Mathematical structures in logic
Exercise class 3
Heyting algebras, Boolean algebras

February 20, 2018

1. Recall that in a topological space \((X, \tau)\), for a set \(A \subseteq X\), the **interior of** \(A\) and the **closure of** \(A\) are defined respectively as the largest open set contained in \(A\) and the smallest closed set that contains \(A\), i.e.:

\[
\text{Int } A = \bigcup \{U : U \in \tau \text{ and } U \subseteq A\}
\]
\[
\text{Cl } A = \bigcap \{C : X \setminus C \in \tau \text{ and } A \subseteq C\}.
\]

Let \((P, \leq)\) be a poset. We know that \(\text{Up}(P)\), the upsets on \(P\), form a topology on \(P\).

(a) Show that this is an Alexandroff topology, i.e. show that the open sets are closed under arbitrary intersections.

(b) Show that for every \(P' \subseteq P\), \(\text{Cl}(P') = \downarrow P'\).

(c) Describe the interior of a set \(P' \subseteq P\).

(Here \(\downarrow P'\) denotes the set \(\{q \in P \mid \exists p \in P', q \leq p\}\).)

2. (Prime filters and maximal filters.) Let \(B\) be a Boolean algebra. Show that:

(a) If \(F\) is a filter on \(B\), then \(I := \{\neg a \in B \mid a \in F\}\) is an ideal on \(B\).

(b) If \(F\) is a filter, then \(B \setminus F\) may not be an ideal on \(B\).

(c) If \(F\) is a prime filter, then \(B \setminus F\) is a prime ideal.

(d) Which of these statements are true for Heyting algebras?

3. Let \(R\) be a pre-order, i.e. transitive and reflexive relation on a set \(X\). Define \(\square_R : \wp(X) \to \wp(X)\) by

\[
\square_R(U) = \{x \in X : R[x] \subseteq U\},
\]

where \(R[x] = \{x' \in X : xRx'\}\).

(a) Show that \((\wp(X), \square_R)\) is an \(\mathbf{S4}\)-algebra\(^1\).

(b) Determine the fixed points of the operator \(\square_R\).

(c) Can you define a finite join preserving function \(\Diamond_R : \wp(X) \to \wp(X)\) in a similar way? What are the fixed points for this operator.

4. (More on Alexandroff spaces and posets)

(a) Let \(f : P \to Q\) be a function between posets \((P, \leq_P)\) and \((Q, \leq_Q)\). Show that \(f\) is order-preserving iff \(f\) is continuous with respect to the topologies \(\text{Up}(P)\) and \(\text{Up}(Q)\).

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\(^1\)Also known as an **interior algebra**.
(b) Let \((P, \leq_P)\) and \((Q, \leq_Q)\) be posets. Characterise the order-preserving maps \(f: P \to Q\) with the property that \(f\) is an open map as a function between the induced Alexandroff spaces.

5. (a) Find an example of a Heyting algebra \(A\) and a subalgebra \(A'\) of \(A\) such that \(A'\) is not a homomorphic image of \(A\).

(b) Find an example of a Heyting algebra that has a homomorphic image \(B\) such that \(B\) is not isomorphic to a subalgebra of \(A\).

\((Hint: \) finite such examples exist.)