1. Suppose \((L, \lor, \land)\) is a lattice. Recall that we defined a relation \(a \leq b\) iff \(a \land b = a\). Now define a relation \(\leq'\) on \(L\) via \(a \leq' b\) iff \(a \lor b = b\). Show that \(\leq = \leq'\).

2. Show that every lattice satisfies:
\[
(x \land y) \lor (x \land z) \leq x \land (y \lor z)
\]

3. Recall that in class we defined on a lattice \((L, \leq)\) \(a \land b := \text{inf}\{a, b\}\) and \(a \lor b := \text{sup}\{a, b\}\). Show that these operations satisfy
- \(x \lor (y \lor z) = (x \lor y) \lor z\)
- \(x \land (y \land z) = (x \land y) \land z\)

4. Funny examples:
   a. Give an example of a lattice \((L, \leq)\) such that no infinite subset \(X \subseteq L\) has a least upper bound.
   b. Consider the poset \((\mathbb{N}, \leq)\). Is this a lattice? Is it complete?
   c. Find an example of a lattice \((L, \leq)\) that contains a subset \(A \subseteq L\) such that \(\text{inf} A \) and \(\text{sup} A\) exist but \(\text{sup} A \neq \text{inf} A\) and \(\text{sup} A \leq \text{inf} A\).
   d. Find an example of a poset where \(\text{inf} \emptyset\) does not exist.
   e. Give an example of a lattice \((A, \leq)\) and a subset \(B\) of \(A\) such that \((B, \leq \mid_{B \times B})\) is a lattice, but \(B\) is not a sublattice of \(A\).

5. Let \(f : (L, \leq) \to (L', \leq')\) and \(g : (L', \leq') \to (L, \leq)\) be order-preserving maps between the lattices \((L, \leq)\) and \((L', \leq')\) such that \(g(f(x)) = x\) for all \(x \in L\) and \(f(g(y)) = y\) for all \(y \in L'\). Show that \(f\) and \(g\) establish a lattice isomorphism between \(L\) and \(L'\).

6. Which of the following lattices are modular and which of them are distributive?

![Diagrams of lattices](image-url)
7. If \((X, \leq)\) is a partial order, then the covering relation \(\prec\) on \(X\) is defined as

\[ x \prec y \iff x < y \& \forall z \in X (x < z \leq y \implies z = y). \]

Given two partial orders \((P, \leq_P)\) and \((Q, \leq_Q)\) we define a relation \(\leq\) on \(P \times Q\) via \((p, q) \leq (p', q')\) iff \(p \leq_P p'\) and \(q \leq_Q q'\).

a. Prove that \(\leq\) is a partial order on \(P \times Q\).

b. Prove that \((p, q) \prec (p', q')\) iff

\((p = p'\) and \(q \prec_Q q'\)) or \((p \prec_P p'\) and \(q = q'\)).

8. Prove that the absorption laws imply \(a \land a = a\) and \(a \lor a = a\).

9. Find all posets with 4 elements. (Hint: There are 16 up to isomorphism.)