Introduction to Modal Logic. Exercise class 2

13 September 2018

Exercise 1. Consider the binary modality U ('until') with the following semantics

$$\mathbb{M}, s \Vdash \phi \,\mathrm{U}\, \psi \text{ iff } \begin{cases} \text{there is a } t \text{ such that } Rst \;\&\; \mathbb{M}, t \Vdash \phi, \text{ and} \\ \text{for every } u \text{ such that } Rsu \;\&\; Rut \text{ it holds } \mathbb{M}, u \Vdash \psi. \end{cases}$$

Is U expressible in the language of basic modal logic? And in the language of basic temporal logic?
Hint: consider the models in [BdRV, Exercise 2.2.4].

Exercise 2. Consider the modality $\circ$ with the following semantics

$$\mathbb{M}, s \Vdash \circ\phi \iff \exists t \in W \; (sRt \;\&\; \neg(tRt) \;\&\; \mathbb{M}, t \Vdash \phi).$$

Is $\circ$ expressible in the language of basic modal logic?

Exercise 3. Let $\mathbb{M} = (W, R, V)$ be a Kripke model, and let X be a subset of W. We define $\mathbb{M}_X$ as the restricted model (X, R_X, V_X), where $R_X := R \cap (X \times X)$ and $V_X(p) := V(p) \cap X$. We call $X \subseteq W$ *hereditary* if $s \in X$ and Rst imply $t \in X$; in this case we say that $\mathbb{M}_X$ is a *generated submodel* of $\mathbb{M}$.

(1) Show that $\Delta_X := \{(x, x) \mid x \in X\}$ is a bisimulation between $\mathbb{M}_X$ and $\mathbb{M}$ iff X is hereditary.

(2) Show that if f is a bounded morphism from $\mathbb{M}$ to $\mathbb{M}'$, then the set $f[W] := \{f(s) \mid s \in W\}$ is a hereditary subset of W'.

Exercise 4. A *bounded morphism*[1] between two frames $\mathbb{F} = (W, R)$ and $\mathbb{F}' = (W', R')$ is a map $f : W \to W'$ such that, for all $s, t \in W$ and $t' \in W'$:
(forth) Rst implies $R'f(s)f(t)$;
(back) $R'f(s)t'$ implies the existence of a $t \in W$ with Rst and $f(t) = t'$.
Now let f be such a bounded morphism.

(1) Show that for any valuation V' on $\mathbb{F}'$ one can find a valuation V on $\mathbb{F}$ such that f (or rather, its graph $\{(s, f(s)) \mid s \in W\}$) is a bisimulation between the models $(\mathbb{F}, V)$ and $(\mathbb{F}', V')$.

(2) Show that if f is surjective, then $\mathbb{F} \Vdash \phi$ implies $\mathbb{F}' \Vdash \phi$, for any modal formula ϕ.

[1] Some authors call bounded morphisms *p-morphisms*.

(3) Prove that irreflexivity is not modally definable. That is, show that there is no modal formula ϕ such that ϕ is valid on exactly the frames with an irreflexive accessibility relation.

Exercise 5. Which of the following frame properties are preserved (reflected) by the operations of forming generated subframes, bounded morphic images, disjoint unions?

(1) reflexivity;

(2) transitivity;

(3) irreflexivity;

(4) converse seriality ($\forall x \exists y\, Ryx$);

(5) having cardinality at least n, for some natural number n;

(6) having cardinality at most n, for some natural number n.

Exercise 6. Show that the following frame properties cannot be defined in the basic modal language:

(1) converse seriality;

(2) having cardinality at least n, for some natural number n;

(3) having cardinality at most n, for some natural number n;

(4) acyclicity: 'there is no finite path (of non-zero length) from any point to itself'.

Exercise 7. Find frames $\mathbb{F}_1 = (W_1, R_1)$ and $\mathbb{F}_2 = (W_2, R_2)$ and a formula ϕ such that

$$\mathbb{F}_1 \Vdash \phi \quad \text{and} \quad \mathbb{F}_2 \nVdash \phi.$$

Furthermore, find valuations V_1 and V_2 on $\mathbb{F}_1$ and $\mathbb{F}_2$, respectively, such that

$$(\mathbb{F}_1, V_1), w_1 \leftrightarroweq (\mathbb{F}_2, V_2), w_2,$$

for all $w_1 \in W_1$ and all $w_2 \in W_2$.