Introduction to Modal Logic
Exercise class 5

October 4, 2018

Examples.

- In the formula $\neg p$, the proposition p appears *negatively* because it appears under the scope of a negation;
- In the formula $\neg\neg p$, the proposition p appears *positively* because it appears under the scope of an even number of negations;
- In the formula $\neg(p \vee \neg p)$, the left occurrence of proposition p appears *negatively*, while the right occurrence appears positively.

Exercise 1. Consider as primitive connectives $\vee$, $\neg$, $\bot$ and $\Diamond$. Let p be a propositional letter that occurs in φ. Define by induction on φ: The occurrence of p is positive (negative).

Definition 1. A formula φ is called positive (negative) in p if all occurrences of p are positive (negative).

A formula φ is called upward monotone (respectively downward monotone) in p if for every frame $\mathbb{F}$, every point w and every pair of assignments V and V' such that

$$\left.\begin{array}{ll} V(p) \subseteq V'(p) & \\ V(q) = V'(q) & \text{for } q \neq p \end{array}\right\}$$

it holds

$$(\mathbb{F}, V), w \Vdash \varphi \Rightarrow (\mathbb{F}, V'), w \Vdash \varphi$$
$$\big(\text{resp. } (\mathbb{F}, V'), w \Vdash \varphi \Rightarrow (\mathbb{F}, V), w \Vdash \varphi\big)$$

Exercise 2 (Blackburn et al. 3.5.3)**.**

- Show that if φ is positive in p then it is upward monotone in p, and if it is negative in p then it is downward monotone in p.
- What about the converse? If φ upward (downward) monotone in p does it follow that φ is positive (negative) in p?

Exercise 3. Compute the first order correspondents of the following closed formulas.

(1) $\Box\bot$;

(2) $\Box\Diamond\top \to \Diamond\Box\top$;

(3) $\Diamond\Box\bot \to \Box\Diamond\bot$;

(4) $\Box\Diamond\top \to \Box\bot$;

(5) $\Diamond_P(\Diamond_F\Box_P\bot \to \Box_F\bot)$.

Exercise 4. Recall that a *partial order* on a set W is a binary relation $R \subseteq W^2$ which is reflexive, transitive and anti-symmetric.

(1) Are partial orders modally definable?

(2) Are partial orders modally definable within the class of finite Kripke frames?

Exercise 5 (Blackburn et al. 3.3.3)**.** Consider the language with three diamonds, $\Diamond_1$, $\Diamond_2$ and $\Diamond_3$. For each of the frame conditions on the corresponding accessibility relations below, find out whether it is modally definable or not:

(1) R_1 is the union of R_2 and R_3,

(2) R_1 is the intersection of R_2 and R_3,

(3) R_1 is the complement of R_2.

Exercise 6 (Blackburn et al. 3.2.3)**.** Consider the basic temporal language. Recall that a frame $\mathbb{F} = (W, R_F, R_P)$ for this language is called *bidirectional* if R_P is the converse of R_F.

(1) Show that among the finite bidirectional frames, the formula $\Box_F(\Box_F q \to q) \to \Box_F q$ together with its converse $\Box_P(\Box_P q \to q) \to \Box_P q$ defines the transitive and irreflexive frames.

(2) Show that among the bidirectional frames that are transitive, irreflexive, and satisfy $\forall xy(R_F xy \vee x = y \vee R_P xy)$, these same formulas define the finite frames.

(3) Is there a finite set of formulas in the basic modal language that has these same definability properties?