ONE-STEP HEYTING ALGEBRAS AND HYPERSEQUENT CALCULI WITH THE BOUNDED PROOF PROPERTY

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Abstract. We investigate proof-theoretic properties of hypersequent calculi for intermediate logics using algebraic methods. More precisely, we consider a new weakly analytic subformula property (the bounded proof property) of such calculi. Despite being strictly weaker than both cut-elimination and the subformula property this property is sufficient to ensure decidability of finitely axiomatised calculi. We introduce one-step Heyting algebras and establish a semantic criterion characterising calculi for intermediate logics with the bounded proof property and the finite model property in terms of one-step Heyting algebras. Finally, we show how this semantic criterion can be applied to a number of calculi for well-known intermediate logics such as $\mathbf{LC}, \mathbf{KC}$ and $\mathbf{BD}_2$.

Keywords: intermediate logics, hypersequent calculi, bounded proof property, finite model property, finite duality.

1. Introduction

Having a well-behaved proof system for a logic, e.g., intermediate, modal, substructural etc., can help determine various desirable properties of this logic such as consistency, decidability, interpolation etc. Gentzen-style sequent calculi have for a long time played a pivotal role in proof theory [46] and proving admissibility of the cut-rule has been one of the main techniques for establishing good proof theoretic properties of sequent calculi. However, for various non-classical logics finding a cut-free sequent calculus can be a difficult task, even when the logic in question has a very simple semantics. In fact, in many cases no such calculus seems to exist. In the 1980's Pottinger [42] and Avron [4] introduced hypersequent calculi for handling certain modal and relevance logics. Hypersequents are nothing more than finite (multi)sets of sequents. Nevertheless, they give rise to simple cut-free calculi for many logics for which no ordinary cut-free calculus has been found. Since then many cut-free hypersequent calculi for various modal and intermediate logics have been developed [5, 21, 20, 38, 24, 41]. However, establishing cut-elimination for Gentzen-style sequent or hypersequent calculi by syntactic means can be very cumbersome. Although the basic idea behind syntactic proofs of cut-elimination is simple, each individual calculus will need its own proof of cut-elimination and proofs obtained for one calculus do not necessarily transfer easily to other—even very similar—calculi.

Despite the fact that having a cut-free calculi for a given logic entails various desirable properties it can be argued that cut-free derivations in themselves are not very natural, see e.g., [28, 16]. For example, in the worst case, insisting on cut-free derivations may lead to an exponential blow-up; even for simple propositional logics, see e.g., [46, Thm. 5.2.13]. In fact, for some purposes it is sufficient to know that the cut-rule can be restricted to some well-behaved set of formulas. These two considerations, namely, (i) that cut-free calculi are difficult to construct and that (ii) in practise cut-free derivations may be unfeasible, motivates us to consider a proof-theoretic property, weaker than cut-elimination, ensuring that the cut-rule can be restricted to the set of formulas of implicational degree not exceeding that of the formulas in the premise or the conclusion.

Semantic proofs of cut-elimination have been known since at least 1960 [44], but in recent years general and more systematic approaches to constructing cut-free calculi for various non-classical logics have been developed. For example, [41] provide general methods for obtaining cut-free calculi for larger classes of modal logics based on their frame semantics, and in [22, 23] an algebraic approach connecting cut-elimination with closure under MacNeille completion can be found. One of the attractive features of these approaches is that it allows one to establish cut-elimination for large classes of logics in a uniform way. Moreover, [22, 23] also provide algebraic criteria determining when cut-free (hyper)sequent calculi for a given substructural logic can be obtained.[1] This algebraic approach suggests that algebraic semantics can be used to detect other desirable features of a proof system.

We take a somewhat different approach to connecting algebra and proof theory than the one found in [22, 23]. Our approach can be seen as arising from the careful investigation of the structure of finitely generated free algebras. The free algebra of a propositional logic encodes a lot of information about the logic. For instance, it is well known that the finitely generated free algebras constitute a powerful tool when it comes to establishing meta-theoretical properties for various propositional logics such as interpolation, definability, admissibility of rules etc. In [31] it was shown how to construct finitely generated free Heyting algebras as (chain) colimits of finite distributive lattices. In [32] a similar construction for finitely generated free modal algebras was presented; showing how these algebras arise as colimits of finite Boolean algebras.[2] The intuition behind these constructions is that one builds the finitely generated free algebra in stages by freely adding the Heyting implication (or in the case of modal algebras the modal operator) step by step. Lately this construction has received renewed attention in [17, 11] (for Heyting algebras) and in [15, 33, 27, 35, 14] (for modal algebras).

It was realised in [12] that the so-called *modal one-step algebras* arising as consecutive pairs of algebras in the colimit construction of finitely generates free modal algebras can be used to characterise a certain weak analytic subformula property of proof systems for modal logics. This property—called *the bounded proof property*—holds for an axiom system Ax if for every finite set of formulas $\Gamma \cup \{\varphi\}$ of modal depth[3] at most n such that Γ entails φ over Ax there exists a derivation in Ax witnessing this in which all the formulas have modal depth at most n. We write $\Gamma \vdash^n_{\mathrm{Ax}} \varphi$ if this is the case. With this notation the bounded proof property may be expressed as

$$\Gamma \vdash_{\mathrm{Ax}} \varphi \implies \Gamma \vdash^n_{\mathrm{Ax}} \varphi,$$

for all $n \in \omega$ and all sets of formulas $\Gamma \cup \{\varphi\}$ of depth at most n. Even though this is a fairly weak property it does, e.g., bound the search space when searching for proofs and thus it ensures decidability of calculi with a finite axiomatisation. Furthermore, having this property might serve as an indication of robustness of the axiom system in question. In this way it is like cut-elimination although in general it is much weaker.

In light of the original colimit construction of finitely generated free Heyting algebras it seems natural to ask if one can adapt the work of [12] to the setting of intuitionistic logic and its consistent extensions, i.e., *intermediate logics*. That is, we ask if it is possible to formulate the bounded proof property for intuitionistic logic and define a notion of one-step Heyting algebras which can characterise proof systems of intermediate logics with the bounded proof property.

