# Algebraic and topological semantics for inquisitive logic via choice-free duality*

Nick Bezhanishvili[1], Gianluca Grilletti[1] and Wesley H. Holliday[2]

[1] ILLC, University of Amsterdam
[2] University of California, Berkeley

**Abstract.** We introduce new algebraic and topological semantics for inquisitive logic. The algebraic semantics is based on special Heyting algebras, which we call *inquisitive algebras*, with propositional valuations ranging over only the $\neg\neg$-fixpoints of the algebra. We show how inquisitive algebras arise from Boolean algebras: for a given Boolean algebra $B$, we define its *inquisitive extension* $H(B)$ and prove that $H(B)$ is the unique inquisitive algebra having $B$ as its algebra of $\neg\neg$-fixpoints. We also show that inquisitive algebras determine Medvedev's logic of finite problems. In addition to the algebraic characterization of $H(B)$, we give a topological characterization of $H(B)$ in terms of the recently introduced choice-free duality for Boolean algebras using so-called upper Vietoris spaces (UV-spaces) [2]. In particular, while a Boolean algebra $B$ is realized as the Boolean algebra of compact regular open elements of a UV-space dual to $B$, we show that $H(B)$ is realized as the algebra of compact open elements of this space. This connection yields a new topological semantics for inquisitive logic.

## 1 Introduction

The inquisitive logic `InqB` [7] is an extension of propositional logic that encompasses logical relations between *questions* in addition to statements. To define `InqB`, Ciardelli et al. [6] introduced a semantics based on states of partial information, called *support* semantics, which generalizes the standard truth-based semantics of propositional logic. In [4], connections between this semantics and several intermediate logics—including Medvedev's logic ML [10] and the Kreisel-Putnam logic KP [3, p. 148]—were studied: in particular, `InqB` can be characterized as the logic of general intuitionistic Kripke models based on Medvedev's frames for which the valuations of atomic propositions are principal upsets. Even though the algebraic structures arising from this characterization have been considered in the literature [8], a proper algebraic and topological semantics for inquisitive logic is still missing. The aim of this paper is to fill this gap.

After reviewing inquisitive logic and some topological preliminaries in Section 2, we start in Section 3 with an algebraic semantics for inquisitive logic

* Postprint of May 2020. This version updates the published version in Proceedings of the 26th Workshop on Logic, Language, Information and Computation (WoLLIC 2019) so that the definition of UV-spaces (Definition 2.6) matches that of [2].

based on Heyting algebras with propositional valuations ranging over only the $\neg\neg$-fixpoints of the algebra. The Kripke semantics for inquistive logic can be seen as a particular instance of this algebraic semantics: for $F$ a Medvedev frame, the algebra $\mathrm{Up}_p(F)$ of principal upsets of $F$ is the algebra of $\neg\neg$-fixpoints of the Heyting algebra $\mathrm{Up}(F)$ of all upsets of $F$. For our algebraic semantics, we motivate restricting attention to only special Heyting algebras, which we call *inquisitive algebras*, of which $\mathrm{Up}(F)$ for a Medvedev frame $F$ is an example.

We show how inquisitive algebras arise from Boolean algebras: for a given Boolean algebra $B$, we define in Section 4.1 its *inquisitive extension* $H(B)$ and prove in Section 4.2 that $H(B)$ is the unique inquisitive algebra having $B$ as its algebra of $\neg\neg$-fixpoints. We also show that inquisitive algebras determine Medvedev's logic. In addition to the algebraic characterization of $H(B)$ in Section 4.2, we give a topological characterization of $H(B)$ in Section 4.3 in terms of the recently introduced choice-free duality for Boolean algebras using so-called upper Vietoris spaces (UV-spaces) [2], which we review in Section 2.2. In particular, while a Boolean algebra $B$ is realized as the Boolean algebra of compact regular open elements of a UV-space dual to $B$, we show that $H(B)$ is realized as the algebra of compact open elements of this space.

The topological characterization of $H(B)$ leads in Section 5 to a new topological semantics for inquisitive logic based on UV-spaces. As an additional benefit, we obtain a new topological semantics for Medvedev's logic.

We conclude in Section 6 with some directions for future research. Several appendices contain proofs deferred in the main text.

## 2 Preliminaries

### 2.1 Inquisitive Logic

In this section, we introduce the syntax and the world-based semantics of inquisitive logic and present some basic results used throughout the paper. Further details can be found in [5, 7].

Fix a set AP of atomic propositions.

**Definition 2.1.** *The set $\mathcal{L}$ of inquisitive formulas (over* AP*) is defined by the following grammar:*

$$\phi \ := \ \bot \mid p \mid (\phi \wedge \phi) \mid (\phi \to \phi) \mid (\phi \mathbin{\vee\!\!\!\vee} \phi)$$

*where $p \in \mathrm{AP}$. We define $\neg\phi := \phi \to \bot$ and $(\phi \vee \psi) := \neg(\neg\phi \wedge \neg\psi)$.*

The standard propositional language is the $\mathbin{\vee\!\!\!\vee}$-free fragment of our language. We will refer to formulas in this fragment as *classical formulas*.

The intuitive interpretation of classical formulas is the same as in propositional logic. For example, the formula $p \vee \neg p$ is interpreted as the (tautological) statement "$p$ holds or $p$ does not hold." The role of the new connective $\mathbin{\vee\!\!\!\vee}$, called *inquisitive disjunction*, is to introduce *questions* in the logic. For example, the intuitive reading of the formula $p \mathbin{\vee\!\!\!\vee} \neg p$ is the question "Does $p$ hold?" This intuition is formalized by the standard support semantics for this language [6].

**Definition 2.2.** *Let $\mathcal{W}$ be a set of valuations for* AP (*i.e., functions from* AP *to* $\{0,1\}$). *We recursively define the* support relation $\vDash$ *for formulas in* $\mathcal{L}$ *by:*

$$\begin{array}{lcl} \mathcal{W} \vDash \bot & \iff & \mathcal{W} = \emptyset \\ \mathcal{W} \vDash p & \iff & \forall w \in \mathcal{W}.\ w(p) = 1 \\ \mathcal{W} \vDash \phi \wedge \psi & \iff & \mathcal{W} \vDash \phi \text{ and } \mathcal{W} \vDash \psi \\ \mathcal{W} \vDash \phi \to \psi & \iff & \forall \mathcal{V} \subseteq \mathcal{W}.\ [\text{ if } \mathcal{V} \vDash \phi \text{ then } \mathcal{V} \vDash \psi\ ] \\ \mathcal{W} \vDash \phi \mathbin{\vee\!\!\!\vee} \psi & \iff & \mathcal{W} \vDash \phi \text{ or } \mathcal{W} \vDash \psi. \end{array}$$

A set $\mathcal{W}$ of valuations is interpreted as a *state of partial information*: we know that the actual state of affairs is represented by one of the valuations in $\mathcal{W}$, but we do not know by which one. The information available is enough to assert that a statement holds if every $w \in \mathcal{W}$ agrees on the statement being true. Under this interpretation, every $\mathcal{W}$ supports a tautology such as $p \vee \neg p$ (cf. Lemma 2.3 below). When it comes to questions, the information available solves a question if every $w \in \mathcal{W}$ agrees on *the same* solution. For example "Does $p$ hold?", represented by $p \mathbin{\vee\!\!\!\vee} \neg p$, is solved in $\mathcal{W}$ if $w(p) = 1$ for every $w \in \mathcal{W}$ *or* $w(p) = 0$ for every $w \in \mathcal{W}$, that is, if $\mathcal{W} \vDash p \mathbin{\vee\!\!\!\vee} \neg p$.

The following lemma can be proven by a straightforward induction.

