1. Let $X$ be a Stone space. Consider the map $\varepsilon : X \rightarrow X_{\text{Clop}(X)}$ (where $X_{\text{Clop}(X)}$ is the Stone space dual to $\text{Clop}(X)$) defined by $\varepsilon(x) = \{U \in \text{Clop}(X) : x \in U\}$.

(a) Show that $\varepsilon$ is well-defined.
(b) Show that $\varepsilon$ is continuous, i.e., that for each clopen in $X_{\text{Clop}(X)}$ its $\varepsilon$-pre-image is clopen in $X$.
(c) Show that $\varepsilon$ is injective.
(d) Show that $\varepsilon$ is surjective.
(e) Deduce that $\varepsilon$ is open (i.e. $\varepsilon[U]$ is open for each open set $U$) and hence a homeomorphism.

(Hint for surjectivity: The following characterization of compactness might be useful: a space $X$ is compact if and only if for any family $\mathcal{C}$ of closed sets with the finite intersection property we have $\bigcap \mathcal{C} \neq \emptyset$. Note that $\mathcal{C} = \{C_i : i \in I\}$ has the finite intersection property iff for any finite $J \subseteq I$ the intersection $\bigcap \{C_i : i \in J\} \neq \emptyset$.)

2. Let $B$ be a Boolean algebra and $X_B$ its dual Stone space. Let $(\text{Fil}(B), \subseteq)$ be the poset of filters on $B$ and let $(\text{Cl}(X_B), \subseteq)$ be the poset of closed subsets of $X_B$. Show that there is an order-reversing bijection between $(\text{Fil}(B), \subseteq)$ and $(\text{Cl}(X_B), \subseteq)$. Can you say something similar about the poset of ideals on $B$?

3. Let $X$ be a topological space. A set $U \subseteq X$ is called regular open if $U = \text{Int}(\text{Cl}(U))$. Let $\mathcal{RO}(X)$ be the set of all regular open subsets of $X$.

(a) Show that $\mathcal{RO}(X)$ is a BA where
   - $U \wedge V = U \cap V$,
   - $U \vee V = \text{Int}(\text{Cl}(U \cup V))$,
   - $\neg U = \text{Int}(X \setminus U)$.

   You may assume that $\mathcal{RO}(X)$ is a distributive lattice.
(b) Show that $\mathcal{RO}(X)$ is complete. (Hint: $\bigwedge_{i \in I} U_i = \text{Int}(\text{Cl}(\bigcap_{i \in I} U_i))$).
(c) Show that $\mathcal{RO}(\mathbb{R})$ has no atoms, $\mathbb{R}$ is the real line with the standard interval topology.

Additional exercises

4. The aim of this exercise is to understand a duality of complete and atomic Boolean algebras and sets. This duality is closely related to Stone duality, but still differs from it.
A Boolean algebra $B$ is called atomic, if given $a \neq 0$ in $B$, there exists an atom $b \in B$ such that $b \leq a$. Let $\text{CABA}$ be the class of complete and atomic Boolean algebras. Let also $\text{Set}$ be the class of all sets. To each set $X$ we associate the powerset Boolean algebra $\mathcal{P}(X)$. To each complete and atomic Boolean algebra $B$ we associate the set $\text{At}(B)$ of its atoms. Show that

(a) For every atomic $B$ and every $a \in B$ we have $a = \bigvee \{ x \in \text{At}(B) : x \leq a \}$.

(b) Every complete and atomic Boolean algebra $B$ is isomorphic to $\mathcal{P}(\text{At}(B))$.

(c) Every set $X$ is bijective to $\text{At}(\mathcal{P}(X))$.

5. For people who know (want to learn a bit more) category theory. Let $\text{Set}$ be the category of sets and functions and let $\text{CABA}$ be the category of complete atomic Boolean algebras and complete Boolean algebra homomorphisms. Prove that the correspondence between $\text{Set}$ and $\text{CABA}$ from (4), is part of a dual equivalence $\text{Set}^{\text{op}} \cong \text{CABA}$, i.e.

(a) Show that $\mathcal{P} : \text{Set} \to \text{CABA}$ and $\text{At} : \text{CABA} \to \text{Set}$ are contravariant functors. (What are the action on morphisms?)

(b) Show that the this isomorphisms from HW 4, exercise 1 are natural, i.e. show that for complete atomic Boolean algebra $B$ the isomorphisms $\eta_B : B \to \mathcal{P}(\text{At}(B))$ are components of a natural transformation $\eta : \text{Id}_{\text{CABA}} \Rightarrow \mathcal{P} \circ \text{At}$ and similarly, for every set $X$, the bijections $\mu_X : X \to \text{At}(\mathcal{P}(X))$ are components of a natural transformation $\mu : \text{Id}_{\text{Set}} \Rightarrow \text{At} \circ \mathcal{P}$.

So you need to show that for every complete Boolean homomorphism $f \in \text{Hom}_{\text{CABA}}(B,C)$ and every map $g \in \text{Hom}_{\text{Set}}(X,Y)$ the following diagrams commute.

\[
\begin{array}{ccc}
B & \xrightarrow{f} & C \\
\downarrow{\eta_B} & & \downarrow{\eta_C} \\
\mathcal{P}(\text{At}(B)) & \xrightarrow{\mathcal{P}(\text{At}(f))} & \mathcal{P}(\text{At}(C))
\end{array}
\quad
\begin{array}{ccc}
X & \xrightarrow{g} & Y \\
\downarrow{\mu_X} & & \downarrow{\mu_Y} \\
\text{At}(\mathcal{P}(X)) & \xrightarrow{\text{At}(\mathcal{P}(g))} & \text{At}(\mathcal{P}(Y))
\end{array}
\]