In order to do this one first needs to choose a proof theoretic framework for which to ask this question. In this respect there are two remarks to be made. First of all as any use of *modus ponens* will evidently make the bounded proof property with respect to implications fail, we will have to consider proof systems different from natural deduction or Hilbert-style proof systems. Therefore, a Gentzen-style sequent calculus might be a better option. In these systems modus ponens is replaced with the cut-rule which for good systems can be eliminated or at least restricted to a well-behaved fragment of the logic in question. Secondly, as mentioned in the beginning of the introduction, ordinary sequent calculi are often ill-suited when it comes to giving well-behaved calculi for concrete intermediate logics, in that they generally do not admit cut-elimination. Therefore, keeping up with the recent trend in proof theory of non-classical logics, we base our approach on hypersequent calculi. This makes our results more general and more importantly allows us to consider more interesting examples of proof systems for intermediate logics. This approach is also in line with [13] where the results of [12] are generalised to the framework of multi-conclusion rule systems for modal logics.

We define a notion of one-step Heyting algebras and develop a theory of these algebras parallel to the theory of one-step modal algebras [12]. We show that just as in the modal case the bounded proof property for intuitionistic hypersequent calculi can be characterised algebraically using one-step Heyting algebras. We also develop a notion of intuitionistic one-step frames dual to that of one-step Heyting algebras. Finally, we test the obtained criterion for the bounded proof property on a number of examples of hypersequent calculi for intermediate logics.

[1] However, these criteria only cover the lower levels ($\mathcal{N}_2$ and $\mathcal{P}_3$) of the substructural hierarchy of [22].

[2] The basic idea of constructing finitely generated free modal algebras in an incremental way is in some sense already present in [30] and [1]. Note that [1] is based on a talk given at the BCTCS already in 1988.

[3] Recall that the *modal depth* of a formula φ is the maximal number of nestings of modalities occurring in φ.

The paper is organised as follows. In Section 2 we recall hypersequent calculi for intermediate logics, and define the bounded proof property for such calculi. In Section 3 we introduce one-step Heyting algebras and one-step intuitionistic frames and in Section 4 we provide a semantic characterisation of the bounded proof property in terms of these algebras and frames. Finally, Section 5 discusses a number of examples of calculi for intermediate logics with and without the bounded proof property.

2. Hypersequent calculi and universal classes of Heyting algebras

Let Prop be a set of propositional variables and let $Form(\mathsf{Prop})$ denote the set of formulas determined by the following grammar:

$$\varphi ::= \bot \mid p \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi, \qquad p \in \mathsf{Prop}.$$

Note that any function $\sigma\colon \mathsf{Prop} \to Form(\mathsf{Prop})$ may be extended to a function $\sigma\colon Form(\mathsf{Prop}) \to Form(\mathsf{Prop})$ in the evident way. Such functions are called *substitutions*. Given a (multi)set of formulas Γ and a substitution σ we let $\Gamma\sigma$ denote the (multi)set $\{\sigma(\varphi) \colon \varphi \in \Gamma\}$.[4]

In this paper we shall be concerned with so-called intermediate logics, i.e., consistent extensions of intuitionistic logic. We therefore recall the definition of the intuitionistic propositional calculus

Definition 2.1. The *intuitionistic propositional calculus* (**IPC**) is the smallest set of formulas containing the formulas

$$\begin{aligned}
& p \to (q \to p), \\
& (p \to (q \to r)) \to ((p \to q) \to (p \to r)), \\
& (p \wedge q) \to p, \\
& (p \wedge q) \to q, \\
& p \to (p \vee q), \\
& q \to (p \vee q), \\
& (p \to r) \to ((q \to r) \to ((p \vee q) \to r)) \\
& \bot \to p,
\end{aligned}$$

and closed under the following two inference rules

$$\frac{\varphi \qquad \varphi \to \psi}{\psi}\ (\mathrm{MP}) \qquad\qquad \frac{\varphi}{\sigma(\varphi)}\ (\mathrm{Subst})$$

where σ is any substitution.

A consistent set of formulas $L \supseteq \mathbf{IPC}$ closed under the inference rules (MP) and (Subst) is called an *intermediate logic*.

We define the *implicational degree* $d(\varphi)$ of a formula φ by the following recursion: $d(\bot) = 0$ and $d(p) = 0$ for all $p \in \mathsf{Prop}$. Moreover,

$$d(\varphi \wedge \psi) = d(\varphi \vee \psi) = \max\{d(\varphi), d(\psi)\} \quad \text{and} \quad d(\varphi \to \psi) = \max\{d(\varphi), d(\psi)\} + 1.$$

For $n \in \omega$ we let $Form_n(\mathsf{Prop})$ denote the subset of $Form(\mathsf{Prop})$ consisting of formulas of implicational degree at most n. The following observation about $Form_n(\mathsf{Prop})$ will be crucial later on.

Proposition 2.2. *If* Prop *is a finite set of propositional letters and* $n \in \omega$*, then the set* $Form_n(\mathsf{Prop})$ *of formulas of implicational degree at most* n*, modulo provable equivalence, is finite.*

Proof. Letting $Form_{-1}(\mathsf{Prop}) = \mathsf{Prop} \cup \{\bot, \top\}$, it is straightforward to verify that every formula in $Form_n(\mathsf{Prop})$ is equivalent to a formula of the form $\bigvee_{i=1}^{m} \bigwedge_{j=1}^{n_i} \psi_{ij}$ with $\psi_{ij} \in Form_{n-1}(\mathsf{Prop})$. Consequently, if Prop is a finite set of propositional letters $Form_n(\mathsf{Prop})$ must be finite for each $n \in \omega$. ☐

[4] In case Γ is a multiset $\sigma(\varphi)$ should be counted according to the multiplicity of $\varphi \in \Gamma$.

A *sequent* is a pair of finite (possible empty) multisets of formulas written as $\Gamma \Rightarrow \Delta$ and a *hypersequent* is a finite multiset of hypersequents written as

$$\Gamma_1 \Rightarrow \Delta_1 \ \mid \ \dots \ \mid \ \Gamma_n \Rightarrow \Delta_n.$$

The sequents $\Gamma_k \Rightarrow \Delta_k$, for $k \in \{1, \dots, n\}$, are called the *components* of the hypersequent.

