SAHLQVIST ALGORITHM

BASED ON THE NOTES BY IAN HODKINSON

A boxed atom is a modal formula of the form $\Box^n p$, for some $n \in \mathbb{N}$, where $p$ is a propositional variable, and $\Box^n p$ is defined by the rule: $\Box^0 p = p$, $\Box^1 p = \Box p$, $\Box^{n+1} p = \Box(\Box^n p)$, $n \in \mathbb{N}$.

A Sahlqvist antecedent is built from $\bot$, $\top$ and boxed atoms by applying $\Diamond$ and $\land$.

A simple Sahlqvist formula is a modal formula of the form $\varphi \rightarrow \psi$, where $\varphi$ is a Sahlqvist antecedent and $\psi$ is a positive formula.

A Sahlqvist formula is built from simple Sahlqvist formulas by applying $\Box$ and $\lor$.

Theorem (Sahlqvist correspondence) For any Sahlqvist formula $\varphi$, there is a corresponding first-order sentence that holds in a frame iff $\varphi$ is valid in the frame.

This sentence can be obtained from $\varphi$ by a simple Sahlqvist algorithm. For simplicity we will consider only the case of simple Sahlqvist formulas.

Let $\varphi$ be a simple Sahlqvist formula.

(1) Identify boxed atoms in the antecedent.

(2) Draw the picture that discusses the minimal valuation that makes the antecedent true. Name the worlds involved by $t_0, \ldots, t_n$.

(3) Work out the minimal valuation i.e., get a first-order expression for it in terms of the named worlds.

(4) Work out the standard translation of $\varphi$. Use the names you fixed for the variables that correspond to $\Diamond$'s in the antecedent.

(5) Pull out the quantifiers that bind $t_i$ variables in the antecedent to the front. For this use the equivalences

$$\exists x \alpha(x) \land \beta \Leftrightarrow \exists x (\alpha(x) \land \beta),$$

$$\exists x \alpha(x) \rightarrow \beta \Leftrightarrow \forall x (\alpha(x) \rightarrow \beta).$$

(6) Replace all the predicates $P(x), Q(x)$, etc., with the first-order expression corresponding to the minimal valuation.
(7) Simplify, if possible.

(8) Add $\forall x$ (binding the free variable of the standard translation) to the resulting first-order formula to obtain the global first-order correspondent.

We will look at a few examples.

Let $\varphi = \Box p \to p$.

The diagram:

The minimal valuation is $V(p) = \{z : Rxz\}$.

The standard translation of $\varphi$ is $\forall y(Rxy \to P(y)) \to P(x)$.

Replace $P(y)$ with $Rxy$ and $P(x)$ with $Rxx$. We obtain $\forall y(Rxy \to Rxy) \to Rxx$.

This is equivalent to $Rxx$. By adding $\forall x$ we obtain the global first-order correspondent $\forall x Rxx$ reflexivity!

Let $\varphi = \Box p \to \Box \Box p$.

The diagram:

The minimal valuation is $V(p) = \{z : Rxz\}$.

The standard translation of $\varphi$ is

$$\forall y(Rxy \to P(y)) \to \forall z(Rxz \to \forall u(Rzu \to P(u)))$$

Replace $P(y)$ with $Rxy$ and $P(u)$ with $Rxu$. We obtain

$$\forall y(Rxy \to Rxy) \to \forall z(Rxz \to \forall u(Rzu \to Rxu))$$

This is equivalent to

$$\forall z(Rxz \to \forall u(Rzu \to Rxu))$$

which is equivalent to

$$\forall z \forall u(Rxz \land Rzu \to Rxu)$$
By adding $\forall x$ we obtain the global first-order correspondent
\[
\forall x \forall z \forall u (Rxz \land Rzu \rightarrow Rxu)
\] transitivity!

Let $\varphi = \Box \Box p \rightarrow \Box p$.

The diagram:

The minimal valuation is $V(p) = \{z : \exists v(Rxv \land Rvz)\}$. The standard translation of $\varphi$ is
\[
\forall y(Rxy \rightarrow \forall z(Ryz \rightarrow P(z))) \rightarrow \forall u(Rxu \rightarrow P(u))
\]

 Replace $P(u)$ with $\exists v(Rxv \land Rvu)$. In the antecedent we can replace $P(z)$ with the minimal valuation, but let us note that the instantiation of the standard translation of boxed atoms always gives us a tautology.

