Introduction to Modal Logic. Exercise class 1

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The language of *basic modal logic* is given as follows.

Definition 1. The *formulas* of the basic modal language are given by the following grammar:

$$\phi ::= p \mid \bot \mid \neg\phi \mid \phi \vee \phi \mid \Diamond\phi$$

where p ranges over a given set of *propositional variables*. Next to the standard Boolean abbreviations $\top$, $\wedge$, $\rightarrow$ we will also use the operator $\Box := \neg\Diamond\neg$.

The *Kripke semantics* of modal logic is given by the following definition.

Definition 2. A *(Kripke) frame* is a pair $\mathbb{F} = (W, R)$ consisting of a non-empty set W and a binary *accessibility relation* $R \subseteq W \times W$. Elements of W will be called *(possible) worlds*, *states* or *points*.

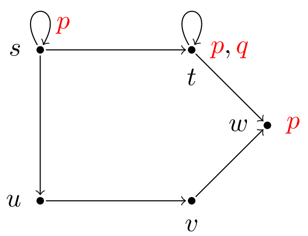
A *(Kripke) model* is a triple $\mathbb{M} = (W, R, V)$ such that (W, R) is a Kripke frame and V is a *valuation*, i.e., V maps propositional variables to subsets of W.

Given a model $\mathbb{M}$ we define the notion of a modal formula being *true* or *satisfied* in $\mathbb{M}$ at a state s by the following induction:

$$\begin{array}{lll}
\mathbb{M}, s \Vdash p & \text{iff} & s \in V(p) \\
\mathbb{M}, s \Vdash \bot & & \text{never} \\
\mathbb{M}, s \Vdash \neg\phi & \text{iff} & \text{not } \mathbb{M}, s \Vdash \phi \\
\mathbb{M}, s \Vdash \phi \vee \psi & \text{iff} & \mathbb{M}, s \Vdash \phi \text{ or } \mathbb{M}, s \Vdash \psi \\
\mathbb{M}, s \Vdash \Diamond\phi & \text{iff} & \mathbb{M}, t \Vdash \phi, \text{ for some } t \in W \text{ with } Rst.
\end{array}$$

A formula is *globally true* in a model $\mathbb{M}$ if it is true at every state in $\mathbb{M}$, notation: $\mathbb{M} \Vdash \phi$; a formula is *satisfiable* in $\mathbb{M}$ if it is true in at least one state in $\mathbb{M}$.

Exercise 1. Consider the model $\mathbb{M} = (W, R, V)$ below, where $W = \{s, t, u, v, w\}$, R is as indicated by the arrows in the picture, and V is given by $V(p) = \{s, t, w\}$ and $V(q) = \{t\}$.



(1) Show that

   (a) $\mathbb{M}, s \Vdash \Diamond(q \wedge \Diamond q)$
   (b) $\mathbb{M}, w \Vdash \Box\bot$
   (c) $\mathbb{M}, s \Vdash \Box\Diamond\Diamond p$
   (d) $\mathbb{M}, s \nVdash \Box\Box\Box p$

(2) Show that

   (a) $\mathbb{M} \nVdash \Diamond\Diamond\Box\bot$
   (b) $\mathbb{M} \Vdash q \to \Diamond q$
   (c) $\mathbb{M} \Vdash \Diamond\Box p \vee \Box\Diamond p$

(3) Can you find a valuation V' on (W, R) such that, with $\mathbb{M}'$ being the resulting model:

   (a) $\mathbb{M}', t \nVdash p \to \Diamond p$
   (b) $\mathbb{M}', s \nVdash \Box\Diamond\Diamond p$
   (c) $\mathbb{M}' \Vdash \Diamond\Diamond\Box\bot$

Definition 3. A formula ϕ is *satisfiable* if it is satisfiable in some model, and *valid* if it is globally true in every model.

Exercise 2. Which of the following formulas are satisfiable? Which ones are valid?