We will let lower case letters $s, s_0, s_1, \dots$ denote sequents while upper case letters $G, H, S, S_0, S_1, \dots$ will denote hypersequents. Note that the notion of implicational degree extends to sequents and hypersequents as follows:

$$d(\Gamma \Rightarrow \Delta) = \max\{d(\varphi)\colon \varphi \in \Gamma \cup \Delta\} \quad \text{and} \quad d(s_1 \mid \dots \mid s_n) = \max\{d(s_k)\colon 1 \leq k \leq n\}.$$

Furthermore, the implicational degree of a finite set of hypersequents will be the maximal implication degree of the hypersequents in that set.

Recall that a *Heyting algebra* is a bounded distributive lattice $(A, \wedge, \vee, 0, 1)$ with an additional binary operation $\rightarrow$ satisfying

$$a \wedge c \leq b \iff c \leq a \rightarrow b.$$

Given a set Prop of propositional letters and a Heyting algebra $\mathfrak{A} = (A, \wedge, \vee, \rightarrow, 0, 1)$ a function $v\colon \mathsf{Prop} \rightarrow A$ is called a *valuation on* $\mathfrak{A}$. Such a valuation extends to a function $v\colon Form(\mathsf{Prop}) \rightarrow A$ in the evident way. A formula φ is said to be *true in* a Heyting algebra $\mathfrak{A}$ under a valuation v iff $v(\varphi) = 1$. Furthermore, a formula is said to be *valid* in $\mathfrak{A}$ if it is true under all valuations v on $\mathfrak{A}$. Using the well-known Lindenbaum-Tarski construction we obtain completeness for **IPC** with respect to the Heyting algebra semantics, see e.g. [18, Thm. 7.21]. Thus φ is a theorem of **IPC** iff it is valid in all Heyting algebras.

The above definitions may easily be extended to (hyper)sequents. To be precise: We say that a sequent $\Gamma \Rightarrow \Delta$ is true in $\mathfrak{A}$ under a valuation v, written $(\mathfrak{A}, v) \vDash \Gamma \Rightarrow \Delta$, if $v(\bigwedge \Gamma) \leq v(\bigvee \Delta)$, and we say that a hypersequent $\Gamma_1 \Rightarrow \Delta_1 \ \mid \ \dots \ \mid \ \Gamma_n \Rightarrow \Delta_n$ is true in $\mathfrak{A}$ under a valuation v, if $(\mathfrak{A}, v) \vDash \Gamma_k \Rightarrow \Delta_k$ for some $k \in \{1, \dots, n\}$. Finally, we say that $\mathfrak{A}$ *validates a sequent* (or *hypersequent*) if it is true under all valuations.

2.1. **Hypersequent proofs and hypersequent calculi.** A *hypersequent rule* (in the language of intuitionistic logic) is a pair consisting of a finite set of hypersequents $\{S_1, \dots, S_n\}$, called the *premises*, and a single hypersequent S, called the *conclusion*. We write hypersequent rules as

$$\frac{S_1 \ \dots \ S_m}{S}\,(r)$$

We define the degree of a hypersequent rule as $\max\{d(S), d(S_1), \dots, d(S_m)\}$.

Remark 2.3. Note that the definition of a hypersequent rule excludes any rule in an extended language, e.g., with quantifiers, modalities or a co-implication. However, as we are here only interested in propositional logics between **IPC** and **CPC** this limitation is of no concern to us.

Given a Heyting algebra $\mathfrak{A}$ and a hypersequent rule (r) we say that $\mathfrak{A}$ validates (r) if for each valuation v on $\mathfrak{A}$ we have that the conclusion S is true in $\mathfrak{A}$ under v if all the premisses S_j are true in $\mathfrak{A}$ under v.

Definition 2.4. Let $\{S, S_1, \dots, S_n\}$ be a set of hypersequents and let

$$\frac{S'_1 \ \dots \ S'_n}{S'}\,(r)$$

be a hypersequent rule. We say that a hypersequent S is *obtained from* $S_1, \dots, S_n$ *by an application of the rule* (r), if there exist a substitution σ and a hypersequent G such that S is of the kind $G \mid S'\sigma$ and S_i is of the kind $G \mid S'_i\sigma$ for $i \in \{1, \dots, n\}$.[5] Where given a substitution σ and a sequent $s = \Gamma \Rightarrow \Delta$ we let $s\sigma$ denote the sequent $\Gamma\sigma \Rightarrow \Delta\sigma$. Similarly given a hypersequent $S = s_1 \mid \dots \mid s_n$ we let $S\sigma$ denote the hypersequent $s_1\sigma \mid \dots \mid s_n\sigma$.

In this way uniform substitution and external weakening are taken into account in the definition of rule application.

We here present the rules for a multi-succedent hypersequent calculus for **IPC**.

[5]Due to the presence of the external weakening rule (ew) (see Definition 2.5 below), this is the same as saying that S_i is of the kind $G_i \mid S'_i\sigma$ and that S is of the kind $G \mid S'\sigma$ for some $G \supseteq \bigcup_{i=1}^n G_i$.

Definition 2.5 (see also [24])**.** The calculus $\mathbf{HLJ'}$ consists of the following rules.

Axioms:

$$\frac{}{p \Rightarrow p}\,(\text{init}) \qquad \frac{}{\bot \Rightarrow}\,(l\bot)$$

External structural rules:

$$\frac{\Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,(ec) \qquad \frac{\Gamma' \Rightarrow \Delta'}{\Gamma' \Rightarrow \Delta' \mid \Gamma \Rightarrow \Delta}\,(ew)$$

Internal structural rules:

$$\frac{\Gamma \Rightarrow p, p, \Delta}{\Gamma \Rightarrow p, \Delta}\,(ric) \qquad \frac{\Gamma, p, p \Rightarrow \Delta}{\Gamma, p \Rightarrow \Delta}\,(lic)$$

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, p \Rightarrow \Delta}\,(liw) \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow p, \Delta}\,(riw)$$

