EXERCISE CLASS 29-11-2018: PDL

(1) Consider a set of **PDL** formulas Σ and let $A \in At(\Sigma)$ be an atom over Σ. Show that:

(a) For every $\varphi \in \neg\mathrm{FL}(\Sigma)$, exactly one of φ and $\sim\varphi$ is in A,

(b) For every $\varphi \vee \psi \in \neg\mathrm{FL}(\Sigma)$, $\varphi \vee \psi \in A$ iff $\varphi \in A$ or $\psi \in A$,

(c) For every $\langle \pi_1; \pi_2 \rangle \varphi \in \neg\mathrm{FL}(\Sigma)$, $\langle \pi_1; \pi_2 \rangle \varphi \in A$ iff $\langle \pi_1 \rangle \langle \pi_2 \rangle \varphi \in A$,

(d) For every $\langle \pi_1 \cup \pi_2 \rangle \varphi \in \neg\mathrm{FL}(\Sigma)$, $\langle \pi_1 \cup \pi_2 \rangle \varphi \in A$ iff $\langle \pi_1 \rangle \varphi$ or $\langle \pi_2 \rangle \varphi \in A$.

(2) (From 2017 Final Exam) For a program π and a natural number k we let

$$\langle \pi \rangle^0 p = p, \quad \langle \pi \rangle^1 p = \langle \pi \rangle p, \quad \langle \pi \rangle^{k+1} p = \langle \pi \rangle \langle \pi \rangle^k p.$$

For each $n \geq 0$ we define a normal extension $\mathbf{PDL}^n$ of **PDL** as follows. $\mathbf{PDL}^n$ is the least normal extension of **PDL** that contains all the instances of the formula:

$$\langle \pi^* \rangle p \leftrightarrow \bigvee_{i=0}^{n} \langle \pi \rangle^i p.$$

(a) Show that the logic $\mathbf{PDL}^n$ is sound with respect to regular frames $(W, \{R_\pi\}_{\pi \in \Pi})$ such that $R_{\pi^*} = \bigcup_{k=0}^n R_\pi^k$, where

$$R_\pi^0 := \{(w, w) \mid w \in W\}, \quad R_\pi^1 := R_\pi, \quad \text{and} \quad R_\pi^{k+1} := R_\pi \circ R_\pi^k.$$

(b) Show that $\mathbf{PDL} = \bigcap_{n \geq 0} \mathbf{PDL}^n$.

(Hint: For the right to left inclusion use the fact that **PDL** has the finite model property.)

(3) (From 2016 Final Exam) Let $(\omega\text{-}^*)$ be the following rule:

$$\text{If } \vdash \varphi \to [\pi]^n \psi \text{ for each } n \in \mathbb{N}, \text{ then } \vdash \varphi \to [\pi^*]\psi.$$

Recall that $[\pi]^0 p = p$ and $[\pi]^{n+1} = [\pi][\pi]^n p$.

We say that $(\omega\text{-}^*)$ is *valid* on a frame (W, R_π, R_{π^*}) if for any valuation V

$$(W, R_\pi, R_{\pi^*}, V) \Vdash \varphi \to [\pi]^n \psi \text{ for each } n \in \mathbb{N} \text{ implies } (W, R_\pi, R_{\pi^*}, V) \Vdash \varphi \to [\pi^*]\psi.$$

Let (W, R_π, R_{π^*}) be a (not necessarily regular) frame. Show that we have

$$R_{\pi^*} \subseteq (R_\pi)^* \text{ iff } (\omega\text{-}^*) \text{ is valid on } (W, R_\pi, R_{\pi^*}).$$

(4) Explain why all the axioms of **PDL**, in the standard axiomatisation, with the exception of Segerberg's induction axiom,

$$[\pi^*](p \to [\pi]p) \to (p \to [\pi^*]p)$$

are canonical.

(5) Show that the finite models from [Def. 4.84, BdRV] used in the **PDL** completeness proof can be obtain (up to isomorphism) via certain filtrations of the canonical model for **PDL**.

ADDITIONAL EXERCISES

(6) Is **PDL** a decidable logic?

(7) ($*$) Let Σ be a non-empty finite set of **PDL**-formulas. Show that $\vdash_{\mathbf{PDL}} \bigvee_{A \in At(\Sigma)} \widehat{A}$.

(8) ($**$) Let φ be a formula in the language of **PDL** and let $\Sigma = \{\varphi\}$. Show that $\neg\mathrm{FL}(\Sigma)$ is finite. *Hint: This is not so easy.*