**Lemma 2.3.**

1. *For every $\phi$, $\emptyset \vDash \phi$;*
2. *If $\mathcal{W} \vDash \phi$ and $\mathcal{V} \subseteq \mathcal{W}$, then $\mathcal{V} \vDash \phi$;*
3. *If $\alpha$ is a classical formula, $\mathcal{W} \vDash \alpha$ iff $\forall w \in \mathcal{W}.\ w(\alpha) = 1$.*[3]

This lemma tells us that for a given $\mathcal{W}$ and $\phi$, the set $[\![\phi]\!]^{\mathcal{W}} := \{\mathcal{V} \subseteq \mathcal{W} \mid \mathcal{V} \vDash \phi\}$ is a non-empty $\subseteq$-downset; and moreover, if $\phi$ is classical, it is a principal downset. These observations suggest the following connection with Medvedev's logic ML—recall that ML is the logic of *Medvedev frames*, which are Kripke frames of the form $(\mathcal{P}_0(W), \supseteq)$ for $W$ a finite set, where $\mathcal{P}_0(W) = \{V \subseteq W \mid V \neq \emptyset\}$.

**Lemma 2.4 ([4, Proposition 2.2.2]).** *Let $\mathcal{W}$ be a set of valuations and consider the intuitionistic Kripke model $(\mathcal{P}_0(\mathcal{W}), \supseteq, V)$ where*

$$V(p) = \mathcal{P}_0(\{w \in \mathcal{W} \mid w(p) = 1\}).$$

*Then for every formula $\phi \in \mathcal{L}$, we have*[4]

$$\mathcal{W} \vDash \phi \quad \iff \quad (\mathcal{P}_0(\mathcal{W}), \supseteq, V) \Vdash \phi.$$

If $\mathcal{W}$ is finite, then $(\mathcal{P}_0(\mathcal{W}), \supseteq)$ is a Medvedev frame; and $V(p)$ has to be a principal upset of this frame. Moreover, if we are interested in the validity of a fixed formula $\phi(p_1, \ldots, p_n)$, we can restrict our attention to sets of valuations over $p_1, \ldots, p_n$, which are always finite. Thus, we obtain the following.

[3] Here we consider the standard extension of valuations over atomic propositions to arbitrary propositional formulas.

[4] Under the intuitionistic semantics, we interpret $\mathbin{\vee\!\!\!\vee}$ as the intuitionistic disjunction.

**Proposition 2.5.** $\texttt{InqB}$ *is the logic of the class of intuitionistic Kripke models*

$$\{(\mathcal{P}_0(X), \supseteq, V) \mid X \text{ is finite and } V(p) \text{ is principal for all } p \in \mathrm{AP}\}.$$

In [5, Sec. 3.1] a sound and complete natural deduction system for $\texttt{InqB}$ is presented, which is equivalent to the following Hilbert style system:

| | | | |
|---|---|---|---|
| **Axioms** | *IPC:* | Axioms of IPC. | |
| | *KP:* | $(\neg\phi \to \psi \vee\!\!\!\vee \chi) \to (\neg\phi \to \psi) \vee\!\!\!\vee (\neg\phi \to \chi)$ | for every $\phi, \psi, \chi \in \mathcal{L}$. |
| | *DNE:* | $\neg\neg p \to p$ | for every $p \in \mathrm{AP}$. |
| **Rules** | *MP:* | $\phi, \phi \to \psi / \psi$. | |

### 2.2 UV-spaces

In this section, we recall the basic constructions of the choice-free duality for Boolean algebras recently developed in [2]. They will be used in Sections 4.3 and 5, where we introduce a topological semantics for inquisitive logic.

Recall that for any poset $(X, \leq)$, we define

$$\texttt{Cl}_{\leq}(U) = \{x \in X \mid \exists y \geq x.\ y \in U\}, \tag{1}$$

$$\texttt{Int}_{\leq}(U) = X \setminus \texttt{Cl}_{\leq}(X \setminus U) = \{x \in X \mid \forall y \geq x.\ y \in U\}. \tag{2}$$

We call a set $U$ *$\leq$-regular open* if $U = \texttt{Int}_{\leq}\texttt{Cl}_{\leq}(U)$. Let $X$ be a topological space and $\leq$ its specialization order. Let $\mathcal{RO}(X)$ be the collection of $\leq$-regular open subsets of $X$. Let $\mathsf{CO}(X)$ denote the collection of compact open subsets of $X$. Finally, let $\mathsf{CO}\mathcal{RO}(X) = \mathsf{CO}(X) \cap \mathcal{RO}(X)$.

**Definition 2.6.** *An* upper Vietoris space *(UV-space) is a $T_0$ space $X$ such that:*

1. $\mathsf{CO}\mathcal{RO}(X)$ *is closed under* $\cap$ *and* $\texttt{Int}_{\leq}(X \setminus \cdot)$ *and forms a basis for* $X$*;*
2. *every proper filter in* $\mathsf{CO}\mathcal{RO}(X)$ *is* $\mathsf{CO}\mathcal{RO}(x) = \{U \in \mathsf{CO}\mathcal{RO}(X) \mid x \in U\}$ *for some* $x \in X$.

Given a UV-space $X$ the set $\mathsf{CO}\mathcal{RO}(X)$ forms a Boolean algebra, where $\wedge$ is the intersection, $\vee$ is $\texttt{Int}_{\leq}\texttt{Cl}_{\leq}$ of the union, and $\neg$ is $\texttt{Int}_{\leq}$ of the set-theoretic complement. It was observed in [2] that $\mathsf{CO}\mathcal{RO}(X)$ coincides with the set of compact regular open (in the topology of $X$) subsets of $X$. Conversely, for a Boolean algebra $B$ we consider the set $UV(B)$ of all proper filters of $B$ and define a topology generated by $\{\widehat{a} \mid a \in B\}$, where $\widehat{a} = \{x \in UV(B) \mid a \in x\}$. Then $UV(B)$ is a UV-space, where the specialization order is the inclusion order of filters, and $B$ is isomorphic to the algebra $\mathsf{CO}\mathcal{RO}(UV(B))$. This correspondence can be extended to a full (choice-free) duality of the category of Boolean algebras and the category of UV-spaces [2]. The name "upper Vietoris" refers to the fact that, assuming the Axiom of Choice, the UV-dual of a Boolean algebra $B$ is homeomorphic to the space of closed subsets of the Stone dual of $B$ equipped with the upper Vietoris topology (for a choice-free version of this, see [2]).

## 3 Algebraic semantics via inquisitive algebras

In this section, we define inquisitive algebras and a semantics for `InqB` via these algebras. We start with the following well-known result (see, e.g., [9, p. 51]).

**Proposition 3.1.** *For any Heyting algebra $H$, let $H_{\neg\neg} = \{\neg\neg x \mid x \in H\}$. Then:*

1. *$H_{\neg\neg}$ forms a bounded $\{\wedge, \rightarrow\}$-subalgebra of $H$;*
2. *$H_{\neg\neg}$ forms a Boolean algebra with join given by $a \vee_{H_{\neg\neg}} b = \neg\neg(a \vee_H b)$.*

*Example 3.2.* Let $B$ be a *complete* Boolean algebra and consider the Heyting algebras $\mathrm{Dw}_0(B)$ and $\mathrm{Dw}_p(B)$ of its non-empty and principal downsets, respectively. The latter is isomorphic to $B$, with the join in $\mathrm{Dw}_p(B)$ given by $\{a\}^{\downarrow} \vee \{b\}^{\downarrow} = \neg\neg(\{a\}^{\downarrow} \cup \{b\}^{\downarrow}) = \{a \vee_B b\}^{\downarrow}$, where $U^{\downarrow}$ is the downset generated by $U$. Then as shown in Appendix A:

$$\mathrm{Dw}_p(B) = (\mathrm{Dw}_0(B))_{\neg\neg}. \tag{3}$$

*Example 3.3.* Let $B$ be a Boolean algebra—not necessarily complete—and let $\mathrm{Dw}_{fg}(B)$ be the set of finitely generated downsets of $B$. Then as shown in Appendix A:

$$\mathrm{Dw}_p(B) = (\mathrm{Dw}_{fg}(B))_{\neg\neg}. \tag{4}$$

Elements of $\mathrm{Dw}_{fg}(B)$ can be represented in a special way that will be useful for later results. The proof of the next lemma is straightforward.