Logical rules:

$$\frac{\Gamma \Rightarrow p, \Delta \qquad \Gamma, q \Rightarrow \Delta}{\Gamma, p \to q \Rightarrow \Delta}\,(l\to) \qquad \frac{\Gamma, p \Rightarrow q}{\Gamma \Rightarrow p \to q}\,(r\to)$$

$$\frac{\Gamma, p, q \Rightarrow \Delta}{\Gamma, p \wedge q \Rightarrow \Delta}\,(l\wedge) \qquad \frac{\Gamma \Rightarrow p, \Delta \qquad \Gamma \Rightarrow q, \Delta}{\Gamma \Rightarrow p \wedge q, \Delta}\,(r\wedge)$$

$$\frac{\Gamma, p \Rightarrow \Delta \qquad \Gamma, q \Rightarrow \Delta}{\Gamma, p \vee q \Rightarrow \Delta}\,(l\vee) \qquad \frac{\Gamma \Rightarrow p, q, \Delta}{\Gamma \Rightarrow p \vee q, \Delta}\,(r\vee)$$

The cut rule:

$$\frac{\Gamma \Rightarrow p, \Delta \qquad p, \Sigma \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Pi, \Delta}\,(cut)$$

Since in this paper we are only interested in the implicational degree of formulas occurring in a derivation we may in fact take the meta-variables for contexts $\Gamma, \Delta, \Sigma, \Pi$ etc. to be single propositional letters. Consequently, we do not have to rely on rule schemes but can restrict attention to single rules.

Remark 2.6. Thus, with the exception of the external structural rules, the rule of the calculus $\mathbf{HLJ'}$ are the same as for the multi-succedent sequent calculus $\mathbf{LJ'}$ for **IPC**. The essential difference is that the definition of rule application has been changed so as to fit the framework of hypersequents. Consequently, only after adding additional hypersequent rules will it be possible to derive sequents not already derivable in $\mathbf{LJ'}$.

As we will only be interested in calculi for intermediate logics we shall understand by a *hypersequent calculus* any collection of hypersequent rules extending the calculus $\mathbf{HLJ'}$. This means that rules such as external contraction and the cut-rule belong to every hypersequent calculus even though they may be eliminable. Of course, this can no longer be guaranteed when additional rules are added.

If $\mathscr{S} \cup \{S\}$ is a set of hypersequents and HC is a hypersequent calculus we say that S is *derivable* (or *provable*) *from* $\mathscr{S}$ *over* HC, written $\mathscr{S} \vdash_{\mathrm{HC}} S$, if there exists a finite sequence of hypersequents $S_1, \ldots, S_m$ such that S_m is the hypersequent S and for all $k \in \{1, \ldots, m-1\}$ either S_k belongs to $\mathscr{S}$ or S_k is obtained by applying a rule from HC to some subset of $\{S_1, \ldots, S_{k-1}\}$. If moreover there is $n \in \omega$ such that $d(S_k) \leq n$ for all $k \in \{1, \ldots, m\}$ we write $\mathscr{S} \vdash^n_{\mathrm{HC}} S$.

Note that applying substitutions to hypersequents in $\mathscr{S}$ are not allowed. Thus $\vdash_{\mathrm{HC}}$ denotes the global consequence relation, in the sense that the members of $\mathscr{S}$ will be taken as axioms, i.e., leaves in a derivation tree.

Definition 2.7. A hypersequent rule $(S_1, \dots S_n)/S$ is *derivable* in a hypersequent calculi HC if

$$\{S_1, \dots, S_n\} \vdash_{\mathrm{HC}} S.$$

Two hypersequent calculi HC and HC$'$ are *equivalent* if all the rules of HC are derivable in HC$'$ and vice versa.

Note that if HC and HC$'$ are equivalent then for all finite sets $\mathscr{S} \cup \{S\}$ of hypersequents we have that

$$\mathscr{S} \vdash_{\mathrm{HC}} S \quad \text{iff} \quad \mathscr{S} \vdash_{\mathrm{HC}'} S.$$

The next proposition will be used throughout the paper.

Proposition 2.8. *Any hypersequent rule is equivalent (in* **HLJ$'$***) to a finite set of hypersequent rules all having single component hypersequents as premisses.*

Proof. Let $(r) = (S_1, \dots, S_n)/S$ be given and let m_1 be the number of components of S_1, say $S_1 = s_{11} \mid \dots \mid s_{1m_1}$. We show that (r) is equivalent to the set of rules

$$\frac{s_{1k} \ S_2 \ \dots \ S_n}{S} \ (r_{1k})$$

for $k \in \{1, \dots, m_1\}$. The following derivation shows that S is indeed derivable from $\{S_1, \dots, S_n\}$ using the rules $(r_{1k})_{k=1}^{m_1}$.

$$\dfrac{\dfrac{\dfrac{S_1 \ \dots \ S_n}{s_{12} \mid \dots \mid s_{1m_1} \mid S} \ (r_{11}) \qquad S_2 \ \dots \ S_n}{s_{13} \mid \dots \mid s_{1m_1} \mid S \mid S} \ (r_{12})}{s_{13} \mid \dots \mid s_{1m_1} \mid S} \ (ec)$$

$$\vdots$$

$$\dfrac{\dfrac{s_{1m_1} \mid S \qquad\qquad S_2 \ \dots \ S_n}{S \mid S} \ (r_{1m_1})}{S} \ (ec)$$

Conversely, using external weakening it follows all of the rules $(r_{1k})_{k=1}^{m_1}$ are derivable from the rule (r). Applying this procedure n times yields a finite set of hypersequent rules with single component hypersequents as premisses equivalent to (r). $\square$

In order to establish soundness and completeness of derivability of hypersequent rules with respect to Heyting algebras we will need the following facts.

Lemma 2.9. *Let $\mathscr{S} \cup \{S\}$ be a set of hypersequents and let s be a sequent. Then for every hypersequent calculus* HC *we have that*

$$(\mathscr{S} \cup \{s\} \vdash_{\mathrm{HC}} S \quad \text{and} \quad \mathscr{S} \vdash_{\mathrm{HC}} s \mid S) \quad \Longrightarrow \quad \mathscr{S} \vdash_{\mathrm{HC}} S.$$

Proof. Assuming that $\mathscr{S} \vdash_{\mathrm{HC}} s \mid S$ we see that for any hypersequent S' if $\mathscr{S} \cup \{s\} \vdash_{\mathrm{HC}} S'$, then, by induction on the length of a derivation witnessing this, we must have that $\mathscr{S} \vdash_{\mathrm{HC}} S' \mid S$. Therefore, if $\mathscr{S} \vdash_{\mathrm{HC}} s \mid S$ and $\mathscr{S} \cup \{s\} \vdash_{\mathrm{HC}} S$ we may conclude that $\mathscr{S} \vdash_{\mathrm{HC}} S \mid S$, whence by applying external contraction we obtain that $\mathscr{S} \vdash_{\mathrm{HC}} S$, as desired. $\square$

We then introduce a variant of the well-known Lindenbaum-Tarski construction.

