EXERCISE CLASS 12-12-2018:
NEIGHBORHOOD SEMANTICS

(1) Call an NBD-model $M = (W, N, V)$ an augmented NBD-model if the neighborhood of each point is a complete filter, i.e., $N(w)$ is a non-empty, upwards closed set which is closed under arbitrary intersections, for every $w \in W$.

(a) Let $M = (W, R, V)$ be a Kripke model. Define an NBD-model $M = (W, N, V)$ such that for each $w \in W$ and each modal formula $\varphi$ we have

$$M, w \models \varphi \iff M, w \models \varphi \quad (*)$$

(b) Let $M = (W, N, V)$ be an augmented NBD-model. Define a Kripke model $M = (W, R, V)$ such that for each $w \in W$ and each modal formula $\varphi$, $(*)$ holds.

(c) Is it possible to find $M$ as in point 1b for an arbitrary $M$?

(2) Define $E \oplus \gamma$ as the smallest set of formulas containing $E$, $\gamma$ and closed under MP and US. Prove that

(a) $EM = E \oplus (\Box(p \land q) \rightarrow \Box p \land \Box q)$ is the smallest minimal modal logic containing $E$ and closed under the rule $RM$

$$RM \frac{p \rightarrow q}{\Box p \rightarrow \Box q}$$

(b) $EN = E \oplus (\Box \top)$ is the smallest minimal modal logic containing $E$ and closed under Necessitation.

(c) $EMCN = E \oplus (\Box(p \land q) \rightarrow \Box p \land \Box q) \oplus (\Box p \land \Box q \rightarrow \Box(p \land q) \oplus (\Box \top)$ is the in fact the modal logic $K$.

(3) Define the modality $\langle \rangle$ as follows,

$$M, w \models \langle \varphi \rangle \iff \exists X \in N(w), \ X \subseteq \{\varphi\}_M$$

Prove that $\langle \rangle$ validates the axiom $(M)$ and that $\langle \rangle$ and $\Box$ coincide on monotone NBD-frames.

(4) Let $L$ be a modal logic. Given a formula in the language of basic modal logic $\varphi$ define

$$|\varphi|_L := \{\Gamma \in M_L : \varphi \in \Gamma\},$$

where $M_L$ is the set of maximal $L$-consistent sets. Show that for formulas $\varphi$ and $\psi$,

$$|\varphi|_L \subseteq |\psi|_L \iff \models_L \varphi \rightarrow \psi.$$
(d) The logic $K$ is sound and complete with respect to the class of (augmented) filter models.

Hint: For item (c) consider the canonical model $\mathcal{M}_{EM} = (W_{EM}, N_{EM}, V_{EM})$ for $EM$ and for each $w \in W$, consider the new neighborhood function

$$N^\text{min}_{EM}(w) := \{X \in \wp(W_{EM}) : \exists Y \in N^\text{min}_{EM}(w) (Y \subseteq X)\}.$$