MATHEMATICAL STRUCTURES IN LOGIC 2018
FINAL EXAM

- Deadline: April 9 at 19:00.
- The exam can be submitted electronically (in a single pdf-file!) to Saul Fernandez (`saul.fdez.glez@gmail.com`) and Nick Bezhanishvili (`N.Bezhanishvili@uva.nl`).
- Grading is from 0 to 40 points.
- Good luck!

(1) (10pt) Let $\mathbf{LC} = \mathbf{IPC} + (\varphi \to \psi) \vee (\psi \to \varphi)$.

(a) Show that subdirectly irreducible **LC**-algebras (that is, HAs validating **LC**) are chains with a second greatest element.

(b) Show that **LC** has the finite model property.

(c) Characterize the lattice of subvarieties of the variety LC of all **LC**-algebras. (Hint: prove that every finite chain is a subalgebra of any infinite subdirectly irreducible **LC**-algebra.)

(2) (10pt) Let $(X, \leq)$ be a non-empty Priestley space.

(a) Show that the set of maximal points of $(X, \leq)$ is non-empty. (Hint: use Zorn's Lemma[1] and compactness of X.)

(b) Let $(X, \leq)$ be a Priestley space. Show that for each closed down-set U and $x \notin U$, there is a clopen down-set V such that $U \subseteq V$ and $x \notin V$.

Use this fact to show that for clopen up-sets U and V if the greatest clopen up-set C such that $C \cap U \subseteq V$ exists, then $C = (\downarrow (U \setminus V))^c$. (You are not allowed to use Esakia duality for Heyting algebras. You can use the fact that for each closed set $F \subseteq X$ the set $\downarrow F$ is closed, Tutorial 5(2).)

(c) Give an example of a Stone space X and a partial order $\leq$ on X such that $\uparrow x$ is a closed set for each $x \in X$, but $(X, \leq)$ is not a Priestley space. (Hint: it might be useful to work with the two-point compactification of $\mathbb{N}$. That is, consider the space $\mathbb{N} \cup \{\infty_1, \infty_2\}$, whose topology is generated by the set

$$\mathcal{S} = \{\text{finite subsets of } \mathbb{N}, \text{cofinite subsets of } \mathbb{N} \text{ with } \{\infty_1, \infty_2\}, E \cup \{\infty_1\}, O \cup \{\infty_2\}\},$$

where E is the set of even numbers and O is the set of odd numbers. In other words we are taking the least topology containing $\mathcal{S}$).

[1] Recall that Zorn's Lemma states that if every chain in a nonempty poset has an upper bound, then the poset has a maximal element.

(3) (10pt) Let $\mathcal{B} = (B, \Box)$ be an $\mathsf{S4}$-algebra. A filter $F \subseteq B$ is called a *modal filter* if for each $a \in B$ we have

$$a \in F \;\Rightarrow\; \Box a \in F.$$

(a) Show that there is one-to-one correspondence between the congruences of $\mathcal{B}$ and modal filters of $\mathcal{B}$. (You can assume the correspondence between Boolean congruences and filters.)

(b) Let (X, R) be the $\mathsf{S4}$-space dual to $\mathcal{B}$. Characterize modal filters of $\mathcal{B}$ in dual terms. (You can assume the correspondence between filters of B and closed subsets of X.)

(c) Give a dual characterisation of subdirectly irreducible $\mathsf{S4}$-algebras. (Consult Homework 5(2) for a dual characterization of subdirectly irreducible Heyting algebras.)

(4) (10pt) Recall that an element $a \neq 0$ of a Heyting algebra A is *completely join prime*, if $a \leq \bigvee S$ for $S \subseteq A$, implies $a \leq s$ for some $s \in S$. (Alternatively A is completely join prime generated if for each $b \neq 0$, there is a completely join prime element $a \in A$ such that $a \leq b$.)

(a) Let $(X, \leq)$ be a poset and let $\mathcal{H}$ be the Heyting algebra of all upsets of X. Show that

(i) $\mathcal{H}$ is complete,

(ii) $\mathcal{H}$ is completely join prime generated i.e., every element a of $\mathcal{H}$ is generated by completely join prime elements.

(b) Show that if a HA A is complete and completely join prime generated, then A is isomorphic to the Heyting algebra of all upsets for some poset $(X, \leq)$. (Hint: check the proof of the fact that every complete and atomic BA is isomorphic to the powerset BA.)

(c) Let A be a HA and X its dual Esakia space. Show that $a \in A$ is completely join prime iff $\varphi(a) = {\uparrow} x$ such that $\{x\}$ is a clopen set. (Note that the join if infinitely many clopen up-sets is not necessarily their union, see Homework 4(2)).

(d) (*) (**Bonus +3pt**) Can you characterize when A is completely join prime generated in (purely order-topological) terms of its dual X? (It might be useful to work with a set $X_0 = \{x \in X : \{x\} \text{ and } {\uparrow} x \text{ are clopen}\}$.)