Proposition 2.10. *For every hypersequent calculus* HC *and every set of hypersequents $\mathscr{S} \cup \{S\}$ such that $\mathscr{S} \nvdash_{\mathrm{HC}} S$ there exists a Heyting algebra $\mathfrak{LT}_{\mathrm{HC}}(\mathscr{S}, S)$ validating* HC *and a valuation on $\mathfrak{LT}_{\mathrm{HC}}(\mathscr{S}, S)$ under which all the hypersequents $\mathscr{S}$ are true but the hypersequent S is not.*

Proof. Let Prop be the set of propositional letters occurring in $\mathscr{S} \cup \{S\}$ and let $\widetilde{\mathscr{S}}$ be a maximal set of hypersequents, based on $Form(\mathsf{Prop})$, extending $\mathscr{S}$ such that $\widetilde{\mathscr{S}} \nvdash_{\mathrm{HC}} S$. By Zorn's Lemma such a set always exists. Then define an equivalence relation $\approx$ on the set of formulas $Form(\mathsf{Prop})$ as follows:

$$\varphi \approx \psi \iff \widetilde{\mathscr{S}} \vdash_{\mathrm{HC}} \Rightarrow \varphi \leftrightarrow \psi.$$

Since HC extends a hypersequent calculus of **IPC** one may readily verify that $\mathfrak{LT}_{\mathrm{HC}}(\mathscr{S}, S) = Form(\mathsf{Prop})/{\approx}$ is a Heyting algebra.

We observe that by the maximality of $\widetilde{\mathscr{S}}$, Lemma 2.9 together with the assumption that $\widetilde{\mathscr{S}} \nvdash_{\mathrm{HC}} S$ yields that

$$(1) \qquad \widetilde{\mathscr{S}} \vdash_{\mathrm{HC}} s_1 \mid \ldots \mid s_m \mid S \implies \widetilde{\mathscr{S}} \vdash_{\mathrm{HC}} s_i \text{ for some } 1 \leq i \leq m,$$

for all sequents $s_1, \ldots, s_m$. For suppose not, then in particular $\widetilde{\mathscr{S}} \nvdash_{\mathrm{HC}} s_1$ and therefore by the maximality of $\widetilde{\mathscr{S}}$ we can conclude that $\widetilde{\mathscr{S}} \cup \{s_1\} \vdash_{\mathrm{HC}} S$. So by Lemma 2.9 we must have that $\widetilde{\mathscr{S}} \vdash_{\mathrm{HC}} s_2 \mid \ldots \mid s_m \mid S$. Thus after repeating this argument m times we obtain $\widetilde{\mathscr{S}} \vdash_{\mathrm{HC}} S$, in direct contradiction with the initial assumption.

Observe that from (1) and external weakening it follows that if $\widetilde{\mathscr{S}} \vdash_{\mathrm{HC}} s_1 \mid \ldots \mid s_m$ then $\widetilde{\mathscr{S}} \vdash_{\mathrm{HC}} s_i$ for some $i \in \{1, \ldots, m\}$. From this it is easy to verify that $\mathfrak{LT}_{HC}(\mathscr{S}, S)$ validates all the rules of HC.

Finally, we claim that under the valuation determined by sending propositional variables to their respective equivalence classes of the equivalence relation $\approx$, the algebra $\mathfrak{LT}_{HC}(\mathscr{S}, S)$ makes all the hypersequents from $\mathscr{S}$ true but does not make the hypersequent S true. This, however, is evident. ☐

Remark 2.11. One could initially be tempted to believe that the construction in the proof of Proposition 2.10 will yield free algebras for the universal class of Heyting algebras validating the calculus HC. However, this is not the case. Indeed, there exist universal classes of algebras without free algebras [37, Cor. 3]. To see why the construction fails to produce free algebras note that $\varphi \approx \psi$ in the above construction (taking $\mathscr{S} = \emptyset$) does not imply that the corresponding terms are identified in all Heyting algebras validating HC, only that they may consistently (relative to HC) be identified. Consequently, given a Heyting algebra $\mathfrak{A}$ validating HC and a function $v \colon \mathsf{Prop} \to \mathfrak{A}$, i.e., a valuation on $\mathfrak{A}$, it can not be ensured that $[\varphi]_{\approx} \mapsto v(\varphi)$ will be a well-defined homomorphism from $\mathfrak{LT}_{\mathrm{HC}}(\mathscr{S}, S)$ to $\mathfrak{A}$. This is because the equivalence relation depends both on the hypersequent S and and on some maximally consistent set $\widetilde{\emptyset}$ such that $\widetilde{\emptyset} \nvdash_{\mathrm{HC}} S$.

Proposition 2.12 (Algebraic soundness and completeness)**.** *Let* HC *be a hypersequent calculus and let* (r) *be a hypersequent rule. Then the following are equivalent:*

(1) *The rule* (r) *is derivable in* HC*;*
(2) *All Heyting algebras validating* HC *also validates* (r)*.*

Proof. That item 1 implies item 2 follows from a straightforward induction on the length of derivations of rules. That item 2 implies item 1 is an immediate consequence of Proposition 2.10. ☐

2.2. **Hypersequents calculi, multi-conclusion rules and universal classes of Heyting algebras.** Given a hypersequent calculus HC we obtain an intermediate logic $\Lambda(\mathrm{HC}) := \{\varphi \in Form(\mathsf{Prop}) \colon \vdash_{\mathrm{HC}} \Rightarrow \varphi\}$. We say that a hypersequent calculus HC is a calculus for an intermediate logic L if $\Lambda(\mathrm{HC}) = L$. This means that derivability relations $\vdash_L$ and $\vdash_{\mathrm{HC}}$ coincides for sequents in the sense that

$$(2) \qquad \vdash_L \bigwedge \Gamma \to \bigvee \Delta \quad \text{iff} \quad \vdash_{\mathrm{HC}} \Gamma \Rightarrow \Delta$$

holds for all sequents $\Gamma \Rightarrow \Delta$.

