MATHEMATICAL STRUCTURES IN LOGIC 2018
HOMEWORK 4

- Deadline: March 6 — at the **beginning** of the tutorial class.
- In exceptional cases homework can be submitted electronically (in a single pdf-file!) to Saul Fernandez (`saul.fdez.glez@gmail.com`).
- Grading is from 0 to 100 points.
- Discussion of problems is allowed, but each student should submit a homework they themselves have written.
- Good luck!

(1) (40pt) The aim of this exercise is to understand the connection between filters and congruences of a Boolean algebra.

Given a Boolean algebra A and a filter $F \subseteq A$, define a relation $\sim_F$ by

$$a \sim_F b \text{ iff } a \leftrightarrow b \in F.$$

Conversely, given a congruence $\sim$ on A define $F_\sim$ by

$$F_\sim = [1]_\sim.$$

(a) $\sim_F$ is a congruence. Show the $\vee$-clause of this statement. That is, show that if $a \sim_F b$ and $c \sim_F d$, then $a \vee c \sim_F b \vee d$.

(b) Show that $\sim$ is equal to $\sim_{F_\sim}$.

(c) Show that F is equal to $F_{\sim_F}$.

(d) Show that $F \subseteq F'$ implies that $\sim_F$ is a subset of $\sim_{F'}$.

(2) (20pt) Let A be a BA and X its dual Stone space. Let $S, T \subseteq A$ be such that $\bigvee S$ and $\bigwedge T$ exist. Show that

(a) $\varphi(\bigvee S) = \mathrm{Cl}(\bigcup\{\varphi(a) : a \in S\})$,

(b) $\varphi(\bigwedge T) = \mathrm{Int}(\bigcap\{\varphi(a) : a \in T\})$.

(3) (20pt) A topological space X is called *extremally disconnected* if the closure of every open set in X is open (hence clopen since the closure is always closed). Prove that a Boolean algebra A is complete if and only if the Stone space X dual to A is extremally disconnected.

(Hint: use exercise 2.)

(4) (20pt) Recall that in a topological space X a point $x \in X$ is *isolated* if $\{x\}$ is open.

  (a) Show that there is a one-to-one correspondence between the atoms of a BA A and the isolated points of its dual Stone space X. (Hint: show that $a \in A$ is an atom iff $\varphi(a)$ is a singleton.)

  (b) Let $\mathrm{Iso}_X := \{x \in X : x \text{ is an isolated point}\}$. Show that if A is atomic (see the tutorial sheet for the definition), then Iso_X is dense in X.

  Recall that $Y \subseteq X$ is dense if the closure of Y is X. (Hint: use exercise 2.)