**Lemma 3.4.** *Every downset $D \in \mathrm{Dw}_{fg}(B)$ can be represented in a unique way as $D = \{a_1, \ldots, a_n\}^{\downarrow}$ with $a_i \nleq a_j$ for $i \neq j$.*

We now define an algebraic semantics for inquisitive logic by restricting the interpretations of atoms to $H_{\neg\neg}$, as in the definition of *inquisitive validity* below. We will denote the meet, join, and implication in a Heyting algebra with the same symbols used for the connectives of our language, $\wedge$, $\vee\!\!\!\vee$, and $\rightarrow$.

**Definition 3.5 (Algebraic semantics).**

*Let $H$ be a Heyting algebra and $V : \mathrm{AP} \rightarrow H$. For each $\phi \in \mathcal{L}$, we define $[\![\phi]\!]^{H,V} \in H$ recursively as follows:*

$$\begin{aligned} [\![\bot]\!]^{H,V} &= \bot & [\![\phi \wedge \psi]\!]^{H,V} &= [\![\phi]\!]^{H,V} \wedge [\![\psi]\!]^{H,V} \\ [\![p]\!]^{H,V} &= V(p) & [\![\phi \vee\!\!\!\vee \psi]\!]^{H,V} &= [\![\phi]\!]^{H,V} \vee\!\!\!\vee [\![\psi]\!]^{H,V} \\ & & [\![\phi \rightarrow \psi]\!]^{H,V} &= [\![\phi]\!]^{H,V} \rightarrow [\![\psi]\!]^{H,V} . \end{aligned}$$

*Let $H, V \vDash \phi$ mean that $[\![\phi]\!]^{H,V} = \top$.*

*A formula $\phi$ is* intuitionistically valid in $H$ *if for every $V : \mathrm{AP} \rightarrow H$, we have $[\![\phi]\!]^{H,V} = \top$. Let* $\mathrm{IntLog}(H)$ *be the set of formulas intuitionistically valid in $H$. A formula is* intuitionistically valid *if it is intuitionistically valid in every Heyting algebra.*

*A formula $\phi$ is* inquisitively valid in $H$ *if for every $V : \mathrm{AP} \rightarrow H_{\neg\neg}$, we have $[\![\phi]\!]^{H,V} = \top$. Let* $\mathrm{InqLog}(H)$ *be the set of formulas inquisitively valid in $H$. A formula is* inquisitively valid *if it is inquisitively valid in every Heyting algebra.*

From now on we write $[\![\phi]\!]$ instead of $[\![\phi]\!]^{H,V}$ if $H$ and $V$ are clear from context. Some properties of the semantics are straightforward to prove. For example:

**Lemma 3.6.** *If $\phi$ does not contain the symbol $\vee\!\!\!\vee$, then $[\![\phi]\!] \in H_{\neg\neg}$.*

It is immediate that every intuitionistic theorem is an inquisitve validity. And since the image of the valuations is restricted to $H_{\neg\neg}$, the formula $\neg\neg p \to p$ is also valid. But it is not the case that $\neg\neg\phi \to \phi$ is valid for every $\phi \in \mathcal{L}$, as Example 3.7 shows, so the set of validities is not closed under uniform substitution.

*Example 3.7.* Consider $H = \mathrm{Dw}_{fg}(\mathcal{P}(W))$ for a finite set $W$ with at least two elements. Notice that $H = \mathrm{Dw}_0(\mathcal{P}(W)) \cong \mathrm{Dw}(\mathcal{P}_0(W))$. In this case the algebraic semantics boils down to the support semantics for inquisitive logic (cf. Lemma 2.4).

Given $A \subseteq W$, one can easily verify that $\neg\neg\{A\}^{\downarrow} = \{A\}^{\downarrow}$ and consequently $\neg\neg p \to p \in \mathrm{InqLog}(H)$. On the other hand, for $A, B \subseteq W$ we have $\neg\neg\{A, B\}^{\downarrow} = \{A \cup B\}^{\downarrow}$ and thus $\neg\neg(p \vee\!\!\!\vee q) \to (p \vee\!\!\!\vee q) \notin \mathrm{InqLog}(H)$.

A natural question to ask is for which Heyting algebra $H$ we have $\mathtt{InqB} \subseteq \mathrm{InqLog}(H)$. The following obvious lemma gives a partial answer to this question. We call $H$ a KP-*algebra* if $H$ validates KP.

**Lemma 3.8.** *If $H$ is a* KP*-algebra, then* $\mathtt{InqB} \subseteq \mathrm{InqLog}(H)$.

Combining Lemma 3.8 with the fact that the standard support semantics is a special case of our algebraic semantics (see Example 3.7), we obtain the following:

**Proposition 3.9.** *The set of formulas valid on* KP*-algebras is exactly the set of* `InqB` *validities.*

However, arbitrary KP-algebras are somewhat "too big" for our semantics. For example, if $H = \mathrm{Dw}_0(B)$ for a complete Boolean algebra $B$, then no matter what valuation we consider, the semantic value $[\![\phi]\!]$ of a formula $\phi$ must be an element of the subalgebra generated by $\mathrm{Dw}_p(B)$, that is, $\mathrm{Dw}_{fg}(B)$. This observation can be formalized as follows.

**Lemma 3.10.** *Let $H$ be a Heyting algebra and $H'$ the subalgebra of $H$ generated by $H_{\neg\neg}$. Then:*

1. $(H')_{\neg\neg} = H_{\neg\neg}$;
2. *for every valuation $V : \mathrm{AP} \to H_{\neg\neg}$ and formula $\phi$ we have $[\![\phi]\!]^{H,V} = [\![\phi]\!]^{H',V}$;*
3. *if $H$ is a* KP*-algebra, so is $H'$.*

Thus, without loss of generality, we can restrict attention to algebras in which $H_{\neg\neg}$ generates $H$.

**Definition 3.11.** *A Heyting algebra $H$ is* regularly generated *if it is generated by $H_{\neg\neg}$.*

In fact, we can motivate one more restriction on the class of algebras we consider. As in Subsection 2.1, formulas of `InqB` are interpreted as sentences (statements or questions) and the support semantics agrees with this interpretation. For example, a question $p \vee\!\!\!\vee \neg p$ ("Does $p$ hold?") is supported in an information model iff either $p$ ("$p$ holds") or $\neg p$ ("$p$ does not hold") is supported in the model. However, this is not necessarily the case in the algebraic setting: for example, a Boolean algebra $B$ is trivially a regularly generated KP-algebra, since $B_{\neg\neg} = B$; and $[\![p \vee\!\!\!\vee \neg p]\!] = \top$ regardless of the value of $[\![p]\!]$ and $[\![\neg p]\!]$.

This motivates us to recall the following standard definition [3, p. 455].

**Definition 3.12.** *A Heyting algebra $H$ is* well connected *if for all $a, b \in H$, if $a \vee\!\!\!\vee b = 1$, then $a = 1$ or $b = 1$.*

Thus, we finally arrive at our definition of the class of inquisitive algebras.