Given a hypersequent calculus HC the class $\mathcal{U}(\mathrm{HC})$ of Heyting algebras validating HC will evidently be a universal class. Conversely, given a universal class U of Heyting algebras, determined by a set of universal sentences Φ, we obtain a hypersequent calculus $\mathcal{HC}(U)$ by adding for each universal sentence $\sigma = \forall \underline{x} (\bigwedge_{k=1}^{m} (\varphi_k(\underline{x}) = 1) \implies \bigvee_{l=1}^{m} (\psi_l(\underline{x}) = 1)) \in \Phi$ the rule

$$\frac{\Rightarrow \varphi_1 \ldots \Rightarrow \varphi_n}{\Rightarrow \psi_1 \mid \ldots \mid \Rightarrow \psi_m}\,(r_\sigma)$$

to the hypersequent calculus $\mathbf{HLJ'}$. Of course, in concrete instances appropriate invertible rules of $\mathbf{HLJ'}$ may then be applied to obtain a cleaner version of the rule (r_σ). Here we are tacitly identifying quantifier-free formulas in the language of Heyting algebras and propositional formulas in the language of intuitionistic logic[6].

Using Proposition 2.12 it is easy to verify that $\mathcal{U}(\mathcal{HC}(U)) = U$ and that $\mathcal{HC}(\mathcal{U}(\mathrm{HC}))$ will be equivalent to HC. Thus we have a one-to-one correspondence between hypersequent calculi for intermediate logics (modulo equivalence) and universal classes of Heyting algebras.

Similarly we obtain a correspondence between multi-conclusion rules [39, 9] and hypersequent calculi. Given a multi-conclusion rule $(r) = (\varphi_1, \ldots, \varphi_n)/(\psi_1, \ldots, \psi_m)$ we obtain a hypersequent rule:

$$\frac{\Rightarrow \varphi_1 \ldots \Rightarrow \varphi_n}{\Rightarrow \psi_1 \mid \ldots \mid \Rightarrow \psi_m} \ (r_H)$$

Again, in concrete instances appropriate invertible rules of $\mathbf{HLJ'}$ may then be applied to obtain a cleaner version of the rule (r_H).

Conversely, given a hypersequent rule with single component premises

$$\frac{\Gamma_1 \Rightarrow \Delta_1 \ldots \Gamma_n \Rightarrow \Delta_n}{\Sigma_1 \Rightarrow \Pi_1 \mid \ldots \mid \Sigma_m \Rightarrow \Pi_m} \ (r)$$

we obtain a multi-conclusion rule:

$$\frac{\bigwedge \Gamma_1 \to \bigvee \Delta_1, \ldots, \bigwedge \Gamma_n \to \bigvee \Delta_n}{\bigwedge \Sigma_1 \to \bigvee \Pi_1, \ldots, \bigwedge \Sigma_m \to \bigvee \Pi_m} \ (r_M)$$

Example 2.13. Consider the universal clause

$$(\sigma) \qquad \forall x \forall y\ (x \wedge y \leq 0 \implies x \leq 0 \text{ or } y \leq 0).$$

This corresponds to the single rule:

$$\frac{p \wedge q \Rightarrow \bot}{p \Rightarrow \bot \mid q \Rightarrow \bot} \ (r_\sigma)$$

Finally, the rule (r_σ) may be transformed into the multi-conclusion rule

$$\frac{\neg(p \wedge q)}{\neg p, \neg q} \ ((r_\sigma)_M)$$

Evidently a Heyting algebra validates a multi-conclusion rule (resp. hypersequent rule) iff it validates the corresponding hypersequent rule (resp. multi-conclusion rule). Since by Proposition 2.8 every hypersequent calculus is equivalent to one only consisting of rules with single component premisses this yields (modulo equivalence) a correspondence between multi-conclusion consequence relations and hypersequent calculi. Thus, for the purposes of axiomatising intermediate logics hypersequent calculi and multi-conclusion consequence relations may be used interchangeably.

2.3. **The bounded proof property.** We say that a hypersequent calculus HC has the *bounded proof property* if whenever $\mathscr{S} \cup \{S\}$ is a set of hypersequents of implicational degree at most n such that $\mathscr{S} \vdash_{\mathrm{HC}} S$ then $\mathscr{S} \vdash^n_{\mathrm{HC}} S$, i.e., there exists a proof witnessing $\mathscr{S} \vdash_{\mathrm{HC}} S$ consisting only of hypersequents of degree at most n. The bounded proof property is thus a very weak form of analyticity[7] in the sense that having the bounded proof property ensures that some backward proof search strategy will be applicable. However, having the bounded proof property will indicate some kind of robustness of the hypersequent calculus in question. For instance the subformula property will entail the bounded proof property. Therefore, if a hypersequent calculus enjoys cut-elimination it will also, under mild additional assumptions, have the subformula property and hence the bounded proof property. Finally, as in the modal case [12, 13], having the bounded proof property will ensure that the derivability relation $\vdash_{\mathrm{HC}}$ is decidable, given that HC consists of finitely many rules. This is due to the fact that by Proposition 2.2 for a given finite set of propositional variables Prop there are only finitely many non-equivalent formulas in Prop of implicational degree at most n.

[6]Formally this is done by fixing a bijection between the set of variables of the first-order language and the set of propositional letters which may then be extended in the evident way.

[7]Recall that a calculus is *analytic* if it enjoys the subformula property, i.e., derivations using only subformulas can always be found.