(1) $\Box\top$

(2) $\Diamond p \to \Diamond\Diamond p$

(3) $(\Box p \wedge \Diamond q) \to \Diamond(p \wedge q)$

(4) $\Box\Diamond p \to \Diamond\Box p$

(5) $\Diamond\Box p \to \Box\Diamond p$

(6) $\Diamond(p \vee q) \to (\Diamond p \vee \Diamond q)$

Definition 4. A formula ϕ is *valid* on a frame $\mathbb{F}$ if ϕ is globally true in the model $(\mathbb{F}, V)$, for every valuation V, and *satisfiable* in $\mathbb{F}$ if it is satisfiable in the model $(\mathbb{F}, V)$ for some valuation V.

Poly-modal logic is the version of modal logic where, instead of just one modal diamond $\Diamond$, there is a family $\{\Diamond_I \mid i \in I\}$, indexed by some set I.

Definition 5. Given a set $\{\Diamond_i \mid i \in I\}$ of modal diamonds, we define the associated set of modal formulas by the following grammar:

$$\phi ::= p \mid \bot \mid \neg\phi \mid \phi \vee \phi \mid \Diamond_i \phi$$

This language is interpreted in the obvoius way by poly-modal Kripke structures, which generalize the mono-modal structures in that they have an accessibility relation R_i for *each* diamond $\Diamond_i$. In particular, the semantic clause for the modality $\Diamond_i$ is as follows:

$$\mathbb{M}, s \Vdash \Diamond_i \phi \text{ iff } \mathbb{M}, t \Vdash \phi, \text{ for some } t \in W \text{ with } R_i st.$$

Exercise 3. Let $\mathbb{B} = (B, R_1, R_2)$ the following *binary tree frame*. B is the set of strings of 0s and 1s, and the relations R_1 and R_2 are defined by

$$\begin{array}{lll} R_1 st & \text{iff} & t = s0 \text{ or } t = s1 \\ R_2 st & \text{iff} & t \text{ is a proper initial segment of } s. \end{array}$$

Which of the following formulas are valid on $\mathbb{B}$:

(1) $(\Diamond_1 p \wedge \Diamond_1 q) \to \Diamond_1(p \wedge q)$

(2) $(\Diamond_1 p \wedge \Diamond_1 q \wedge \Diamond_1 r) \to \big(\Diamond_1(p \wedge q) \vee \Diamond_1(p \wedge r) \vee \Diamond_1(q \wedge r)\big)$

(3) $(\Diamond_2 p \wedge \Diamond_2 q \wedge \Diamond_2 r) \to \big(\Diamond_2(p \wedge q) \vee \Diamond_2(p \wedge r) \vee \Diamond_2(q \wedge r)\big)$

(4) $\Box_2(p \to \Box_1 p) \to (\Box_1 p \to \Box_2 p)$.

Exercise 4. Let $\mathbb{F} = (W, R)$ be a Kripke frame. Prove the following:

(1) $\mathbb{F} \Vdash p \to \Diamond p$ iff R is reflexive;

(2) $\mathbb{F} \Vdash \Diamond\Diamond p \to \Diamond p$ iff R is transitive.

A special case of a poly-modal logic is *temporal logic*.

Definition 6. The basic temporal language is built using two modal diamonds, $\Diamond_F$ and $\Diamond_P$ (often written as F and P, respectively).

The intended semantics of this language consists of so-called *bidirectional structures*, where the accessibility relation associated with the 'past' operator $\Diamond_P$ is the *converse* of the relation associated with the 'future' operator $\Diamond_F$.

Exercise 5. Let $\mathbb{F} = (W, R_F, R_P)$ be a bidirectional temporal frame. Show that $\mathbb{F} \Vdash q \to \Box_F \Diamond_P q$.

A bidirectional frame $\mathbb{F}$ is usually simply denoted as $\mathbb{F} = (W, R)$, where we implicitly understand that $R_F = R$ and $R_P = R^{\smile}$ (the converse of R).

Exercise 6. (*) Let $\mathbb{Q} = (Q, <)$ and $\mathbb{R} = (R, <)$ be the (bidirectional) frames given by the (strict) orderings of, respectively, the rational and the real numbers. Give a formula ϕ in the basic temporal language such that $\mathbb{R} \Vdash \phi$ but $\mathbb{Q} \not\Vdash \phi$. (Hint: consider a valuation V on $\mathbb{Q}$ with $V(p) = \{q \in Q \mid q < r\}$, where r is an arbitrary irrational number r.)