EXERCISE CLASS 8-11-2018: CANONICAL MODELS AND THE FINITE MODEL PROPERTY

1. More on completeness and the canonical model

(1) (a) Show that the normal modal logic $\mathbf{KD} := \mathbf{K} + (\Diamond\top)$ is sound and complete with respect to the class of serial Kripke frames, i.e., Kripke frames satisfying the first-order condition $\forall x \exists y(xRy)$.
(b) Show that the normal modal logic $\mathbf{KT} := \mathbf{K} + (\Box p \to p)$ is sound and complete with respect to the class of reflexive Kripke frames.
(c) Show that the normal modal logic $\mathbf{S4} := \mathbf{K} + (\Box p \to p) + (\Box p \to \Box\Box p)$ is sound a complete with respect to the class of reflexive and transitive Kripke frames.
(d) Show that the normal modal logic $\mathbf{KB} := \mathbf{K} + (p \to \Box\Diamond p)$ is sound and complete with respect to the class of symmetric Kripke frames.
(e) ($*$) Show that the normal modal logic $\mathbf{Den} := \mathbf{K} + (\Diamond p \to \Diamond\Diamond p)$ is sound and complete with respect to the class of dense Kripke frames, i.e., Kripke frames satisfying the first-order condition $\forall x \forall z(xRz \implies \exists y(xRy \wedge yRz))$. *Hint: This is not so easy*[1].

(2) A *normal temporal logic* is a normal modal logic (in the basic temporal language) containing the axioms $q \to \Box_F \Diamond_P q$ and $q \to \Box_P \Diamond_F q$. The smallest normal temporal logic is called *the basic temporal logic* and is denoted by $\mathbf{K}_t$.

Show that the basic temporal logic $\mathbf{K}_t$ is sound and complete with respect to bi-directional frames.

(3) Show that if $L = Log(\mathcal{C})$ for some class of (finite) frames, then $L = Log(\mathcal{C}')$ for some class of (finite) rooted frames.

2. Finite model property

(1) Show that the following normal modal logics have the finite model property

(a) The normal modal logic $\mathbf{K}$;
(b) The normal modal logic $\mathbf{KD} := \mathbf{K} + \Diamond\top$,
(c) The normal modal logic $\mathbf{KT} := \mathbf{K} + p \to \Diamond p$,
(d) The normal modal logic $\mathbf{K4} := \mathbf{K} + \Diamond\Diamond p \to \Diamond p$,
(e) The normal modal logic $\mathbf{S4} := \mathbf{KT} + \Diamond\Diamond p \to \Diamond p$,
(f) ($*$) The normal modal logic $\mathbf{S5} := \mathbf{S4} + p \to \Box\Diamond p$,
(g) ($*$) The normal modal logic $\mathbf{S4.2} := \mathbf{S4} + \Diamond\Box p \to \Box\Diamond p$.

(2) Which of the normal modal logics above are decidable?

(3) ($*$) Let, for each $n > 0$ $\mathbf{MT}_n = \Diamond((\Box p_1 \to p_1) \wedge \ldots \wedge (\Box p_n \to p_n))$ and define the normal modal logic,

$$\mathbf{KMT} = \mathbf{K} + \{\mathbf{MT}_n : n \in \mathbb{N}\}.$$

(a) Show that $\mathbf{KMT}$ is sound and complete with respect to the class of Kripke frames satisfying the first-order condition $\forall x \exists y(xRy \,\&\, yRy)$. *Hint: To establish completeness show that for* Γ *any* $\mathbf{KMT}$*-MCS the set* $\{\varphi : \Box\varphi \in \Gamma\} \cup \{\Box\psi \to \psi : \psi \in Fm\}$ *is* $\mathbf{KMT}$*-consistent.*

[1] Given $\mathbf{Den}$-MCSs Γ and Δ such that $\Gamma R^{\mathbf{Den}} \Delta$ You need to show that the set of formulas $\Sigma_0^- \cup \Sigma_1^-$ is $\mathbf{Den}$-consistent, where $\Sigma_0^- := \{\varphi\colon \Box\varphi \in \Gamma\}$ and $\Sigma_1^- := \{\Diamond\psi\colon \psi \in \Delta\}$. To that end you might find it helpful to show that $\vdash_{\mathbf{K}} \Diamond(p \wedge q) \to \Diamond p \wedge \Diamond q$ and that $((p \wedge q) \to r) \to (p \to (q \to r))$ is a propositional tautology.

(b) Show that **KMT** has the finite model property.

(c) Show that **KMT** is decidable.

(d) Show that **KMT** is not finitely axiomatizable. *Hint: For each* $n > 0$ *let* $\mathbb{F}_n = (W, R)$ *be a frame of size* n *with* wRv *iff* $v \neq w$*. Show that* $\mathbb{F}_m \Vdash \mathbf{MT}_n$ *iff* $m > n + 1$.

(e) Show that $(\mathbb{N}, <)$ is a frame for the logic **KMT**. Why is this not a contradiction?