**Definition 3.13 (Inquisitive algebra).** *An* inquisitive algebra *is a regularly generated well-connected* KP*-algebra.*

In the next section, we show how to construct inquisitive algebras from Boolean algebras.

# 4 Inquisitive extension of a Boolean algebra

## 4.1 Construction of the inquisitive extension

We will show that for a given Boolean algebra $B$, there exists a *unique* inquisitive algebra $H$ such that $B$ is isomorphic to $H_{\neg\neg}$. We will construct this $H$ as a quotient of the free Heyting algebra built using elements of $B$ as constants. Consider the set

$$\mathcal{T} = \left\{ t(b_1, \ldots, b_n) \;\middle|\; t \text{ is a term in the signature } \left\{ \dot{\wedge}, \dot{\vee}, \dot{\to}, \dot{\bot}, \dot{\top} \right\} \right\}.$$

We also introduce the shorthand $\dot{\neg} t$ for $t \dot{\to} \dot{\bot}$.

Define the binary relation $\approx$ on $\mathcal{T}$ as the smallest equivalence relation such that:

- $\approx$ respects all Heyting algebra equations (e.g., for commutativity of $\dot{\wedge}$ we require $t_1 \dot{\wedge} t_2 \approx t_2 \dot{\wedge} t_1$);
- $\approx$ respects KP: $\dot{\neg} t_1 \dot{\to} (t_2 \dot{\vee} t_3) \approx (t_1 \dot{\to} t_2) \dot{\vee} (t_1 \dot{\to} t_3)$.
- $\approx$ agrees with the operations on $B$: for $a, b \in B$, $a \dot{\wedge} b \approx a \wedge b$; $a \dot{\to} b \approx a \to b$; $\dot{\bot} \approx \bot$; $\dot{\top} \approx \top$.

$\mathcal{T}/\approx$ has a natural structure of a KP-algebra, with operations defined as

$$[t_1] \wedge [t_2] = [t_1 \dot{\wedge} t_2] \qquad [t_1] \vee\!\!\!\vee [t_2] = [t_1 \dot{\vee} t_2] \qquad [t_1] \to [t_2] = [t_1 \dot{\to} t_2].$$

We call this algebra the *inquisitive extension of $B$* and denote it by $H(B)$. Notice that by construction it is a regularly generated KP-algebra. To simplify the notation, subsequently we will drop the square brackets. By construction, the following universal property holds.

**Lemma 4.1.** *Let $B$ be a Boolean algebra and $H$ a* KP*-algebra such that $B = H_{\neg\neg}$. Then there exists a unique homomorphism $h : H(B) \to H$ such that $h|_B = id_B$. Moreover, if $H$ is regularly generated, then $h$ is surjective.*

*Proof.* Consider the map $f : \mathcal{T} \to H$ defined by the clauses

$$f(b) = b, \text{ for } b \in B \qquad f(t_1 \dot\wedge t_2) = f(t_1) \wedge f(t_2)$$
$$f(t_1 \dot\vee t_2) = f(t_1) \vee\!\!\!\vee f(t_2) \qquad f(t_1 \dot\to t_2) = f(t_1) \to f(t_2).$$

Since $H$ is a KP-algebra and agrees with the operations on $B$, $f$ factors through $H(B)$, and thus we obtain a quotient map $h : H(B) \to H$. Moreover, by construction, $h$ is a Heyting algebra homomorphism.

The image of $B$ is fixed and $H(B)$ is generated by $B$, so uniqueness follows. Moreover, if $H$ is regularly generated, then $h$ is surjective, since $B \subseteq h[H(B)]$ and $B$ generates $H$.

The previous result allows us to understand the structure of the algebra $H(B)$. In particular, elements of $H(B)$ can be represented in a *disjunctive normal form*, corresponding to the normal form of `InqB` formulas (see [5, Prop. 2.4.4]).

**Proposition 4.2.**

1. *Every $x \in H(B)$ can be represented in a unique way as $x = a_1 \vee\!\!\!\vee \ldots \vee\!\!\!\vee a_n$ with $a_1, \ldots, a_n \in B$ and $a_i \not\leq a_j$ for $i \neq j$.*
2. *$H(B) \cong \mathrm{Dw}_{fg}(B)$.*

We will call a representation of $x$ as in item 1 *non-redundant*.

*Proof.* For the proof of item 1, see Appendix B.

For item 2, consider the map $h : H(B) \to \mathrm{Dw}_{fg}(B)$. Since

$$h(a_1 \vee\!\!\!\vee \ldots \vee\!\!\!\vee a_n) = h(a_1) \cup \cdots \cup h(a_n) = \{a_1, \ldots, a_n\}^{\downarrow},$$

$h$ is injective. It is then easy to see that $h$ is an isomorphism.

A direct consequence of Proposition 4.2 is that $H(B)$ is well connected and thus an inquisitive algebra. We can also prove the following interesting property of $H(B)$, which will be useful for later applications.

**Lemma 4.3.** *Let $H'$ be a finitely generated subalgebra of $H(B)$. Then $H'$ is a subalgebra of a finite subalgebra of $H(B)$ of the form $H(B')$, where $B'$ a Boolean subalgebra of $B$.*

*Proof.* Let $a^1_1 \vee\!\!\!\vee \ldots \vee\!\!\!\vee a^1_{k_1}, \ldots, a^n_1 \vee\!\!\!\vee \ldots \vee\!\!\!\vee a^n_{k_n}$ be the non-redundant representations of the generators of $H'$, and let $A$ be the set $A = \{a^i_j \mid i \leq n, j \leq k_i\}$. Let $B'$ be the Boolean subalgebra of $B$ generated by $A$. Notice that this is a finite algebra. Clearly $H' \subseteq H(B') \subseteq H(B)$.

Finally, the isomorphism of Proposition 4.2.2 maps $H(B')$ onto $\mathrm{Dw}_{fg}(B')$—which is finite, since $|\mathrm{Dw}_{fg}(B')|$ is equal to the number of antichains in $B'$. Therefore, $H(B')$ is finite.

The results of this section allow us to draw a strong connection between regularly generated KP-algebras and Medvedev's logic ML.

**Theorem 4.4.** *If $H$ is a regularly generated* KP-*algebra, then $H$ is an* ML-*algebra.*

*Proof.* Let $H$ be a regularly generated KP-algebra. Then, by Lemma 4.1, $H$ is a homomorphic image of some algebra of the form $H(B)$. Thus, it suffices to show that $H(B)$ is an ML-algebra.

It is well known that for every Heyting algebra $A$ and intermediate logic $L$ we have that $A$ is an $L$-algebra iff every finitely generated subalgebra of $A$ is an $L$-algebra. Therefore, by Lemma 4.3, we obtain that $H(B)$ is an ML-algebra iff $H(B')$ is an ML-algebra for every finite Boolean subalgebra $B'$ of $B$.

Thus, we only need to prove the result for algebras of the form $H(B')$ where $B'$ is finite. Then $B' \cong \mathcal{P}(W)$ for some finite set $W$. By Proposition 4.2,

$$H(B') \cong \mathrm{Dw}_{fg}(B') \cong \mathrm{Dw}_{fg}(\mathcal{P}(W)) \cong \mathrm{Dw}_0(\mathcal{P}(W)) \cong \mathrm{Dw}(\mathcal{P}_0(W)),$$

which is exactly the algebra corresponding to the Medvedev frame $(\mathcal{P}_0(W), \supseteq)$. We conclude that $H(B)$ is an ML-algebra and therefore $H$ is also an ML-algebra.

