Mathematical structures in logic
Exercise class 4

Dualities for Boolean algebras

February 27, 2018

1. Let X be a Stone space. Consider the map $\varepsilon : X \to X_{\mathsf{Clop}(X)}$ (where $X_{\mathsf{Clop}(X)}$ is the Stone space dual to $\mathsf{Clop}(X)$) defined by $\varepsilon(x) = \{U \in \mathsf{Clop}(X) : x \in U\}$.

   (a) Show that ε is well-defined.

   (b) Show that ε is continuous, i.e., that for each clopen in $X_{\mathsf{Clop}(X)}$ its ε-pre-image is clopen in X.

   (c) Show that ε is injective.

   (d) Show that ε is surjective.

   (e) Deduce that ε is open (i.e. $\varepsilon[U]$ is open for each open set U) and hence a homeomorphism.

   (*Hint for surjectivity*: The following characterization of compactness might be useful: a space X is compact if and only if for any family $\mathcal{C}$ of closed sets with the finite intersection property we have $\bigcap \mathcal{C} \neq \emptyset$. Note that $\mathcal{C} = \{C_i : i \in I\}$ has the finite intersection property iff for any finite $J \subseteq I$ the intersection $\bigcap\{C_i : i \in J\} \neq \emptyset$.)

2. Let B be a Boolean algebra and X_B its dual Stone space. Let $(\mathsf{Fil}(B), \subseteq)$ be the poset of filters on B and let $(\mathsf{Cl}(X_B), \subseteq)$ be the poset of closed subsets of X_B. Show that there is an order-reversing bijection between $(\mathsf{Fil}(B), \subseteq)$ and $(\mathsf{Cl}(X_B), \subseteq)$. Can you say something similar about the poset of ideals on B?

3. Let X be a topological space. A set $U \subseteq X$ is called *regular open* if $U = \mathrm{Int}(\mathrm{Cl}(U))$. Let $\mathcal{RO}(X)$ be the set of all regular open subsets of X.

   (a) Show that $\mathcal{RO}(X)$ is a BA where
      - $U \wedge V = U \cap V$,
      - $U \vee V = \mathrm{Int}(\mathrm{Cl}(U \cup V))$,
      - $\neg U = \mathrm{Int}(X \setminus U)$.

      You may assume that $\mathcal{RO}(X)$ is a distributive lattice.

   (b) Show that $\mathcal{RO}(X)$ is complete. (Hint: $\bigwedge_{i \in I} U_i = \mathrm{Int}(\mathrm{Cl}(\bigcap_{i \in I} U_i))$.

   (c) Show that $\mathcal{RO}(\mathbb{R})$ has no atoms, $\mathbb{R}$ is the real line with the standard interval topology.

Additional exercises

4. The aim of this exercise is to understand a duality of complete and atomic Boolean algebras and sets. This duality is closely related to Stone duality, but still differs from it.

A Boolean algebra B is called *atomic*, if given $a \neq 0$ in B, there exists an atom $b \in B$ such that $b \leq a$. Let **CABA** be the class of complete and atomic Boolean algebras. Let also **Set** be the class of all sets. To each set X we associate the powerset Boolean algebra $\mathcal{P}(X)$. To each complete and atomic Boolean algebra B we associate the set $At(B)$ of its atoms. Show that

(a) For every atomic B and every $a \in B$ we have $a = \bigvee\{x \in \mathrm{At}(B) : x \leq a\}$.

(b) Every complete and atomic Boolean algebra B is isomorphic to $\mathcal{P}(\mathrm{At}(B))$.

(c) Every set X is bijective to $At(\mathcal{P}(X))$.

5. *For people who know (want to learn a bit more) category theory.* Let **Set** be the category of sets and functions and let **CABA** be the category of complete atomic Boolean algebras and complete Boolean algebra homomorphisms. Prove that the correspondence between **Set** and **CABA** from (4), is part of a dual equivalence $\mathbf{Set}^{\mathrm{op}} \cong \mathbf{CABA}$, i.e.

   (a) Show that $\mathcal{P} : \mathbf{Set} \to \mathbf{CABA}$ and $\mathrm{At} : \mathbf{CABA} \to \mathbf{Set}$ are contravariant functors. (What are the action on morphisms?)

   (b) Show that the this isomorphisms from HW 4, exercise 1 are natural, i.e. show that for complete atomic Boolean algebra B the isomorphims $\eta_B : B \to \mathcal{P}(\mathrm{At}(B))$ are components of a natural transformation $\eta : \mathrm{Id}_{\mathbf{CABA}} \Rightarrow \mathcal{P} \circ \mathrm{At}$ and similarly, for every set X, the bijections $\mu_X : X \to \mathrm{At}(\mathcal{P}(X))$ are components of a natural transformation $\mu : \mathrm{Id}_{\mathbf{Set}} \Rightarrow \mathrm{At} \circ \mathcal{P}$.

   So you need to show that for every complete Boolean homomorphism $f \in \mathrm{Hom}_{\mathbf{CABA}}(B, C)$ and every map $g \in \mathrm{Hom}_{\mathbf{Set}}(X, Y)$ the following diagrams commute.

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[d, "\eta_B"'] & C \arrow[d, "\eta_C"] \\ \mathcal{P}(\mathrm{At}(B)) \arrow[r, "\mathcal{P}(\mathrm{At}(f))"'] & \mathcal{P}(\mathrm{At}(C)) \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "g"] \arrow[d, "\mu_X"'] & Y \arrow[d, "\mu_Y"] \\ \mathrm{At}(\mathcal{P}(X)) \arrow[r, "\mathcal{P}(\mathrm{At}(g))"'] & \mathrm{At}(\mathcal{P}(Y)) \end{tikzcd}$$