As follows from the definition, checking whether or not a hypersequent calculus HC enjoys the bounded proof property we are required to check for every $n \in \omega$ if $\mathscr{S} \vdash_{\mathrm{HC}} S$ entails $\mathscr{S} \vdash^{n}_{\mathrm{HC}} S$ for each set $\mathscr{S} \cup \{S\}$ of hypersequents of degree at most n. However, for each such set $\mathscr{S} \cup \{S\}$ of hypersequents, by replacing the inner most implications $\varphi \to \psi$ in the formulas of $\mathscr{S} \cup \{S\}$ with fresh variables, say $p_{\varphi\psi}$, and adding appropriate premisses we obtain a set of hypersequents $\mathscr{S}' \cup \{S'\}$ of degree $n-1$ with the property

$$\mathscr{S} \vdash_{\mathrm{HC}} S \iff \mathscr{S}' \vdash_{\mathrm{HC}} S'.$$

Moreover, if $\mathscr{S}' \vdash^{n-1}_{\mathrm{HC}} S'$ is witnessed by a derivation $\mathscr{D}$, then we obtain a derivation witnessing $\mathscr{S} \vdash^{n}_{\mathrm{HC}} S$ by replacing all the fresh variables $p_{\varphi\psi}$ occurring in $\mathscr{D}$ with the corresponding formulas $\varphi \to \psi$ of degree one. Thus, if we know that HC satisfies the bounded proof property for all hypersequents of degree at most $n-1$, then it must also satisfy it for hypersequents of degree at most n. Using this idea we may show that the bounded proof property is completely determined by the degree 1 case.

Proposition 2.14. *A hypersequent calculus* HC *has the bounded proof property iff for each set* $\mathscr{S} \cup \{S\}$ *consisting of hypersequents of degree at most* 1*, we have*

$$\mathscr{S} \vdash_{\mathrm{HC}} S \quad \textit{iff} \quad \mathscr{S} \vdash^{1}_{\mathrm{HC}} S.$$

Proof. The left-to-right direction is evident.

For the converse implication let $\mathscr{S} \cup \{S\}$ be a set of hypersequents of degree at most n. We define a sequence of triples $(\mathscr{S}_i, S_i, \sigma_i)_{i=0}^{n-1}$ such that

(i) $\mathscr{S}_i \cup \{S_i\}$ is a set of hypersequents of degree at most $n-i$ and σ_i is a substitution such that $d(\sigma_i(\chi)) \leq d(\chi) + 1$ for all formulas χ occurring in $\mathscr{S}_i \cup \{S_i\}$;
(ii) $S_{i+1}\sigma_{i+1} = S_i$;
(iii) $\mathscr{S}_{i+1}\sigma_{i+1}$ equals $\mathscr{S}_i$ union some set of sequents of the form $\chi \Rightarrow \chi$;
(iv) $\mathscr{S}_{i+1} \vdash_{\mathrm{HC}} S_{i+1} \iff \mathscr{S}_i \vdash_{\mathrm{HC}} S_i$.

Let $\mathscr{S}_0$ be $\mathscr{S}$, S_0 be S and let σ_0 be the identity substitution. Now assume that the triple $(\mathscr{S}_i, S_i, \sigma_i)$ has been defined. Then for each subformula of the form $\varphi \to \psi$ with $d(\varphi) = d(\psi) = 0$ occurring in some formula of some sequent of some hypersequent in $\mathscr{S}_i \cup \{S_i\}$ we introduce a fresh variable $p_{\varphi\psi}$ and replace $\varphi \to \psi$ with $p_{\varphi\psi}$ everywhere. Let $\mathscr{S}'_i$ and S_{i+1} be the result of such replacements. Finally, let

$$\mathscr{S}_{i+1} = \mathscr{S}'_i \cup \{p_{\varphi\psi} \Rightarrow \varphi \to \psi,\ \varphi \to \psi \Rightarrow p_{\varphi\psi}\}_{\varphi \to \psi}.$$

The substitution σ_{i+1} is then defined as $\sigma_{i+1}(p_{\varphi\psi}) = \varphi \to \psi$.

With this definition (i)-(iv) are easily seen to hold.

Now if $\mathscr{S} \vdash_{\mathrm{HC}} S$ then by construction we must have that $\mathscr{S}_{n-1} \vdash_{\mathrm{HC}} S_{n-1}$. Moreover, by item (i) the degree of $\mathscr{S}_{n-1} \cup \{S_{n-1}\}$ is at most 1, hence the initial hypothesis yields $\mathscr{S}_{n-1} \vdash^{1}_{\mathrm{HC}} S_{n-1}$. From items (ii) and (iii) together with the fact that for any hypersequent S we have that $d(S\sigma_{i+1}) \leq d(S)+1$ we observe that if $\mathscr{S}_{n-k} \vdash^{k}_{\mathrm{HC}} S_{n-k}$ then $\mathscr{S}_{n-(k+1)} \vdash^{k+1}_{\mathrm{HC}} S_{n-(k+1)}$, for all $k \in \{0, \ldots, n-1\}$. Consequently, $\mathscr{S}_{n-1} \vdash^{1}_{\mathrm{HC}} S_{n-1}$ entails that $\mathscr{S}_0 \vdash^{n}_{\mathrm{HC}} S_0$. ◻

The polarity (positive or negative) of an occurrence of a subformula in a formula is given by the following recursive definition. The formula φ is positive in φ. The connectives $\wedge$ and $\vee$ preserves polarities while the connective $\to$ preserve polarities in the consequent and reverses polarities in the antecedent. We say that a formula ψ *occurs* in a sequent $\Gamma \Rightarrow \Delta$ if it is a subformula of a formula $\varphi \in \Gamma \cup \Delta$. If ψ occurs positively (resp. negatively) in φ and $\varphi \in \Gamma$ then we count the occurrence of ψ in $\Gamma \Rightarrow \Delta$ as negative (resp. positive) and vice versa if $\varphi \in \Delta$. Finally, ψ is said to occur positively (resp. negatively) in a hypersequent if it occurs positively (resp. negatively) in some component.