**Corollary 4.5.**

$$\begin{aligned} &\mathrm{IntLog}(\{H \mid H \textit{ is a regularly generated } \mathrm{KP}\textit{-algebra}\}) \\ = &\mathrm{IntLog}(\{H(B) \mid B \textit{ is a finite Boolean algebra}\}) \\ = &\mathrm{ML}. \end{aligned}$$

*Proof.* Let $\mathcal{C}_1$ be the class of regularly generated KP-algebras and $\mathcal{C}_2$ the class of $H(B)$'s for a finite Boolean algebra $B$. Firstly, notice that every $H(B)$ is a regularly generated KP-algebra, so $\mathcal{C}_2 \subseteq \mathcal{C}_1$. Consequently $\mathrm{IntLog}(\mathcal{C}_1) \subseteq \mathrm{IntLog}(\mathcal{C}_2)$. Therefore, we just need to prove that $\mathrm{ML} \subseteq \mathrm{IntLog}(\mathcal{C}_1)$ and $\mathrm{IntLog}(\mathcal{C}_2) \subseteq \mathrm{ML}$.

The first inclusion follows directly from Theorem 4.4. For the second inclusion, consider an arbitrary Medvedev frame $(\mathcal{P}_0(W), \supseteq)$—recall that $W$ is finite. As noticed in the proof of Theorem 4.4, the Heyting algebra corresponding to this frame is $\mathrm{Dw}(\mathcal{P}_0(W)) \cong H(\mathcal{P}(W))$. Hence it is isomorphic to an element of $\mathcal{C}_2$. It follows that $\mathrm{IntLog}(\mathcal{C}_2) \subseteq \mathrm{ML}$, as required.

### 4.2 Algebraic characterization of the inquisitive extension

We are now ready to provide our first characterization of $H(B)$.

**Theorem 4.6.** *For a Boolean algebra $B$, its inquisitive extension $H(B)$ is the unique (up to isomorphism) inquisitive algebra such that $H(B)_{\neg\neg}$ is isomorphic to $B$.*

*Proof.* Let $H$ be an inquisitive algebra where $H_{\neg\neg} \cong B$, and fix an isomorphism $g : H_{\neg\neg} \to B$. By Lemma 4.1, there exists a unique morphism $h : H(B) \to H$ such that $h|_{H(B)} = g$, which is surjective since $H$ is regularly generated.

It only remains to show that $h$ is also injective, thus proving that $h$ is an isomorphism. For the proof of injectivity, see Appendix C.

**Corollary 4.7.** *A Heyting algebra $A$ is an inquisitive algebra iff $A$ is isomorphic to $H(A_{\neg\neg})$.*

*Proof.* The right-to-left implication is clear. For the left-to-right, consider an inquisitive algebra $A$. By Theorem 4.6, $H(A_{\neg\neg})$ is isomorphic to any inquisitive algebra with $A_{\neg\neg}$ as the set of $\neg\neg$-fixpoints. In particular, $A \cong H(A_{\neg\neg})$.

We conclude this section with a result analogous to Corollary 4.5 but now for inquisitive logic.

**Corollary 4.8.**

$$\begin{aligned} &\mathrm{InqLog}(\{H \mid H \text{ is a } \mathrm{KP}\text{-algebra}\}) \\ &= \mathrm{InqLog}(\{H(B) \mid B \text{ is a finite Boolean algebra}\}) \\ &= \mathtt{InqB}\,. \end{aligned}$$

*Proof.* By Lemma 3.8, `InqB` is included in the inquisitive logic of the two classes of algebras. For the other inclusion: by Proposition 4.2, given a finite set $W$ we have $H(\mathcal{P}(W)) \cong \mathrm{Dw}(\mathcal{P}_0(W))$. So by Proposition 2.5, the inquisitive logic of the second class of algebras is indeed `InqB`; and since the first class of algebras includes the second, we obtain both equalitites.

### 4.3 Topological characterization of the inquisitive extension

Using the UV-spaces of Section 2.2, we can give a topological realization of $H(B)$, which in the next section will lead to a topological semantics of inquisitive logic. By item 2 of the following theorem, $H(B)$ may be characterized as (isomorphic to) the Heyting algebra of compact open sets of the UV-space dual to $B$.

**Theorem 4.9.** *Let $B$ be a Boolean algebra and $X$ its dual UV-space.*

1. $(\mathsf{O}(X), \subseteq) \cong \mathrm{Dw}_0(B)$.
2. $(\mathsf{CO}(X), \subseteq) \cong \mathrm{Dw}_{fg}(B) \cong H(B)$.

*Proof.* See Appendix D.

For those familiar with Esakia duality for Heyting algebras, we can further exploit Theorem 4.9 to obtain a connection between the choice-free duality for Boolean algebras and Esakia duality. This connection uses the following.

**Proposition 4.10.** *The following function defines an order isomorphism between the set $\mathrm{Spec}(H(B))$ of prime filters of $H(B)$, ordered by inclusion, and the set $\mathrm{Filt}(B)$ of filters of $B$, ordered by inclusion:*

$$\begin{array}{rccc} r: & (\mathrm{Spec}(H(B)), \subseteq) & \to & (\mathrm{Filt}(B), \subseteq) \\ & F & \mapsto & F \cap B \end{array}$$

*Proof.* See Appendix E.

**Proposition 4.11.** *Given $B$ a Boolean algebra, the Esakia space* $\mathrm{Spec}(H(B))$ *dual to $H(B)$ is homeomorphic to the UV-space $UV(B)$ dual to $B$.*

*Proof.* The map $r$ defined in Proposition 4.10 above is a homeomorphism; all the verifications are standard and left to the reader.

In particular, this gives us an alternative proof of Theorem 4.9.2.

The results of this section are summarized in Figure 1.

| **algebras** | **spaces** |
|---|---|
| $B \cong \mathsf{CO}\mathcal{RO}(UV(B))$ | $UV(B)$ |
| $H(B) \cong \mathsf{CO}(UV(B))$ | $\wr\vert$ |
| $H(B) \cong \mathsf{CO}(\mathrm{Spec}(H(B)))$ | $\mathrm{Spec}(H(B))$ |

**Fig. 1.** Summary of results of Section 4.3.

## 5 Topological semantics for inquisitive logic

Theorem 4.9 and Lemma 5.2 allow us to define a topological semantics for `InqB` using the duality based on UV-spaces.

**Definition 5.1 (Topological semantics).**

*Let $X$ be a UV-space and $V : \mathrm{AP} \to \mathsf{CO}\mathcal{RO}(X)$ an atomic valuation. For each inquisitive formula $\phi \in \mathcal{L}$, we define its semantic valuation $[\![\phi]\!]^{X,V} \in \mathsf{CO}(X)$ by recursion as follows*[5]*:*

$$
\begin{aligned}
[\![\bot]\!]^{X,V} &= \emptyset & [\![\phi \wedge \psi]\!]^{X,V} &= [\![\phi]\!]^{X,V} \cap [\![\psi]\!]^{X,V} \\
[\![p]\!]^{X,V} &= V(p) & [\![\phi \vee\!\!\!\vee \psi]\!]^{X,V} &= [\![\phi]\!]^{X,V} \cup [\![\psi]\!]^{X,V} \\
& & [\![\phi \to \psi]\!]^{X,V} &= \mathtt{Int}\left((X \setminus [\![\phi]\!]^{X,V}) \cup [\![\psi]\!]^{X,V}\right).
\end{aligned}
$$

*We adopt the same notational conventions for validity as in Definition 3.5.*

In the Boolean algebra $\mathsf{CO}\mathcal{RO}(X)$, implication is given by $U \to V = \neg U \vee V$ $= \mathtt{Int}_{\leq}\mathtt{Cl}_{\leq}(\mathtt{Int}_{\leq}(X \setminus U) \cup V)$, and it is easy to check that the right-hand side is equal to $\mathtt{Int}_{\leq}((X \setminus U) \cup V)$. By the next result, we can also think in terms of the interior operator `Int` of the main topology, as in Definition 5.1, instead of the interior operator $\mathtt{Int}_{\leq}$ of the order topology.

