EXERCISE CLASS 12-12-2018: NEIGHBORHOOD SEMANTICS

(1) Call an NBD-model $\mathbb{M} = (W, N, V)$ an *augmented NBD-model* if the neighborhood of each point is a complete filter, i.e., $N(w)$ is a non-empty, upwards closed set which is closed under arbitrary intersections, for every $w \in W$.

  (a) Let $\mathcal{M} = (W, R, V)$ be a Kripke model. Define an NBD-model $\mathbb{M} = (W, N, V)$ such that for each $w \in W$ and each modal formula φ we have
$$\mathcal{M}, w \Vdash \varphi \iff \mathbb{M}, w \Vdash \varphi \qquad (*)$$

  (b) Let $\mathbb{M} = (W, N, V)$ be an augmented NBD-model. Define a Kripke model $\mathcal{M} = (W, R, V)$ such that for each $w \in W$ and each modal formula φ, (*) holds.

  (c) Is it possible to find $\mathcal{M}$ as in point 1b for an arbitrary $\mathbb{M}$?

(2) Define $\mathbf{E} \oplus \gamma$ as the smallest set of formulas containing $\mathbf{E}$, γ and closed under MP and US. Prove that

  (a) $\mathbf{EM} = \mathbf{E} \oplus (\ \Box(p \wedge q) \to \Box p \wedge \Box q\ )$ is the smallest minimal modal logic containing $\mathbf{E}$ and closed under the rule RM
$$\text{RM}\ \frac{p \to q}{\Box p \to \Box q}$$

  (b) $\mathbf{EN} = \mathbf{E} \oplus (\ \Box\top\ )$ is the smallest minimal modal logic containing $\mathbf{E}$ and closed under Necessitation.

  (c) $\mathbf{EMCN} = \mathbf{E} \oplus (\ \Box(p \wedge q) \to \Box p \wedge \Box q\ ) \oplus (\ \Box p \wedge \Box q \to \Box(p \wedge q)\ ) \oplus (\ \Box\top\ )$ is the in fact the modal logic $\mathbf{K}$.

(3) Define the modality $\langle\ ]$ as follows,
$$\mathbb{M}, w \Vdash \langle\ ]\varphi \iff \exists X \in N(w),\ X \subseteq [\![\varphi]\!]_{\mathbb{M}}$$
Prove that $\langle\ ]$ validates the axiom (**M**) and that $\langle\ ]$ and $\Box$ coincide on monotone NBD-frames.

(4) Let $\mathbf{L}$ be a modal logic. Given a formula in the language of basic modal logic φ define
$$|\varphi|_{\mathbf{L}} := \{\Gamma \in M_{\mathbf{L}} \colon \varphi \in \Gamma\},$$
where $M_{\mathbf{L}}$ is the set of maximal $\mathbf{L}$-consistent sets. Show that for formulas φ and ψ,
$$|\varphi|_{\mathbf{L}} \subseteq |\psi|_{\mathbf{L}} \iff \vdash_{\mathbf{L}} \varphi \to \psi.$$

(5) Show that

  (a) The logic **EC** is sound and complete with respect to the class of neighborhood frames that are closed under intersections;

  (b) The logic **EN** is sound and complete with respect to the class of neighborhood frames that contains the unit;

  (c) ($*$) The logic **EM** is sound and complete with respect to the class of monotone neighborhood frames;

(d) The logic $\mathbf{K}$ is sound and complete with respect to the class of (augmented) filter models.

Hint: For item (c) consider the canonical model $\mathcal{M}_{\mathbf{EM}} = (W_{\mathbf{EM}}, N_{\mathbf{EM}}, V_{\mathbf{EM}})$ for $\mathbf{EM}$ and for each $w \in W$, consider the new neighborhood function

$$N^{\mathrm{mm}}_{\mathbf{EM}}(w) \coloneqq \{X \in \wp(W_{\mathbf{EM}}) \colon \exists Y \in N^{\mathrm{min}}_{\mathbf{EM}}(w)\ (Y \subseteq X)\}.$$