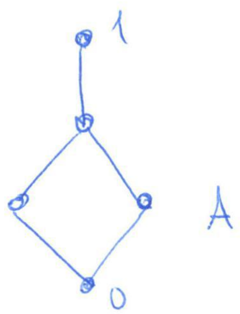
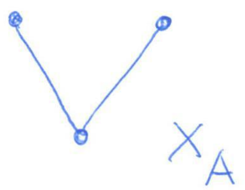
THE LATTICE OF SUBARIETIES OF A FINITELY GENERATED VARIETY

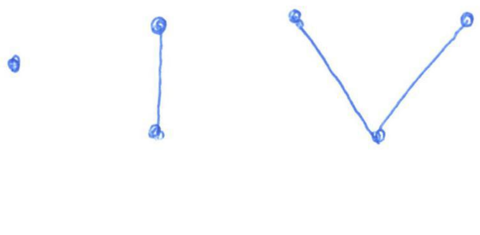
Let A be a finite HA shown in the figure below.



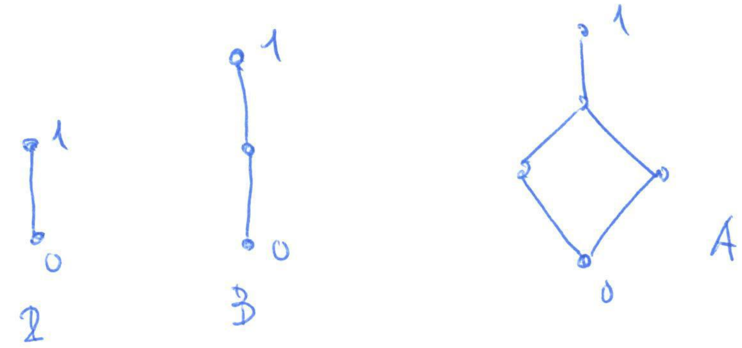
We will now describe the lattice of subvarieties of $\mathsf{V} = \mathbf{Var}(A)$. By Birkhoff's theorem, every variety is generated by subdirectly irreducible algebras. Our first task is to characterize all s.i. algebras in V. Since the variety of HAs is congruence distributive (why?), by Jónsson's lemma, for every s.i. algebra $B \in \mathsf{V}$ we have $B \in \mathbf{PHSP_U}(A)$. As A is finite, $\mathbf{P_U}(A) = \{A\}$. So $B \in \mathbf{PHS}(A)$. As B is s.i., we deduce that $B \in \mathbf{HS}(A)$. It is easy to see that the dual Esakia space X_A of A is the one drawn below (why?).



Now $B \in \mathbf{HS}(A)$ implies that there is a subalgebra C of A such that B is a homomorphic image of C. Dually, there is a p-morphism (bounded morphism, Esakia morphism) from X_A onto X_C and X_B is an up-set of X_C. As B is s.i., X_B is rooted. It is easy to see (why?) that up to isomorphism we can have only three different X_B's (see below).



So we can have only three nontrivial s.i. algebras in V.



It is now easy to see (fill the details) that subsets of this set generate only 3 nontrivial varieties. So the lattice of subvarieties of V is a 4-element chain.