THE LATTICE OF SUBARBITERIES OF A FINITELY GENERATED VARIETY

Let $A$ be a finite HA shown in the figure below.

We will now describe the lattice of subvarieties of $V = \text{Var}(A)$. By Birkhoff’s theorem, every variety is generated by subdirectly irreducible algebras. Our first task is to characterize all s.i. algebras in $V$. Since the variety of HAs is congruence distributive (why?), by Jónsson’s lemma, for every s.i. algebra $B \in V$ we have $B \in \text{PHSP}_U(A)$. As $A$ is finite, $\text{P}_U(A) = \{A\}$. So $B \in \text{PHS}(A)$. As $B$ is s.i., we deduce that $B \in \text{HS}(A)$. It is easy to see that the dual Esakia space $X_A$ of $A$ is the one drawn below (why?).

Now $B \in \text{HS}(A)$ implies that there is a subalgebra $C$ of $A$ such that $B$ is a homomorphic image of $C$. Dually, there is a p-morphism (bounded morphism, Esakia morphism) from $X_A$ onto $X_C$ and $X_B$ is an up-set of $X_C$. As $B$ is s.i., $X_B$ is rooted. It is easy to see (why?) that up to isomorphism we can have only three different $X_B$’s (see below).

So we can have only three nontrivial s.i. algebras in $V$. 

1
It is now easy to see (fill the details) that subsets of this set generate only 3 nontrivial varieties. So the lattice of subvarieties of $V$ is a 4-element chain.