{(x, x) \mid x \in W\}$.

For each modal formula φ let φ^+ be the formula obtained from φ by replacing each subformula of the form $\Diamond\psi$ with $\Diamond\psi \vee \psi$ (we assume that $\bot, \vee, \neg$ and $\Diamond$ are the primitive symbols of the language). More precisely, φ^+ is defined inductively as follows:

- $(\bot)^+ = \bot$;
- $(p)^+ = p$, where p is a propositional letter;
- $(\varphi \vee \psi)^+ = \varphi^+ \vee \psi^+$;
- $(\neg\varphi)^+ = \neg(\varphi^+)$;
- $(\Diamond\varphi)^+ = \Diamond(\varphi^+) \vee \varphi^+$.

Show that for each frame $\mathbb{F} = (W, R)$ and each modal formula φ:

$$\mathbb{F} \Vdash \varphi^+ \text{ iff } \mathbb{F}^r \Vdash \varphi.$$

(2) (30pt) Suppose that $\gamma(x), \delta(x)$ are first–order formulas. We say that $\gamma(x)$ *semantically entails* $\delta(x)$ (Notation: $\gamma(x) \models \delta(x)$) provided that for any model $\mathbb{M} = (W, R, V)$ and any $w \in W$,

$$\mathbb{M} \models \gamma(x)[w] \text{ implies } \mathbb{M} \models \delta(x)[w].$$

We say that $\gamma(x)$ *entails* $\delta(x)$ *along bisimulation* provided that for any models $\mathbb{M} = (W, R, V)$ and $\mathbb{M}' = (W', R', V')$, and any $w \in W$, $w' \in W'$,

$$(\mathbb{M}, w \leftrightarroweq \mathbb{M}', w' \text{ and } \mathbb{M} \models \gamma(x)[w]) \text{ implies } \mathbb{M}' \models \delta(x)[w'].$$

A modal formula φ is called a *modal interpolant* of $(\gamma(x), \delta(x))$ provided that

$$\gamma(x) \models \mathsf{ST}_x(\varphi) \text{ and } \mathsf{ST}_x(\varphi) \models \delta(x).$$

In the following let $\alpha(x)$ and $\beta(x)$ be first–order formulas.

(a) Show that if $\alpha(x)$ entails $\beta(x)$ along bisimulation, then $\alpha(x) \models \beta(x)$.

(b) Suppose that the pair $(\alpha(x), \beta(x))$ has a modal interpolant. Show that $\alpha(x)$ entails $\beta(x)$ along bisimulation.

(c) In fact, the converse of (b) is true (but non-trivial to show), i.e. if $\alpha(x)$ entails $\beta(x)$ along bisimulation then the pair $(\alpha(x), \beta(x))$ has a modal interpolant. Show that this implies (the non-obvious direction of) the van Benthem Characterization Theorem.

(3) (40pt) Show that Grzegorczyk's formula

$$\Box(\Box(p \to \Box p) \to p) \to p$$

defines the class of frames $\mathbb{F} = (W, R)$ satisfying (i) R is reflexive, (ii) R is transitive and (iii) there are no infinite paths $x_0 R x_1 R x_2 R \ldots$ such that for all i we have $x_i \neq x_{i+1}$.