Introduction to Modal Logic. Exercise class 4

27 September 2018

Exercise 1. Compute the standard translations of the following formulas:

(1) $\Diamond p \to \Diamond\Diamond p$;

(2) $\Diamond\Box p \vee \Box p$;

(3) $\Box(\Box p \to q) \vee \Box(\Box q \to p)$;

(4) $(p \wedge \Box(\Diamond p \to \Box q)) \to \Diamond\Box\Box q$;

(5) $\Box(\Box p \to p) \to \Box p$;

(6) $q \to \Box_P \Diamond_F q$.

Exercise 2. Prove that the standard translation is correct, that is:

(1) For all models $\mathbb{M}$, all states $w \in \mathbb{M}$, and all modal formulas φ,

$$\mathbb{M}, w \Vdash \varphi \quad \Leftrightarrow \quad \mathbb{M} \models ST_x(\varphi)[w]\,.$$

(2) For all models $\mathbb{M}$ and all modal formulas φ,

$$\mathbb{M} \Vdash \varphi \quad \Leftrightarrow \quad \mathbb{M} \models \forall x ST_x(\varphi)\,.$$

Exercise 3. Show that Rxx is not equivalent to the standard translation of a formula in the language of basic modal logic.

Exercise 4.

(1) Show that the mapping f defined in the proof of Proposition 2.15 (of Blackburn et al.) is indeed a surjective bounded morphism.

(2) Show that every model is a bounded morphic image of a disjoint union of rooted models.

(3) Deduce from (1) and (2) that every model is a bounded morphic image of a forest (a *forest* is a disjoint union of trees).

Exercise 5. For each $n \in \omega$ find a finite model $\mathbb{M}_n = (W, R, V)$ and a world $w \in W$ such that the unravelling of $\mathbb{M}$ at w is based on a tree in which every node has exactly n immediate successors.

Exercise 6. Let (W, R) and (W', R') be Kripke frames. A map $f : W \to W'$ is called a *homomorphism* if for each $w, v \in W$ we have Rwv implies $R'f(w)f(v)$. Is validity of modal formulas preserved under surjective homomorphisms? In other words, if a modal formula φ is valid in (W, R) and if $f : W \to W'$ is a surjective homomorphism, is φ valid in (W', R')? If yes, provide a proof, if not give a counter-example.

Exercise 7. $(^*)$ Show that if a modal formula φ is satisfiable, then it is satisfiable in a finite tree.