**Lemma 5.2.** *Given $A, B \in \mathsf{CO}(X)$, $\mathtt{Int}((X \setminus A) \cup B) = \mathtt{Int}_{\leq}((X \setminus A) \cup B)$.*

*Proof.* See Appendix F.

[5] Notice that Theorem 4.9 ensures that $[\![\phi \to \psi]\!]^{X,V} \in \mathsf{CO}(X)$.

**Corollary 5.3.** *The set of formulas valid on UV-spaces under this semantics is exactly the set of theorems of* $\mathtt{InqB}$.

*Proof.* Let $X$ be a UV-space. By Theorem 4.9, $\mathsf{CO}(X) \cong H(\mathsf{CO}\mathcal{RO}(X))$. Moreover, by [2], every Boolean algebra is isomorphic to one of the form $\mathsf{CO}\mathcal{RO}(X)$. Combining this result with Corollary 4.8, we obtain:

$$\text{InqLog}(\{X \mid X \text{ a UV-space}\}) = \text{InqLog}(\{H(B) \mid B \text{ a Boolean algebra}\}) = \mathtt{InqB}\,.$$

We conclude this section by pointing out a connection with Medvedev's logic ML. UV-spaces can be used to give a new topological semantics for ML in a way analogous to inquisitive logic, namely by allowing valuations to range over $\mathsf{CO}$-sets in Definition 5.1—and not only $\mathsf{CO}\mathcal{RO}$-sets.

**Corollary 5.4.** ML *is sound and complete with respect to the topological semantics presented above.*

*Proof.* This follows directly from Corollary 4.5 and Theorem 4.9.

## 6 Conclusion

In this paper, we introduced algebraic and topological semantics for inquisitive logic and connected them via choice-free duality for Boolean algebras [2]. This opens up new avenues for further research, three of which we will briefly mention.

The main results of this paper are concerned with KP-algebras, since the KP-axiom is essential for inquisitive logic. However, one could consider the more general case of arbitrary (regularly generated) Heyting algebras and study the corresponding generalized inquisitive logics.

Another generalization to consider is to replace the double negation nucleus $\neg\neg$ with an arbitrary (perhaps definable) nucleus on a Heyting algebra. Of course, the algebra of fixed points of such a nucleus will no longer be Boolean. This yields the nuclear semantics for "inquisitive intuitionistic logic" in [1]. How to characterize inquisitive extensions in that setting and what topological duality to use for their representation remain open problems.

Finally, just as in the case of intermediate and modal logics, where algebraic semantics and duality provide tools for studying lattices of these logics, we hope that this newly developed algebraic semantics and duality will open the door for investigations of lattices of inquisitive logics.

## References


1. G. Bezhanishvili and W. H. Holliday. Inquisitive intuitionistic logic. Manuscript, 2019.
2. N. Bezhanishvili and W. H. Holliday. Choice-free Stone duality. *Journal of Symbolic Logic*, Forthcoming.
3. A. Chagrov and M. Zakharyaschev. *Modal logic*, volume 35 of *Oxford Logic Guides*. The Clarendon Press, New York, 1997.

4. I. Ciardelli. Inquisitive semantics and intermediate logics. MSc Thesis, University of Amsterdam, 2009.
5. I. Ciardelli. *Questions in logic*. PhD thesis, Institute for Logic, Language and Computation, University of Amsterdam, 2016.
6. I. Ciardelli, J. Groenendijk, and F. Roelofsen. Inquisitive semantics: A new notion of meaning. *Language and Linguistics Compass*, 7(9):459–476, 2013.
7. I. Ciardelli, J. Groenendijk, and F. Roelofsen. *Inquisitive Semantics*. Oxford University Press, Oxford, 2018.
8. S. Frittella, G. Greco, A. Palmigiano, and F. Yang. Structural multi-type sequent calculus for inquisitive logic. *CoRR*, abs/1604.00936, 2016.
9. P. T. Johnstone. *Stone Spaces*, volume 3 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1982.
10. Yu. T. Medvedev. Interpretation of logical formulas by means of finite problems. *Soviet Mathematics Doklady*, 7(4):857–860, 1966.


# A Examples 1 and 2

*Proof of* (3).
Given a complete Boolean algebra $B$, we show that $\mathrm{Dw}_p(B) = (\mathrm{Dw}_0(B))_{\neg\neg}$. First, if we consider a principal downset, we have

$$\neg\{b\}^{\downarrow} = \{a \in B \mid a \wedge b = \bot\} = \{\neg b\}^{\downarrow} \quad\Longrightarrow\quad \neg\neg\{b\}^{\downarrow} = \{b\}^{\downarrow}.$$

So $\mathrm{Dw}_p(B) \subseteq (\mathrm{Dw}_0(B))_{\neg\neg}$. For the other inclusion, it suffices to show that $\neg D$ is principal for every downset $D$. We have

$$\neg D = \{a \in B \mid \forall d \in D.\ a \wedge d \leq \bot\} \subseteq \left\{\bigvee \neg D\right\}^{\downarrow}.$$

On the other hand, $\bigvee \neg D \in \neg D$, since for every $e \in D$, we have

$$e \wedge \bigvee \neg D = \bigvee\{e \wedge a \mid \forall d \in D.\ a \wedge d \leq \bot\} = \bigvee\{\bot\} = \bot.$$

It follows that $\neg D = \{\bigvee \neg D\}^{\downarrow}$. Thus, $\neg D$ is principal.

*Proof of* (4).
Given a Boolean algebra $B$, we show that $\mathrm{Dw}_p(B) = (\mathrm{Dw}_{fg}(B))_{\neg\neg}$. The inclusion $\mathrm{Dw}_p(B) \subseteq (\mathrm{Dw}_{fg}(B))_{\neg\neg}$ is proved as above. For the other inclusion it suffices to show that for any $b_1, \dots, b_n \in B$, $\neg\{b_1, \dots, b_n\}^{\downarrow}$ is principal. This follows from the equalities

$$\neg\{b_1, \dots, b_n\}^{\downarrow} = \{a \in B \mid \forall i \leq n.\ a \wedge b_i = \bot\} = \{\neg b_1 \wedge \cdots \wedge \neg b_n\}^{\downarrow}.$$

# B Proof of Proposition 4.2

We divide the proof in two steps: proving that every element $x \in H(B)$ can be written in the form $x = b_1 \vee\!\!\!\vee \dots \vee\!\!\!\vee b_m$ with $b_1, \dots, b_m \in B$; and proving that from this form we can obtain a non-redundant representation.

For the first part: since $H(B)$ is the quotient of the set $\mathcal{T}$ of terms, we can proceed by induction on $t \in \mathcal{T}$.

