Exercise 1. Consider the binary modality $U$ ('until') with the following semantics

$$M, s \models \phi U \psi \text{ iff } \begin{cases} \text{there is a } t \text{ such that } Rst & \text{and } M, t \models \phi, \\ \text{for every } u \text{ such that } Rsu & \text{it holds } M, u \models \psi. \end{cases}$$

Is $U$ expressible in the language of basic modal logic? And in the language of basic temporal logic?

Hint: consider the models in [BdRV, Exercise 2.2.4].

Exercise 2. Consider the modality $\circ$ with the following semantics

$$M, s \models \phi \circ \psi \iff \exists t \in W (sRt \land \neg (tRt) \land M, t \models \phi).$$

Is $\circ$ expressible in the language of basic modal logic?

Exercise 3. Let $M = (W, R, V)$ be a Kripke model, and let $X$ be a subset of $W$. We define $M_X$ as the restricted model $(X, R_X, V_X)$, where $R_X := R \cap (X \times X)$ and $V_X(p) := V(p) \cap X$. We call $X \subseteq W$ hereditary if $s \in X$ and $Rst$ imply $t \in X$; in this case we say that $M_X$ is a generated submodel of $M$.

(1) Show that $\Delta_X := \{(x, x) \mid x \in X\}$ is a bisimulation between $M_X$ and $M$ iff $X$ is hereditary.

(2) Show that if $f$ is a bounded morphism from $M$ to $M'$, then the set $f[W] := \{f(s) \mid s \in W\}$ is a hereditary subset of $W'$.

Exercise 4. A bounded morphism\(^1\) between two frames $F = (W, R, V)$ and $F' = (W', R')$ is a map $f : W \to W'$ such that, for all $s, t \in W$ and $t' \in W'$:

(forth) $Rst$ implies $R'f(s)f(t)$;

(back) $R'f(s)t'$ implies the existence of a $t \in W$ with $Rst$ and $f(t) = t'$.

Now let $f$ be such a bounded morphism.

(1) Show that for any valuation $V'$ on $F'$ one can find a valuation $V$ on $F$ such that $f$ (or rather, its graph $\{(s, f(s)) \mid s \in W\}$) is a bisimulation between the models $(F, V)$ and $(F', V')$.

(2) Show that if $f$ is surjective, then $F \models \phi$ implies $F' \models \phi$, for any modal formula $\phi$.

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\(^1\)Some authors call bounded morphisms $p$-morphisms.
(3) Prove that irreflexivity is not modally definable. That is, show that there is no modal formula φ such that φ is valid on exactly the frames with an irreflexive accessibility relation.

**Exercise 5.** Which of the following frame properties are preserved (reflected) by the operations of forming generated subframes, bounded morphic images, disjoint unions?

1. reflexivity;
2. transitivity;
3. irreflexivity;
4. converse seriality (∀x∃y Ryx);
5. having cardinality at least n, for some natural number n;
6. having cardinality at most n, for some natural number n.

**Exercise 6.** Show that the following frame properties cannot be defined in the basic modal language:

1. converse seriality;
2. having cardinality at least n, for some natural number n;
3. having cardinality at most n, for some natural number n;
4. acyclicity: ‘there is no finite path (of non-zero length) from any point to itself’.

**Exercise 7.** Find frames \( F_1 = (W_1, R_1) \) and \( F_2 = (W_2, R_2) \) and a formula \( \phi \) such that

\[ F_1 \models \phi \quad \text{and} \quad F_2 \not\models \phi. \]

Furthermore, find valuations \( V_1 \) and \( V_2 \) on \( F_1 \) and \( F_2 \), respectively, such that

\[ (F_1, V_1), w_1 \models (F_2, V_2), w_2, \]

for all \( w_1 \in W_1 \) and all \( w_2 \in W_2 \).