INTRODUCTION TO MODAL LOGIC
EXERCISE CLASS 7:
NORMAL MODAL LOGICS AND HILBERT STYLE DERIVIATIONS

18 OCTOBER, 2018

(1) Define *Sahlqvist antecedent* as the one built from $\bot$, $\top$, boxed atoms and negative formulas by applying $\Diamond$ and $\wedge$. *Simple Sahlqvist formulas*[1] and *Sahlqvist formulas* are defined as in the notes using this new definition of a Sahlqvist antecedent.

(a) (3.7.4 Blackburn et al.) Suppose $\varphi \to \psi$ is a simple Sahlqvist formula in this new sense. Show that the formula $\Box(\varphi \to \psi)$ is locally equivalent (on frames) to a simple Sahlqvist formula, namely to $\Diamond(\varphi \wedge \neg\psi) \to q$, where q does not occur in φ and ψ.

(b) Show that if $\varphi_1 \to \psi_1$ and $\varphi_2 \to \psi_2$ are modal formulas, then $(\varphi_1 \to \psi_1) \vee (\varphi_2 \to \psi_2)$ is locally equivalent (on frames) to the modal formula $(\varphi_1 \wedge \varphi_2) \to (\psi_1 \vee \psi_2)$.

(c) Deduce that every Sahlqvist formula in this new sense is locally equivalent to a simple Sahlqvist formula in this new sense.

(2) Let $\mathcal{C}$ be a class of frames and let $\mathcal{M}$ be a class of models.

(a) Show that

$$\mathsf{Log}(\mathcal{C}) := \{\varphi \colon \forall \mathbb{F} \in \mathcal{C}\ (\mathbb{F} \Vdash \varphi)\}$$

is a normal modal logic.

(b) Is the set of formulas

$$\mathsf{Th}(\mathcal{M}) := \{\varphi \colon \forall \mathbb{M} \in \mathcal{M}\ (\mathbb{M} \Vdash \varphi)\}$$

a normal modal logic?

(3) Let Λ be a normal modal logic and let $\varphi, \varphi', \psi, \psi'$ and χ be formulas in the language of basic modal logic. Show that

(a) If $\varphi \to \psi$ is (a substitution instance of) a propositional tautology, then $\vdash_\Lambda \varphi$ implies $\vdash_\Lambda \psi$.
(b) If $\vdash_\Lambda \varphi$ and $\vdash_\Lambda \psi$ then $\vdash_\Lambda \varphi \wedge \psi$.
(c) If $\vdash_\Lambda \varphi \to \psi$ and $\vdash_\Lambda \psi \to \chi$ then $\vdash_\Lambda \varphi \to \chi$.
(d) If $\vdash_\Lambda \varphi \to \psi$ and $\vdash_\Lambda \varphi' \to \psi'$ then $\vdash_\Lambda (\varphi \wedge \varphi') \to (\psi \vee \psi')$

(4) Let Λ be a normal modal logic and let φ and ψ be formulas in the language of basic modal logic. Prove that

(a) $\vdash_\Lambda \varphi \to \psi$ implies $\vdash_\Lambda \Box\varphi \to \Box\psi$,

[1] This is now different from the definition of simple Sahlqvist formulas given in Blacknurn et al. 3.47.

(b) $\vdash_\Lambda \varphi \to \psi$ implies $\vdash_\Lambda \Diamond\varphi \to \Diamond\psi$,
(c) $\vdash_\Lambda \Box(\varphi \wedge \psi) \leftrightarrow (\Box\varphi \wedge \Box\psi)$,
(d) $\vdash_\Lambda \Diamond(\varphi \vee \psi) \leftrightarrow (\Diamond\varphi \vee \Diamond\psi)$.

(5) (Equivalent replacement). Let $\varphi[\psi]$ be a formula that contains ψ as a subformula. Let $\varphi[\chi]$ denote the formula where ψ in $\varphi[\psi]$ is replaced with the formula χ. Show that

$$\vdash_\Lambda \psi \leftrightarrow \chi \quad \text{implies} \quad \vdash_\Lambda \varphi[\psi] \leftrightarrow \varphi[\chi],$$

for any normal modal logic Λ.

(6) Show that $\not\vdash_{\mathbf{S4}} p \to \Box\Diamond p$ and that $\not\vdash_{\mathbf{K}} \Box p \vee \Box\neg p$. Recall that $\mathbf{S4} = \mathbf{K} + (\Box p \to p) + (\Box p \to \Box\Box p)$.

(7) A normal modal logic Λ is *Halldén complete* if for every pair of formulas φ and ψ with no common variables we have that

$$\vdash_\Lambda \varphi \vee \psi \quad \text{implies} \quad \vdash_\Lambda \varphi \text{ or } \vdash_\Lambda \psi.$$

Is the normal modal logic **K** Halldén complete? Give proof or counter-example.