EXERCISE CLASS 1-11-2018: NORMAL MODAL LOGICS AND CANONICAL MODELS

(1) Let $\mathcal{C}$ be a class of frames and let $\mathcal{M}$ be a class of models.

(a) Show that

$$\mathsf{Log}(\mathcal{C}) := \{\varphi \colon \forall \mathbb{F} \in \mathcal{C}\ (\mathbb{F} \Vdash \varphi)\}$$

is a normal modal logic. Conclude that **K** is sound w.r.t. the class of all frames.

(b) Is the set of formulas

$$\mathsf{Th}(\mathcal{M}) := \{\varphi \colon \forall \mathbb{M} \in \mathcal{M}\ (\mathbb{M} \Vdash \varphi)\}$$

a normal modal logic?

(2) Show that $\not\vdash_{\mathbf{S4}} p \to \Box\Diamond p$ and that $\not\vdash_{\mathbf{K}} \Box p \vee \Box\neg p$. Recall that $\mathbf{S4} = \mathbf{K} + (\Box p \to p) + (\Box p \to \Box\Box p)$.

(3) A normal modal logic Λ is *Halldén complete* if for every pair of formulas φ and ψ with no common variables we have that

$$\vdash_\Lambda \varphi \vee \psi \quad \text{implies} \quad \vdash_\Lambda \varphi \text{ or } \vdash_\Lambda \psi.$$

Is the normal modal logic **K** Halldén complete? Give proof or counter-example.

(4) (a) Let $\Gamma := \{p, q, p \wedge q, \Box p, \Box q, \Box(p \wedge q)\}$, $\Delta := \{p, \neg q, \Box p\}$, and $\Delta' := \{\Box p, \Box q, \Box(p \wedge q)\}$ be sets of formulas.

(b) Are these sets maximal consistent (in some language)?

(c) Let the relation R' on $\{\Gamma, \Delta, \Delta'\}$ and the valuation V' on $\{\Gamma, \Delta, \Delta'\}$ be as defined on the canonical model. Draw the resulting Kripke model.

(5) Let L be a normal modal logic and let Γ be an L-MCS. Show that

(i) If $\varphi \in \Gamma$ and $\varphi \to \psi \in \Gamma$ then $\psi \in \Gamma$;

(ii) $L \subseteq \Gamma$;

(iii) For every formula φ either $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$;

(iv) For every pair of formulas φ and ψ we have that $\varphi \wedge \psi \in \Gamma$ iff $\varphi \in \Gamma$ and $\psi \in \Gamma$;

(v) For every pair of formulas φ and ψ we have that $\varphi \vee \psi \in \Gamma$ iff $\varphi \in \Gamma$ or $\psi \in \Gamma$.

(6) Let L be a normal modal logic and define a relation R'' on the canonical model for L by

$$R''(\Gamma, \Delta) \quad \text{iff} \quad \forall \varphi(\varphi \in \Delta \implies \Diamond\varphi \in \Gamma),$$

where Γ and Δ are L-MCSs. Show that $R'' = R'$, where R' is the relation

$$R'(\Gamma, \Delta) \quad \text{iff} \quad \forall \varphi(\Box\varphi \in \Gamma \implies \varphi \in \Delta),$$

where Γ and Δ are L-MCSs. Thus, for any normal modal logic L, we may define the canonical relation R^L as either R' or R''.

(7) Show that in the canonical model for **K** (or any other consistent normal modal logic L) there exist (L-)MCSs Γ and Δ that are incomparable (i.e., we have neither $R^L(\Gamma, \Delta)$ nor $R^L(\Delta, \Gamma)$).

(8) Let Γ be a set of formulas (say, in the language of basic modal logic). Prove that if Γ is satisfiable then it is consistent. Can you generalise this to cover L-consistency for an arbitrary normal modal logic L?

(9) Let L be a consistent normal modal logic. Given a world w in an L-model $\mathbb{M}$, show that the set of formulas $\{\varphi \colon \mathbb{M}, w \Vdash \varphi\}$ is an L-MCS.

(10) (*) Show that for every MCS Γ in the canonical model for **K**, there exists a MCS Δ such that $\Delta R^{\mathbf{K}} \Gamma$.