EXERCISE CLASS 22-11-2018:

PDL

(1) Let $F = (W, \{R_\pi\}_{\pi \in \Pi})$ be a frame. Show that for each $\pi, \pi_1, \pi_2 \in \Pi$ we have

(a) $F \vdash \langle \pi_1 \rangle p \leftrightarrow \langle \pi \rangle \langle \pi_2 \rangle p$ iff $R_{\pi_1 \pi_2} = R_{\pi_1} \circ R_{\pi_2}$,

(b) $F \vdash \langle \pi_1 \cup \pi_2 \rangle p \leftrightarrow \langle \pi_1 \rangle p \lor \langle \pi_2 \rangle p$ iff $R_{\pi_1 \cup \pi_2} = R_{\pi_1} \cup R_{\pi_2}$,

(c) If $(R_\pi)^* = R_\pi^*$ then,

\[
F \vdash \langle \pi^* \rangle p \leftrightarrow p \lor \langle \pi \rangle \langle \pi^* \rangle p \quad \text{and} \quad F \vdash [\pi^*](p \rightarrow [\pi]p) \rightarrow (p \rightarrow [\pi^*]p),
\]

(d) If $F \vdash p \lor \langle \pi \rangle \langle \pi^* \rangle p \rightarrow \langle \pi^* \rangle p$, then $(R_\pi)^* \subseteq R_\pi^*$.

(2) Let $M$ be the set of all PDL-MCSs, and $\Sigma$ a set of formulas in the language of PDL. Show that:

(a) $At(\Sigma) = \{\Gamma \cap \neg FL(\Sigma) \mid \Gamma \in M\}$,

(b) If $X \subseteq \neg FL(\Sigma)$ and $X$ is PDL-consistent, then there exists $A \in At(\Sigma)$ such that $X \subseteq A$.

(3) Consider a set of PDL formulas $\Sigma$ and let $A \in At(\Sigma)$ be an atom over $\Sigma$. Show that:

(a) For every $\varphi \in \neg FL(\Sigma)$, exactly one of $\varphi$ and $\sim \varphi$ is in $A$,

(b) For every $\varphi \lor \psi \in \neg FL(\Sigma)$, $\varphi \lor \psi \in A$ iff $\varphi \in A$ or $\psi \in A$,

(c) For every $\langle \pi_1 ; \pi_2 \rangle \varphi \in \neg FL(\Sigma)$, $\langle \pi_1 ; \pi_2 \rangle \varphi \in A$ iff $\langle \pi_1 \rangle \langle \pi_2 \rangle \varphi \in A$,

(d) For every $\langle \pi_1 \cup \pi_2 \rangle \varphi \in \neg FL(\Sigma)$, $\langle \pi_1 \cup \pi_2 \rangle \varphi \in A$ iff $\langle \pi_1 \rangle \varphi$ or $\langle \pi_2 \rangle \varphi \in A$.

(4) (From 2017 Final Exam) For a program $\pi$ and a natural number $k$ we let

\[
\langle \pi \rangle^0 p = p, \quad \langle \pi \rangle^1 p = \langle \pi \rangle p, \quad \langle \pi \rangle^{k+1} p = \langle \pi \rangle \langle \pi \rangle^k p.
\]

For each $n \geq 0$ we define a normal extension $PDL^n$ of PDL as follows. $PDL^n$ is the least normal extension of PDL that contains all the instances of the formula:

\[
\langle \pi^* \rangle p \leftrightarrow \bigsqcup_{i=0}^{n} \langle \pi \rangle^i p.
\]

(a) Show that the logic $PDL^n$ is sound with respect to regular frames $(W, \{R_\pi\}_{\pi \in \Pi})$ such that $R_\pi^* = \bigcup_{k=0}^{n} R_\pi^k$, where

\[
R_0^\pi := \{(w, w) \mid w \in W\}, \quad R_1^\pi := R_\pi, \quad \text{and} \quad R_\pi^{k+1} := R_\pi \circ R_\pi^k.
\]

(b) Show that $PDL = \bigcap_{n \geq 0} PDL^n$.

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1In BdRV $R_{\pi_1} \circ R_{\pi_2}$ is also denoted $R_{\pi_1 ; \pi_2}$. 

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(Hint: For the right to left inclusion use the fact that PDL has the finite model property.)

(5) Explain why all the axioms of PDL, in the standard axiomatisation, with the exception of Segerberg’s induction axiom,

\[
[\pi^*](p \rightarrow [\pi]p) \rightarrow (p \rightarrow [\pi^*]p)
\]

are canonical.

ADDITIONAL EXERCISES

(6) Is PDL a decidable logic?

(7) (*) Let \( \varphi \) be a formula in the language of PDL and let \( \Sigma = \{ \varphi \} \). Show that \( \neg \text{FL}(\Sigma) \) is finite. Hint: This is not so easy.