EXERCISE CLASS 22-11-2018:
PDL

(1) Let $\mathbb{F} = (W, \{R_\pi\}_{\pi\in\Pi})$ be a frame. Show that for each $\pi, \pi_1, \pi_2 \in \Pi$ we have

  (a) $\mathbb{F} \Vdash \langle \pi_1; \pi_2\rangle p \leftrightarrow \langle \pi_1\rangle\langle \pi_2\rangle p$ iff $R_{\pi_1;\pi_2} = R_{\pi_1} \circ R_{\pi_2}$,[1]

  (b) $\mathbb{F} \Vdash \langle \pi_1 \cup \pi_2\rangle p \leftrightarrow \langle \pi_1\rangle p \vee \langle \pi_2\rangle p$ iff $R_{\pi_1\cup\pi_2} = R_{\pi_1} \cup R_{\pi_2}$,

  (c) If $(R_\pi)^* = R_{\pi^*}$ then,

$$\mathbb{F} \Vdash \langle \pi^*\rangle p \leftrightarrow p \vee \langle \pi\rangle\langle \pi^*\rangle p \text{ and}$$
$$\mathbb{F} \Vdash [\pi^*](p \to [\pi]p) \to (p \to [\pi^*]p),$$

  (d) If $\mathbb{F} \Vdash p \vee \langle \pi\rangle\langle \pi^*\rangle p \to \langle \pi^*\rangle p$, then $(R_\pi)^* \subseteq R_{\pi^*}$.

(2) Let $\mathcal{M}$ be the set of all **PDL**-MCSs, and Σ a set of formulas in the language of **PDL**. Show that:

  (a) $At(\Sigma) = \{\Gamma \cap \neg\mathrm{FL}(\Sigma) \mid \Gamma \in \mathcal{M}\}$,

  (b) If $X \subseteq \neg\mathrm{FL}(\Sigma)$ and X is **PDL**-consistent, then there exists $A \in At(\Sigma)$ such that $X \subseteq A$.

(3) Consider a set of **PDL** formulas Σ and let $A \in At(\Sigma)$ be an atom over Σ. Show that:

  (a) For every $\varphi \in \neg\mathrm{FL}(\Sigma)$, exactly one of φ and $\sim\varphi$ is in A,

  (b) For every $\varphi \vee \psi \in \neg\mathrm{FL}(\Sigma)$, $\varphi \vee \psi \in A$ iff $\varphi \in A$ or $\psi \in A$,

  (c) For every $\langle \pi_1; \pi_2\rangle\varphi \in \neg\mathrm{FL}(\Sigma)$, $\langle \pi_1; \pi_2\rangle\varphi \in A$ iff $\langle \pi_1\rangle\langle \pi_2\rangle\varphi \in A$,

  (d) For every $\langle \pi_1 \cup \pi_2\rangle\varphi \in \neg\mathrm{FL}(\Sigma)$, $\langle \pi_1 \cup \pi_2\rangle\varphi \in A$ iff $\langle \pi_1\rangle\varphi$ or $\langle \pi_2\rangle\varphi \in A$.

(4) (From 2017 Final Exam) For a program π and a natural number k we let

$$\langle \pi\rangle^0 p = p, \quad \langle \pi\rangle^1 p = \langle \pi\rangle p, \quad \langle \pi\rangle^{k+1} p = \langle \pi\rangle\langle \pi\rangle^k p.$$

For each $n \geq 0$ we define a normal extension $\mathbf{PDL}^n$ of **PDL** as follows. $\mathbf{PDL}^n$ is the least normal extension of **PDL** that contains all the instances of the formula:

$$\langle \pi^*\rangle p \leftrightarrow \bigvee_{i=0}^{n} \langle \pi\rangle^i p.$$

  (a) Show that the logic $\mathbf{PDL}^n$ is sound with respect to regular frames $(W, \{R_\pi\}_{\pi\in\Pi})$ such that $R_{\pi^*} = \bigcup_{k=0}^n R_\pi^k$, where

$$R_\pi^0 := \{(w, w) \mid w \in W\}, \quad R_\pi^1 := R_\pi, \quad \text{and} \quad R_\pi^{k+1} := R_\pi \circ R_\pi^k.$$

  (b) Show that $\mathbf{PDL} = \bigcap_{n\geq 0} \mathbf{PDL}^n$.

[1] In BdRV $R_{\pi_1} \circ R_{\pi_2}$ is also denoted $R_{\pi_1}; R_{\pi_2}$.

(Hint: For the right to left inclusion use the fact that **PDL** has the finite model property.)

(5) Explain why all the axioms of **PDL**, in the standard axiomatisation, with the exception of Segerberg's induction axiom,

$$[\pi^*](p \to [\pi]p) \to (p \to [\pi^*]p)$$

are canonical.

ADDITIONAL EXERCISES

(6) Is **PDL** a decidable logic?

(7) ($*$) Let φ be a formula in the language of **PDL** and let $\Sigma = \{\varphi\}$. Show that $\neg\mathrm{FL}(\Sigma)$ is finite. *Hint: This is not so easy.*