Exercise 1. Which of the following frame properties are preserved (reflected) by the operations of forming generated subframes, p-morphic images, disjoint unions?

1. reflexivity;
2. transitivity;
3. irreflexivity;
4. converse seriality ($\forall x \exists y R_{yx}$);
5. having cardinality at least $n$, for some natural number $n$;
6. having cardinality at most $n$, for some natural number $n$.

Exercise 2. Show that the following frame properties cannot be defined in the basic modal language:

1. converse seriality;
2. having cardinality at least $n$, for some natural number $n$;
3. having cardinality at most $n$, for some natural number $n$;
4. acyclicity: ‘there is no finite path (of non-zero length) from any point to itself’.

Exercise 3. Let $M = (W, R, V)$ be a model and $\Sigma$ a set of formulas closed under subformulas. Show that the filtrations $M^\Sigma = (W_\Sigma, R^\Sigma, V^\Sigma)$ and $M^f = (W_\Sigma, R^f, V^f)$ are indeed the smallest and the largest filtrations, respectively.\(^1\)

Exercise 4. Compute both the largest and the smallest filtration of the model below through each of the following sets

$\Sigma_1 := \{p, q\}$ \quad $\Sigma_2 := \{p, q, \Diamond p\}$ \quad $\Sigma_3 := \{q\}$.

\(^1\)At the lecture $W_\Sigma$ was denoted by $W^f$. 
Exercise 5. Which of the following properties of frames are preserved by taking suitable filtrations of Kripke models?

1. reflexivity
2. symmetry
3. seriality
4. directedness: \( \forall x_0 x_1 x_2 ((R_{x_0}x_1 \land R_{x_0}x_2) \rightarrow \exists y (R_{x_1}y \land R_{x_2}y)) \)
5. density: \( \forall x_1 x_2 (x_1 R x_2 \rightarrow \exists y (R_{x_1}y \land R_{yx_2})) \)
6. every world sees a reflexive world: \( \forall x \exists y (R_{xy} \land R_{yx}) \)

Exercise 6. \((*)\) Let \( M = (W, R, V) \) be a model with \( R \) a transitive relation and \( \Sigma \) a subformula closed set. Let \( (R^n)^* \) be the transitive closure of smallest filtration relation \( R^n \). Show that \( (W, (R^n)^*, V_f) \) is a transitive filtration of \( M \).