MATHEMATICAL STRUCTURES IN LOGIC 2018
HOMEWORK 1

- Deadline: February 13 — at the **beginning** of the tutorial class.
- In exceptional cases homework can be submitted electronically (in a single pdf-file!) to Saul Fernandez (`saul.fdez.glez@gmail.com`).
- Grading is from 0 to 100 points.
- Discussion of problems is allowed, but each student should submit a homework they themselves have written.
- Good luck!

(1) (30pt) Let $(P, \leq)$ be a poset. Show that if $\sup(A)$ exists for each $A \subseteq P$, then $\inf(B)$ also exists for each $B \subseteq P$, and therefore $(P, \leq)$ is a complete lattice.

(2) (20pt) Give an example of a poset $(P, \leq)$ in which there are three elements x, y, z such that

(a) $\{x, y, z\}$ is an antichain (a set $A \subseteq P$ is an *antichain* if $a \not\leq b$ for distinct $a, b \in A$),

(b) $x \vee y$, $y \vee z$ and $z \vee x$ fail to exist,

(c) $\bigvee\{x, y, z\}$ exists.

It is sufficient to just provide the Hasse diagram for this lattice. (Hint: P will have more than three elements.)

(3) (20pt) Let L be a lattice. We say that a non-zero element $a \in L$ is *join irreducible* if $a = b \vee c$ implies $a = b$ or $a = c$. Let $(P, \leq)$ be a poset. $A \subseteq P$ is an *up-set* if $x \in A$ and $x \leq y$ imply $y \in A$. Let $\mathrm{Up}(P)$ be the set of all up-sets of P.

(a) Show that $(\mathrm{Up}(P), \subseteq)$ is a distributive lattice

(b) Characterize join irreducible elements of $(\mathrm{Up}(P), \subseteq)$ for a finite P.

(4) (30pt)

(a) Show that the lattice $(\mathrm{FinCofin}(\mathbb{N}), \subseteq)$ of finite and cofinite subsets of $\mathbb{N}$, forms a Boolean algebra, which is not complete.

(b) Show that the lattice $(\mathrm{Fin}(\mathbb{N}) \cup \{\mathbb{N}\}, \subseteq)$ of finite subsets of $\mathbb{N}$ (together with $\mathbb{N}$) forms a complete bounded distributive lattice.