- If $x \in B$, then we are done.
- If $x = y \wedge z$, then consider two representations $y = c_1 \vee\!\!\!\vee \ldots \vee\!\!\!\vee c_k$ and $z = d_1 \vee\!\!\!\vee \ldots \vee\!\!\!\vee d_l$. Then

$$x = y \wedge z = (c_1 \vee\!\!\!\vee \ldots \vee\!\!\!\vee c_k) \wedge (d_1 \vee\!\!\!\vee \ldots \vee\!\!\!\vee d_l) = \bigvee\!\!\!\bigvee \{c_i \wedge d_j \mid i \leq k, j \leq l\}.$$

- If $x = y \vee\!\!\!\vee z$, then

$$x = y \vee\!\!\!\vee z = c_1 \vee\!\!\!\vee \ldots \vee\!\!\!\vee c_k \vee\!\!\!\vee d_1 \vee\!\!\!\vee \ldots \vee\!\!\!\vee d_l.$$

- If $x = y \to z$, then

$$\begin{aligned} x = y \to z &= (c_1 \vee\!\!\!\vee \ldots \vee\!\!\!\vee c_k) \to (d_1 \vee\!\!\!\vee \ldots \vee\!\!\!\vee d_l) \\ &= (c_1 \to d_1 \vee\!\!\!\vee \ldots \vee\!\!\!\vee d_l) \wedge \cdots \wedge (c_k \to d_1 \vee\!\!\!\vee \ldots \vee\!\!\!\vee d_l) \\ &= \wedge_{i=1}^{l} \left((c_i \to d_1) \vee\!\!\!\vee \ldots \vee\!\!\!\vee (c_i \to d_l)\right) && \text{(by } KP\text{)} \\ &= \vee_{f:[n]\to[m]} \left(\wedge_{i=1}^{l} (c_i \to d_{f(i)})\right). \end{aligned}$$

For the second part: let $x = b_1 \vee\!\!\!\vee \ldots \vee\!\!\!\vee b_m$ be an arbitrary representation of $x$. If $\forall i, j.\ b_i \not\leq b_j$, then we are done. Otherwise, suppose (without loss of generality) that $b_1 \leq b_2$. Then

$$b_1 \vee\!\!\!\vee b_2 \vee\!\!\!\vee \ldots \vee\!\!\!\vee b_n = b_2 \vee\!\!\!\vee \ldots \vee\!\!\!\vee b_n.$$

Repeating this procedure, we obtain a non-redundant representation of $x$.

## C Proof of Theorem 4.6

It only remained to prove that $h$ is injective. Let $x, y \in H(B)$ and suppose that $h(x) = h(y)$. Let $x = a_1 \vee\!\!\!\vee \ldots \vee\!\!\!\vee a_n$ and $y = b_1 \vee\!\!\!\vee \ldots \vee\!\!\!\vee b_m$ be their non-redundant representations. Then where $\sqcap, \sqcup, \Rightarrow$ are the operations of $H$, we have

$$\begin{aligned} & a_1 \sqcup \cdots \sqcup a_n = b_1 \sqcup \cdots \sqcup b_m \\ \Longrightarrow\ & (a_1 \sqcup \cdots \sqcup a_n) \Leftrightarrow (b_1 \sqcup \cdots \sqcup b_m) = \top \\ \Longrightarrow\ & \begin{cases} \bigsqcup_{f:[n]\to[m]} \bigsqcap_{i\leq n} (a_i \Rightarrow b_{f(i)}) = \top \\ \bigsqcup_{g:[m]\to[n]} \bigsqcap_{j\leq m} (b_j \Rightarrow a_{g(j)}) = \top \end{cases} \\ \Longrightarrow\ & \begin{cases} \exists f : [n] \to [m].\ \bigsqcap_{i\leq n} (a_i \Rightarrow b_{f(i)}) = \top \\ \exists g : [m] \to [n].\ \bigsqcap_{j\leq m} (b_j \Rightarrow a_{g(j)}) = \top \end{cases} && \text{(since } H \text{ is inquisitive)} \\ \Longrightarrow\ & \begin{cases} \forall i \leq n.\ \exists j \leq m.\ (a_i \Rightarrow b_j) = \top \\ \forall j \leq m.\ \exists i \leq n.\ (b_j \Rightarrow a_i) = \top \end{cases} \\ \Longrightarrow\ & \begin{cases} \forall i \leq n.\ \exists j \leq m.\ a_i \leq b_j \\ \forall j \leq m.\ \exists i \leq n.\ b_j \leq a_i \end{cases} && \text{(since } h|_B = id_B\text{)} \\ \Longrightarrow\ & \begin{cases} x \leq y \\ y \leq x \end{cases} \\ \Longrightarrow\ & x = y. \end{aligned}$$

So $h$ is injective and thus an isomorphism, as required.

## D Proof of Theorem 4.9

To prove Theorem 4.9, we will use the following lemma.

**Lemma D.1.** *Let $A = \bigcup_{i\in I} U_i$ and $B = \bigcup_{j\in J} V_j$ be open sets of a UV-space $X$, where $U_i, V_j$ are $\mathsf{CO}\mathcal{RO}$-sets. Then $A \subseteq B$ iff $\forall i \in I.\, \exists j \in J.\, U_i \subseteq V_j$.*

*Proof.* Firstly, we show that every $\mathsf{CO}\mathcal{RO}$-set $U$ is the upset of a singleton: since $\{U\}^{\uparrow}$ is a filter in $\mathsf{CO}\mathcal{RO}(X)$, there exists a point $x$ such that $\{U\}^{\uparrow} = \mathsf{CO}\mathcal{RO}(X)$. It follows that $U = \bigcap \mathsf{CO}\mathcal{RO}(x) = \{x\}^{\uparrow}$.

We can use this to prove the result. Call $x_i$ the generator of $U_i$ for each $i \in I$.

$$\begin{aligned} A \subseteq B \quad &\iff \textstyle\bigcup_{i\in I} U_i \subseteq \bigcup_{j\in J} V_j &&\iff \forall i \in I.\, U_i \subseteq \textstyle\bigcup_{j\in J} V_j \\ &\iff \forall i \in I.\, U_i \subseteq \textstyle\bigcup_{j\in J} V_j &&\iff \forall i \in I.\, x_i \in \textstyle\bigcup_{j\in J} V_j \\ &\iff \forall i \in I.\, \exists j \in J.\, x_i \in V_j &&\iff \forall i \in I.\, \exists j \in J.\, U_i \subseteq V_j. \end{aligned}$$

We are now ready to prove Theorem 4.9.

*Proof of Theorem 4.9.*

For the first part: consider the map $f : O(X) \to \mathrm{Dw}_0(B)$ defined by[6]

$$f\left(\bigcup_{i\in I} \widehat{a_i}\right) = \{a_i \mid i \in I\}^{\downarrow}.$$

To show that $f$ is well defined and order preserving and reflecting, we observe the following equivalences, using Lemma D.1 for the first:

$$\begin{aligned} \bigcup_{i\in I} \widehat{a_i} \subseteq \bigcup_{j\in J} \widehat{b_j} &\iff \forall i \in I.\, \exists j \in J.\, \widehat{a_i} \subseteq \widehat{b_j} \\ &\iff \forall i \in I.\, \exists j \in J.\, a_i \leq b_j \\ &\iff \forall i \in I.\, \exists j \in J.\, \{a_i\}^{\downarrow} \subseteq \{b_j\}^{\downarrow} \\ &\iff \{a_i \mid i \in I\}^{\downarrow} \subseteq \{b_j \mid j \in J\}^{\downarrow}. \end{aligned}$$

Thus, $f$ is also injective. Notice that surjectivity is trivially satisfied. Hence $f$ is an isomorphism.

For the second part: since elements of $\mathsf{CO}(X)$ are exactly the sets of the form $\widehat{a_1} \cup \cdots \cup \widehat{a_n}$ for some $a_1, \ldots, a_n \in B$, we obtain that $f|_{\mathsf{CO}(X)}$ is an isomorphism with range $\mathrm{Dw}_{fg}(B)$, as required.

## E Proof of Proposition 4.10

It is immediate that $r$ is well defined and order preserving. For injectivity, notice that a prime filter $\mathfrak{p}$ of $H(B)$ is completely determined by the elements of $B$ it contains, since for every non-redundant representation $a_1 \vee\!\!\!\vee \ldots \vee\!\!\!\vee a_n$, we have

$$a_1 \vee\!\!\!\vee \ldots \vee\!\!\!\vee a_n \in \mathfrak{p} \iff a_1 \in \mathfrak{p} \text{ or } \ldots \text{ or } a_n \in \mathfrak{p}. \tag{5}$$

[6] Here we are adopting the convention $\{\}^{\downarrow} := \{\bot\}$, so that $f(\emptyset) = \{\bot\}$.

