EXERCISE CLASS 15-11-2017: GENERAL FRAMES

1. General Frames

(1) Let $\mathbb{M} := (\mathbb{F}, V)$ be a model with $\mathbb{F} := (W, R)$ and let $\mathcal{A}_V := \{\overline{V}(\varphi) \colon \varphi \in \text{Form}\}$, where $\overline{V}(\varphi) := \{w \in W \colon \mathbb{M}, w \Vdash \varphi\}$. Show that $\mathfrak{g}_{\mathbb{M}} := (\mathbb{F}, \mathcal{A}_V)$ is a general frame.

(2) Let L be a consistent normal modal logic and $\mathbb{M}^L = (\mathbb{F}^L, V^L)$ its canonical model. Let $\mathfrak{g}^L$ denote the general frame $(\mathbb{F}^L, \mathcal{A}_{V^L})$. Show that for any formula φ,

$$\varphi \in L \iff \mathfrak{g}^L \Vdash \varphi.$$

Conclude that any normal modal logic is sound and complete with respect to a class of general frames.

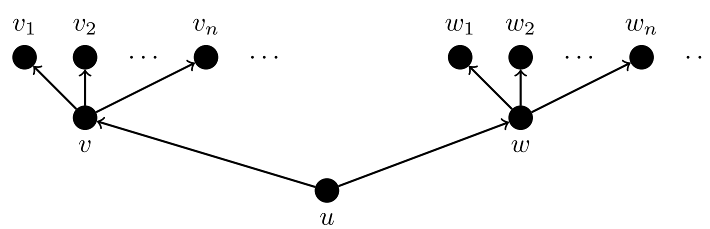
(3) Let $\mathcal{K}$ be a class of general frames. Show that $\text{Log}(\mathcal{K}) := \{\varphi : \forall \mathfrak{g} \in \mathcal{K}\ (\mathfrak{g} \Vdash \varphi\})\}$ is a normal modal logic.

(4) Consider the Kripke frame (W, R) depicted in below. Let $\mathcal{A}$ be the collection of all finite and co-finite subsets of W. Show that

(a) $\mathfrak{g} := (W, R, \mathcal{A})$ is a general frame,

(b) $(W, R), u \not\Vdash \Diamond\Box p \to \Box\Diamond p$,

(c) $\mathfrak{g}, u \Vdash \Diamond\Box p \to \Box\Diamond p$.



(5) Let ω be a symbol with $\omega \notin \mathbb{N}$. Consider $\mathfrak{g} := (\mathbb{N} \cup \{\omega\}, R, \mathcal{A})$ where $R := \{(\omega, x) \colon x \in \mathbb{N} \cup \{\omega\}\} \cup \{(n, m) \colon m < n\}$ and $\mathcal{A}$ is the set of finite subsets of $\mathbb{N}$ and the co-finite subsets of $\mathbb{N} \cup \{\omega\}$ which contain ∞. *Make a drawing*

(a) Show that $\mathfrak{g}$ is a general frame.

(b) Show that $\mathfrak{g} \Vdash \Box(\Box p \to p) \to \Box p$.

(c) Show that $\mathbb{F} := (\mathbb{N} \cup \{\omega\}, R)$ is not a frame for $\mathbf{KL} = \mathbf{K} + \Box(\Box p \to p) \to \Box p$.

(d) Use (a) and (b) to show that the set $\{\Box\varphi \to \varphi \colon \varphi \in \text{Form}\}$ is $\mathbf{KL}$-consistent.

(e) Conclude that the canonical model for $\mathbf{KL}$ contains a reflexive world and therefore that $\mathbf{KL}$ is not canonical.

(6) (BdRV 4.4.3) Show that for any consistent normal modal logic L in the language of basic modal logic $Frm(L) \neq \emptyset$. Conclude that any such consistent normal modal logic is sound with respect to some non-empty class of frames. *Hint: show that either $L \subseteq Log(\bullet)$ or $L \subseteq Log(\circ)$, where $Log(\circ) := \mathbf{K} + p \leftrightarrow \Box p$ and $Log(\bullet) := \mathbf{K} + \Box\bot$.* Why does this not contradict the fact that $\mathbf{K}_t\mathbf{ThoM}$ is a consistent logic with no Kripke frames?