We obtain
\[
\forall u(Rxu \rightarrow \exists v(Rxv \land Rvu))
\]

By adding $\forall x$ we obtain the global first-order correspondent
\[
\forall x \forall u (Rxu \rightarrow \exists v(Rxv \land Rvu))
\] density!

Let $\varphi = \Diamond \Box p \rightarrow p$.

The diagram:

The minimal valuation is $V(p) = \{z : Rtz\}$.

The standard translation of $\varphi$ is
\[
\exists t(Rxt \land \forall z(Rtz \rightarrow P(z))) \rightarrow P(x)
\]

Pull out the existential quantifier in the antecedent. We obtain
\[
\forall t(Rxt \land \forall z(Rtz \rightarrow P(z)) \rightarrow P(x))
\]

Replace $P(z)$ with $Rtz$ and $P(x)$ with $Rtx$. We obtain
\[
\forall t(Rxt \land \forall z(Rtz \rightarrow Rtx) \rightarrow Rtx)
\]
This is equivalent to
\[ \forall t(Rxt \rightarrow Rtx) \]

By adding \( \forall x \) we obtain the global first-order correspondent
\[ \forall x \forall t(Rxt \rightarrow Rtx) \text{ symmetry!} \]

If \( \varphi \) is a Sahlqvist formula, say \( \square(\varphi \rightarrow \psi) \lor \square(\varphi' \rightarrow \psi') \) (where \( \varphi \rightarrow \psi \) and \( \varphi' \rightarrow \psi' \) are simple Sahlqvist formulas), then draw a diagram where outer \( \square \)'s are treated as \( \Diamond \)'s of simple Sahlqvist formulas and \( \lor \) is treated as \( \land \) of simple Sahlqvist formulas.

Let \( \varphi = \square(\square p \rightarrow q) \lor \square(\square q \rightarrow p) \).

The diagram:

The minimal valuation is \( V(p) = \{ z : Rt_1 z \} \) and \( V(q) = \{ z : Rt_2 z \} \).

The standard translation of \( \varphi \) (keeping in mind \( t_1 \) and \( t_2 \)) is
\[ \forall t_1(Rx t_1 \rightarrow (ST_{t_1}(\square p) \rightarrow Q(t_1))) \lor \forall t_2(Rx t_2 \rightarrow (ST_{t_2}(\square q) \rightarrow P(t_2))) \]

Pull out the quantifiers and replace \( Q(t_1) \) with \( Rt_2 t_1 \) and \( P(t_2) \) with \( Rt_1 t_2 \). Note again that the instantiation of the standard translation of boxed atoms gives a tautology.

We obtain
\[ \forall t_1 \forall t_2((Rx t_1 \rightarrow Rt_2 t_1) \lor (Rx t_2 \rightarrow Rt_2 t_1)) \]

which is equivalent to
\[ \forall t_1 \forall t_2((Rx t_1 \land Rx t_2) \rightarrow (Rt_1 t_2 \lor Rt_2 t_1)) \]

By adding \( \forall x \) we obtain the global first-order correspondent
\[ \forall x \forall t_1 \forall t_2((Rx t_1 \land Rx t_2) \rightarrow (Rt_1 t_2 \lor Rt_2 t_1)) \text{ linearity!} \]
Let $\varphi = \Box(\Box p \to p)$.

The diagram:

The minimal valuation is $V(p) = \{z : Rt z\}$.

The standard translation of $\varphi$ (keeping in mind $t$) is

$$\forall t(Rxt \to (ST_t(\Box p) \to P(t)))$$

Replace $P(t)$ with $Rtt$. Note again that the instantiation of the standard translation of boxed atoms gives a tautology.

We obtain

$$\forall t(Rxt \to Rtt)$$

By adding $\forall x$ we obtain the global first-order correspondent

$$\forall x \forall t(Rxt \to Rtt) \text{ every successor is reflexive!}$$