Using this fact, we can also show surjectivity: let $F$ be a filter of $B$ and define $\mathfrak{p}_F$ as the smallest set including $F$ and respecting (5). Then clearly $\mathfrak{p}_F$ is an upset and respects the $\vee\!\!\!\vee$-condition of prime filters. Moreover, it is closed under meets, since

$$\begin{aligned} & a_1 \varovee \ldots \varovee a_n \in \mathfrak{p}_F \text{ and } b_1 \varovee \ldots \varovee b_m \in \mathfrak{p}_F \\ \iff & \exists i.\ a_i \in \mathfrak{p}_F \text{ and } \exists j.\ b_j \in \mathfrak{p}_F \\ \iff & \exists i.\ a_i \in F \text{ and } \exists j.\ b_j \in F \\ \iff & \exists i.\ \exists j.\ a_i \wedge b_j \in F \\ \iff & \exists i.\ \exists j.\ a_i \wedge b_j \in \mathfrak{p}_F \\ \iff & (a_1 \varovee \ldots \varovee a_n) \wedge (b_1 \varovee \ldots \varovee b_m) = \bigvee\!\!\!\bigvee \{a_i \wedge b_j \mid i \leq n, j \leq m\} \in \mathfrak{p}_F. \end{aligned}$$

Since $r(\mathfrak{p}_F) = F$, we also have surjectivity.

## F Proof of Lemma 5.2

To prove Lemma 5.2, we first need to establish some technical results. In the following we denote $X \setminus A$ by $\overline{A}$. For a UV space $X$ and $x, y \in X$, let $x \sqcap y$ be the greatest lower bound of $x$ and $y$ in the specialization order of $X$ [2, Corollary 5.4].

**Lemma F.1.** *Let $U \in \mathsf{C}\mathcal{RO}(X)$ and $x_1, x_2 \in U$. Then $x_1 \sqcap x_2 \in U$.*

*Proof.* By Corollary 5.4 of [2], $U = U \vee U = U \cup \{x \sqcap y \mid x, y \in U\}$.

**Lemma F.2.** *Given $U, V \in \mathsf{C}\mathcal{RO}(X)$, $\mathtt{Int}_{\leq}\left(\overline{U} \cup V\right) = \neg U \vee V$.*

*Proof.*

**Left-to-right inclusion.** Consider an element $x \in \mathtt{Int}_{\leq}\left(\overline{U} \cup V\right)$. If $x \in \neg U \cup V$, then there is nothing to prove; so suppose this is not the case. By Corollary 5.4 of [2], there is a decomposition $x = x_1 \sqcap x_2$ such that $x_1 \in \neg U$ and $x_2 \in U$.

Since $x_2 \notin \overline{U}$ and $x_2 \geq x \in \mathtt{Int}_{\leq}\left(\overline{U} \cup V\right)$, it follows that $x_2 \in V$. So $x \in \{y \sqcap z \mid y \in \neg U, z \in V\} \subseteq \neg U \vee V$, as desired.

**Right-to-left inclusion.** Consider $x \in \neg U \vee V$ and take an arbitrary $w \geq x$. We want to show that $w \in \overline{U} \cup V$.

If $w \in \neg U \cup V \subseteq \overline{U} \cup V$, then there is nothing to prove; so suppose this is not the case. By Corollary 5.4 of [2], we can write $w = w_1 \sqcap w_2$ with $w_1 \in \neg U$ and $w_2 \in V$. In particular, $w_1$ is a successor of $w$ not in $U$, and since $\overline{U}$ is a $\leq$-downset, it follows that $w \in \overline{U} \subseteq \overline{U} \cup V$.

Since $w$ was an arbitrary successor of $x$, it follows $x \in \mathtt{Int}_{\leq}\left(\overline{U} \cup V\right)$.

**Lemma F.3.** *Given $U_i, V_j \in \mathsf{C}\mathcal{RO}(X)$, the following identity holds:*

$$\mathtt{Int}_{\leq}\left(\left(\bigcap_{i=1}^{m} \overline{U_i}\right) \cup \left(\bigcup_{j=1}^{n} V_j\right)\right) = \bigcup_{f:[m]\to[n]} \bigcap_{i=1}^{m} \left(\neg U_i \vee V_{f(i)}\right).$$

*Proof.* By Lemma F.2, the identity is equivalent to

$$\mathtt{Int}_{\leq}\left(\left(\bigcap_{i=1}^{m}\overline{U_i}\right)\cup\left(\bigcup_{j=1}^{n}V_j\right)\right)=\bigcup_{f:[m]\to[n]}\mathtt{Int}_{\leq}\left(\bigcap_{i=1}^{m}\left(\overline{U}_i\cup V_{f(i)}\right)\right).$$

Let $L$ and $R$ be the left-hand side and right-hand side, respectively.

**Right-to-left inclusion.** Consider $x\in R$. This means that:

$$\exists f:[m]\to[n].\ \forall y\geq x.\ y\in\bigcap_{i=1}^{m}\left(\overline{U}_i\cup V_{f(i)}\right).$$

So with fixed $f$ as above, given $y\geq x$, we have:

$$y\in\bigcap_{i=1}^{m}\left(\overline{U}_i\cup V_{f(i)}\right)\subseteq\bigcap_{i=1}^{m}\left(\overline{U}_i\cup\left(\bigcup_{j=1}^{n}V_j\right)\right)=\left(\bigcap_{i=1}^{m}\overline{U_i}\right)\cup\left(\bigcup_{j=1}^{n}V_j\right).$$

As $y$ was an arbitrary successor of $x$, it follows that $x\in L$.

**Left-to-right inclusion**. We will show this step by contradiction. Suppose that $x\notin R$. This means that:

$$\forall f:[m]\to[n].\ \exists y\geq x.\ \exists i\in[m].\ y\notin\overline{U}_i\cup V_{f(i)},$$

or equivalently

$$\exists i\in[m].\ \forall j\in[n].\ \{x\}^{\uparrow}\cap U_i\cap\overline{V}_j\neq\emptyset.$$

Fix an index $k$ instantiating the first quantifier, and consider for each $j\in[n]$ an element $y_j\in\{x\}^{\uparrow}\cap U_k\cap\overline{V}_j$. Define $y=y_1\sqcap\cdots\sqcap y_n$. We have:

- For every $j\in[n]$, $y_j\geq x$, and thus $y\geq x$.
- Since $y_j\in\overline{V}_j$ and $V_j$ is open, it follows that $\mathtt{Cl}(y_j)\subseteq\overline{V}_j$; and consequently $y\in\overline{V}_j$, since $y\leq y_j$.
- Since $y_1,\ldots,y_n\in U_k$, we have $y\in U_k$ (see Lemma F.1).

So it follows that $y\geq x$ and $y\in U_k\cap\overline{V_1}\cap\cdots\cap\overline{V_n}$. Thus in particular $y\notin\left(\bigcap_{i=1}^{m}\overline{U}_i\right)\cup\left(\bigcup_{j=1}^{n}V_j\right)$, from which we obtain $x\notin L$, as desired.

We are now able to prove Lemma 5.2.

*Proof (Proof of Lemma 5.2).* By Lemma F.3, $\mathtt{Int}_{\leq}(\overline{A}\cup B)\in\mathsf{CO}(X)$. Since the order topology is finer than the main topology, we have

$$\mathtt{Int}(\overline{A}\cup B)=\mathtt{Int}\left(\mathtt{Int}_{\leq}(\overline{A}\cup B)\right)=\mathtt{Int}_{\leq}(\overline{A}\cup B).$$