Introduction to Modal Logic. Exercise class 1

6 September 2018

The language of basic modal logic is given as follows.

**Definition 1.** The formulas of the basic modal language are given by the following grammar:

\[ \phi ::= p \mid \bot \mid \neg \phi \mid \phi \lor \psi \mid 3 \phi \]

where \( p \) ranges over a given set of propositional variables. Next to the standard Boolean abbreviations \( \top, \land, \rightarrow \) we will also use the operator \( \Box := \neg 3 \neg \).

The Kripke semantics of modal logic is given by the following definition.

**Definition 2.** A (Kripke) frame is a pair \( \mathcal{F} = (W, R) \) consisting of a non-empty set \( W \) and a binary accessibility relation \( R \subseteq W \times W \). Elements of \( W \) will be called (possible) worlds, states or points.

A (Kripke) model is a triple \( \mathcal{M} = (W, R, V) \) such that \( (W, R) \) is a Kripke frame and \( V \) is a valuation, i.e., \( V \) maps propositional variables to subsets of \( W \).

Given a model \( \mathcal{M} \) we define the notion of a modal formula being true or satisfied in \( \mathcal{M} \) at a state \( s \) by the following induction:

\[
\begin{align*}
\mathcal{M}, s \models p & \quad \text{iff } s \in V(p) \\
\mathcal{M}, s \models \bot & \quad \text{never} \\
\mathcal{M}, s \models \neg \phi & \quad \text{iff not } \mathcal{M}, s \models \phi \\
\mathcal{M}, s \models \phi \lor \psi & \quad \text{iff } \mathcal{M}, s \models \phi \text{ or } \mathcal{M}, s \models \psi \\
\mathcal{M}, s \models 3 \phi & \quad \text{iff } \mathcal{M}, t \models \phi, \text{ for some } t \in W \text{ with } Rst.
\end{align*}
\]

A formula is globally true in a model \( \mathcal{M} \) if it is true at every state in \( \mathcal{M} \), notation: \( \mathcal{M} \models \phi \); a formula is satisfiable in \( \mathcal{M} \) if it is true in at least one state in \( \mathcal{M} \).

**Exercise 1.** Consider the model \( \mathcal{M} = (W, R, V) \) below, where \( W = \{ s, t, u, v, w \} \), \( R \) is as indicated by the arrows in the picture, and \( V \) is given by \( V(p) = \{ s, t, w \} \) and \( V(q) = \{ t \} \).
(1) Show that
(a) $M, s \models \lozenge (q \land \lozenge q)$
(b) $M, w \models \Box \bot$
(c) $M, s \models \Box \lozenge \lozenge p$
(d) $M, s \not\models \Box \Box \Box p$

(2) Show that
(a) $M \not\models \lozenge \lozenge \lozenge \Box \bot$
(b) $M \models q \rightarrow \lozenge q$
(c) $M \models \lozenge \lozenge p \lor \lozenge \lozenge p$

(3) Can you find a valuation $V'$ on $(W, R)$ such that, with $M'$ being the resulting model:
(a) $M', t \not\models p \rightarrow \lozenge p$
(b) $M', s \not\models \lozenge \lozenge \lozenge p$
(c) $M' \models \lozenge \lozenge \Box \bot$

Definition 3. A formula $\phi$ is **satisfiable** if it is satisfiable in some model, and **valid** if it is globally true in every model.

**Exercise 2.** Which of the following formulas are satisfiable? Which ones are valid?

1. $\Box \top$
2. $\lozenge p \rightarrow \lozenge \lozenge p$
3. $(\Box p \land \lozenge q) \rightarrow \lozenge (p \land q)$
4. $\lozenge \lozenge p \rightarrow \Box \lozenge p$
5. $\lozenge \Box p \rightarrow \lozenge \lozenge p$
6. $\lozenge (p \lor q) \rightarrow (\lozenge p \lor \lozenge q)$

Definition 4. A formula $\phi$ is **valid** on a frame $F$ if $\phi$ is globally true in the model $(F, V)$, for every valuation $V$, and **satisfiable** in $F$ if it is satisfiable in the model $(F, V)$ for some valuation $V$.

**Poly-modal** logic is the version of modal logic where, instead of just one modal diamond $\lozenge$, there is a family $\{\lozenge_i \mid i \in I\}$, indexed by some set $I$.

Definition 5. Given a set $\{\lozenge_i \mid i \in I\}$ of modal diamonds, we define the associated set of modal formulas by the following grammar:

$$\phi ::= p \mid \bot \mid \neg \phi \mid \phi \lor \phi \mid \lozenge_i \phi$$

This language is interpreted in the obvious way by poly-modal Kripke structures, which generalize the mono-modal structures in that they have an accessibility relation $R_i$ for each diamond $\lozenge_i$. In particular, the semantic clause for the modality $\lozenge_i$ is as follows:

$$M, s \models \lozenge_i \phi \text{ iff } M, t \models \phi, \text{ for some } t \in W \text{ with } R_i st.$$
Exercise 3. Let $B = (B, R_1, R_2)$ the following binary tree frame. $B$ is the set of strings of 0s and 1s, and the relations $R_1$ and $R_2$ are defined by

\[ R_{1st} \text{ iff } t = s0 \text{ or } t = s1 \]
\[ R_{2st} \text{ iff } t \text{ is a proper initial segment of } s. \]

Which of the following formulas are valid on $B$:

1. $(\Diamond_1 p \land \Diamond_1 q) \rightarrow \Diamond_1 (p \land q)$
2. $(\Diamond_1 p \land \Diamond_1 q \land \Diamond_1 r) \rightarrow (\Diamond_1 (p \land q) \lor \Diamond_1 (p \land r) \lor \Diamond_1 (q \land r))$
3. $(\Diamond_2 p \land \Diamond_2 q \land \Diamond_2 r) \rightarrow (\Diamond_2 (p \land q) \lor \Diamond_2 (p \land r) \lor \Diamond_2 (q \land r))$
4. $\Box_2 (p \rightarrow \Box_1 p) \rightarrow (\Box_1 p \rightarrow \Box_2 p)$.

Exercise 4. Let $F = (W, R)$ be a Kripke frame. Prove the following:

1. $F \models p \rightarrow \Diamond_1 p$ iff $R$ is reflexive;
2. $F \models \Diamond_1 \Diamond_1 p \rightarrow \Diamond_1 p$ iff $R$ is transitive.

A special case of a poly-modal logic is temporal logic.

Definition 6. The basic temporal language is built using two modal diamonds, $\Diamond_F$ and $\Diamond_P$ (often written as $F$ and $P$, respectively).

The intended semantics of this language consists of so-called bidirectional structures, where the accessibility relation associated with the ‘past’ operator $\Diamond_P$ is the converse of the relation associated with the ‘future’ operator $\Diamond_F$.

Exercise 5. Let $F = (W, R_F, R_P)$ be a bidirectional temporal frame. Show that $F \models q \rightarrow \Box_F \Diamond_P q$.

A bidirectional frame $F$ is usually simply denoted as $F = (W, R)$, where we implicitly understand that $R_F = R$ and $R_P = \overline{R}$ (the converse of $R$).

Exercise 6. (*) Let $Q = (Q, <)$ and $R = (R, <)$ be the (bidirectional) frames given by the (strict) orderings of, respectively, the rational and the real numbers. Give a formula $\phi$ in the basic temporal language such that $R \models \phi$ but $Q \not\models \phi$.

(Hint: consider a valuation $V$ on $Q$ with $V(p) = \{ q \in Q \mid q < r \}$, where $r$ is an arbitrary irrational number $r$.)