EXERCISE CLASS 8-11-2018:  
CANONICAL MODELS AND THE FINITE MODEL PROPERTY

1. More on completeness and the canonical model

(a) Show that the normal modal logic $KD := K + (\Diamond \top)$ is sound and complete with respect to the class of serial Kripke frames, i.e., Kripke frames satisfying the first-order condition $\forall x \exists y(xRy)$.

(b) Show that the normal modal logic $KT := K + (\Box p \to p)$ is sound and complete with respect to the class of reflexive Kripke frames.

(c) Show that the normal modal logic $S4 := K + (\Box p \to p) + (\Box p \to \Box \Box p)$ is sound complete with respect to the class of reflexive and transitive Kripke frames.

(d) Show that the normal modal logic $KB := K + (p \to \Box \Box p)$ is sound and complete with respect to the class of symmetric Kripke frames.

(e) (∗) Show that the normal modal logic $Den := K + (\Diamond p \to \Diamond \Diamond p)$ is sound and complete with respect to the class of dense Kripke frames, i.e., Kripke frames satisfying the first-order condition $\forall x \forall z (xRz \implies \exists y (xRy \land yRz))$. Hint: This is not so easy.$^1$

A normal temporal logic is a normal modal logic (in the basic temporal language) containing the axioms $q \to 2F3Pq$ and $q \to 2P3Fq$. The smallest normal temporal logic is called the basic temporal logic and is denoted by $Kt$.

Show that the basic temporal logic $Kt$ is sound and complete with respect to bi-directional frames.

(3) Show that if $L = Log(C)$ for some class of (finite) frames, then $L = Log(C')$ for some class of (finite) rooted frames.

2. Finite model property

(1) Show that the following normal modal logics have the finite model property

(a) The normal modal logic $K$;

(b) The normal modal logic $KD := K + \Diamond \top$,

(c) The normal modal logic $KT := K + p \to \Diamond p$;

(d) The normal modal logic $K4 := K + \Box \Diamond p \to \Diamond p$;

(e) The normal modal logic $S4 := KT + \Diamond \Diamond p \to \Diamond p$;

(f) (∗) The normal modal logic $S5 := S4 + p \to \Box \Diamond p$;

(g) (∗) The normal modal logic $S4.2 := S4 + \Diamond \Box p \to \Box \Diamond p$.

(2) Which of the normal modal logics above are decidable?

(3) (∗) Let, for each $n > 0$, $MT_n = \Diamond ((\Box p_1 \to p_1) \land \ldots \land (\Box p_n \to p_n))$ and define the normal modal logic $KMT := K + \{MT_n : n \in \mathbb{N}\}$.

(a) Show that $KMT$ is sound and complete with respect to the class of Kripke frames satisfying the first-order condition $\forall x \exists y (xRy \& yRy)$. Hint: To establish completeness show that for $\Gamma$ any $KMT$-MCS the set $\{\psi : \Box \psi \in \Gamma\} \cup \{\Box \psi \to \psi : \psi \in Fm\}$ is $KMT$-consistent.

$^1$Given $\text{Den}$-MCSs $\Gamma$ and $\Delta$ such that $\Gamma \models^\text{Den} \Delta$. You need to show that the set of formulas $\Sigma_0^\text{Den} \cup \Sigma_1^\text{Den}$ is $\text{Den}$-consistent, where $\Sigma_0^\text{Den} := \{\varphi : \Box \varphi \in \Gamma\}$ and $\Sigma_1^\text{Den} := \{\Diamond \psi : \psi \in \Delta\}$. To that end you might find it helpful to show that $\vdash_K \Diamond (p \land q) \to \Diamond p \land q$ and that $((p \land q) \to r) \to (p \to (q \to r))$ is a propositional tautology.
(b) Show that $\mathbf{KMT}$ has the finite model property.

(c) Show that $\mathbf{KMT}$ is decidable.

(d) Show that $\mathbf{KMT}$ is not finitely axiomatizable. \textit{Hint: For each }$n > 0$\textit{ let }$F_n = (W, R)$\textit{ be a frame of size }$n$\textit{ with }$wRv$\textit{ iff }$v \neq w.$\textit{ Show that }$F_m \models \mathbf{MT}_n$\textit{ iff }$m > n + 1.$

(e) Show that $(\mathbb{N}, <)$ is a frame for the logic $\mathbf{KMT}$. Why is this not a contradiction?