

**INTRODUCTION TO MODAL LOGIC 2018  
HOMEWORK 5**

- Deadline: November 27 — at the **beginning** of class.
- Grading is from 0 to 100 points.
- Please staple and hand in your homework. Submit electronically only (!) in case of emergency.
- Good luck!

(1) (30pt) (From the 2014 Exam) In the following exercise you can use that the canonical model for **S4.3** is reflexive and transitive.

(a) Show that the canonical model for the modal logic

$$\mathbf{S4.3} = \mathbf{S4} + \Box(\Box p \rightarrow q) \vee \Box(\Box q \rightarrow p)$$

has no branching to the right. Recall that a reflexive Kripke frame has no branching to the right if

$$\forall x \forall y \forall z ((Rxy \wedge Rxz) \rightarrow (Ryz \vee Rzy)).$$

You are not allowed to use Sahlqvist completeness theorem.

(b) Deduce that **S4.3** is sound and complete with respect to reflexive transitive frames with no branching to the right.

(2) (30pt)

(a) Let  $\Sigma$  be a finite subformula closed set. Let  $\mathfrak{M} = (W, R, V)$  be a model such that  $(W, R)$  is a rooted transitive reflexive frame with no branching to the right. Show that a transitive filtration of  $\mathfrak{M}$  through  $\Sigma$  is a rooted reflexive transitive frame with no branching to the right. (Hint: start by showing that if  $r$  is a root of  $\mathfrak{M}$ , then  $[r]$  is a root of the filtrated model  $\mathfrak{M}_\Sigma$ .)

Recall that a reflexive and transitive frame  $(W, R)$  is *rooted* if there is  $x \in W$  such that for each  $y \in W$  we have  $Rxy$ .

(b) Deduce that **S4.3** has the finite model property.

(c) Deduce that **S4.3** is decidable.

(3) (40pt) (Item (a) is from the 2014 Exam)

(a) Show that for any modal formulas  $\varphi$  and  $\psi$  we have

$$\vdash_{\mathbf{K}} \Box\varphi \vee \Box\psi \text{ implies } \vdash_{\mathbf{K}} \varphi \text{ or } \vdash_{\mathbf{K}} \psi.$$

(b) Show that for any modal formulas  $\varphi$  and  $\psi$  we have:

$$\vdash_{\mathbf{S5}} \varphi \rightarrow \Box\psi \text{ iff } \vdash_{\mathbf{S5}} \Diamond\varphi \rightarrow \psi.$$

(c) Show that the above properties do not hold for all normal modal logics. That is, give an example of normal modal logics  $L_1$  and  $L_2$  which do not satisfy (a) and (b), respectively.

(Hint: use soundness and completeness of  $\mathbf{K}$  and  $\mathbf{S5}$  with respect to Kripke frames.)