We say that a hypersequent rule (r) is *reduced* if all the formulas occurring in (r) have implicational degree at most 1. Evidently not all rules will be reduced. However, every rule (r) may be transformed into an equivalent rule (r') which is reduced—of course, there is nothing that guarantees that (r) and (r') will share the same proof-theoretic properties. Consider, for example, the rule

$$\frac{}{\neg q \to p \Rightarrow ((p \to q) \to p) \to p}\ (r_0)$$

of degree 3. As $(p \to q) \to p$ has exactly one occurrence (with negative polarity) we may introduce fresh propositional letter r for this formula and add a new premise $r \Rightarrow (p \to q) \to p$ to obtain the equivalent rule

$$\frac{r \Rightarrow (p \to q) \to p}{\neg q \to p \Rightarrow r \to p}\,(r_1)$$

of degree 2. In (r_1) the formula $(p \to q)$ has exactly one occurrence (again with negative polarity) and $\neg q$ has exactly one occurrence (with positive polarity). We may thus abstract these occurrence away with fresh propositional letters s and t, respectively, to obtain the equivalent rule

$$\frac{r \Rightarrow s \to p \qquad s \Rightarrow p \to q \qquad \neg q \Rightarrow t}{t \to p \Rightarrow r \to p}\,(r_2)$$

of degree 1.

The following proposition shows that such a transformation may always be performed.

Proposition 2.15. *Any hypersequent rule is equivalent to a reduced hypersequent rule.*

Proof. Given a hypersequent rule $(r) = (S_1, \dots, S_m)/S_{m+1}$ of depth $n+1$ with $n \geq 1$ and an *occurrence* of a formula α of degree $n+1$ in (r) the main connective of which is $\to$, we produce an equivalent rule with one less occurrence of the formula α.

Let S_i be the hypersequent with the given occurrence of α and let $\Gamma \Rightarrow \Delta$ be the sequent in S_i with the given occurrence of α. As the formula α is of depth $n+1$ it must be of the form $\varphi \to \psi$ with $\max\{d(\varphi), d(\psi)\} = n$. We introduce a fresh variable p and replace the given occurrence of α in S_i with $p \to \psi$ or $\varphi \to p$, depending on whether $d(\varphi) = n$ or $d(\psi) = n$. If both $d(\varphi)$ and $d(\psi) = n$ we introduce two fresh variables. Let S_i' be the hypersequent resulting from such a replacement. Evidently S_i' has one less occurrence of the formula α than S_i. In case $i \leq m$ let S_i'' be the hypersequent obtained by replacing the sequent $\Gamma \Rightarrow \Delta$ in S_i with the sequent $\varphi \Rightarrow p$ or $p \Rightarrow \psi$ depending on whether $d(\varphi) = n$ or $d(\psi) = n$. In case $i = m+1$ let S_i'' be the hypersequent consisting of the single component hypersequent $p \Rightarrow \varphi$ or $\psi \Rightarrow p$ depending on whether $d(\varphi) = n$ or $d(\psi) = n$.

In this way we obtain a rule

$$\frac{S_1 \ \dots \ S_{i-1}\ S_i'\ S_{i+1}\ \dots\ S_m\ S_i''}{S_{m+1}}\,(r') \qquad \text{or} \qquad \frac{S_1\ \dots\ S_m\ S_{m+1}''}{S_{m+1}'}\,(r')$$

depending on whether $i \leq m$ or $i = m+1$.

By Proposition 2.12 (or by appropriate applications of the cut-rule) this rule must be equivalent to the rule (r).

Continuing this procedure for each occurrence of a formula of degree $n+1$ in (r) we obtain a rule (r_n) of degree n which is equivalent to (r). In this way we obtain a sequence $(r_{n+1}), (r_n), \dots, (r_1)$ of equivalent rules such that $(r_{n+1}) = (r)$ and $d(r_k) = k$, for all $k \in \{1, \dots, n+1\}$. □

Remark 2.16. Note as the above procedure abstracts away one occurrence of a formula of the form $\varphi \to \psi$ at a time, and since we first abstract away outermost occurrences, it is always clear whether to replace the formula occurring negatively or positively in the formula $\varphi \to \psi$. Note further that this procedure works just as well for the other connectives. The only thing particular to the implication is that the polarity is reversed in the antecedent.

In light of Proposition 2.15 we may without loss of generality assume that all hypersequent calculi are reduced, i.e., only consisting of reduced rules. In the following section we shall introduce algebraic structures which may interpret such reduced rules.

3. One-step Heyting algebras

Let bDL denote the category of bounded distributive lattices and bounded lattice homomorphisms. Then a well-known theorem by Birkhoff states that the category bDL_ω of finite bounded distributive lattice is dually equivalent to the category Pos_ω of finite posets and order-preserving maps, for details see e.g., [29, Chap. 5]. This duality is established via the downsets functor $\mathrm{Do}\colon \mathsf{Pos} \to \mathsf{bDL}$ and the functor $J\colon \mathsf{bDL} \to \mathsf{Pos}$ mapping a bounded distributive lattice D to the poset of join-irreducible elements of D. If $f\colon P \to P'$ is an order-preserving map between posets then $\mathrm{Do}(f)\colon \mathrm{Do}(P') \to \mathrm{Do}(P)$ is the preimage function $f^*(U) := f^{-1}(U)$.

If $h\colon D \to D'$ is a homomorphism between finite bounded distributive then h has a left adjoint $h^\flat\colon D' \to D$ given by

$$h^\flat(a') := \bigwedge_{a' \le h(a)} a.$$

We may therefore let $J(h)\colon J(D') \to J(D)$ be $h^\flat \restriction J(D')$.

It is well known that any finite bounded distributive lattice D is in fact a Heyting algebra with Heyting implication defined as

$$a \to b := \bigwedge\{c\colon a \wedge c \le b\}.$$

Therefore the category HA_ω of finite Heyting algebras and Heyting algebra homomorphisms is a (non-full) subcategory of bDL_ω. Let $\mathsf{Pos}^{\text{open}}$ denote the category of posets and open order-preserving maps, where a map between posets $f\colon P \to Q$ is *open* if

$$\forall a \in P\ \forall b \in Q\ (b \le f(a) \implies \exists a' \in P\ (a' \le a \text{ and } f(a') = b)).$$

Theorem 3.1 (Folklore)**.** *The dual equivalence of the categories bDL_ω and Pos_ω restricts to a dual equivalence between the categoires HA_ω and $\mathsf{Pos}_\omega^{open}$.*

We now introduce algebraic structures which may interpret the fragment of intuitionistic logic consisting of formulas of implicational degree at most 1.

Definition 3.2. A *one-step Heyting algebra* is a triple (D_0, D_1, i) such that $i\colon D_0 \to D_1$ is a homomorphism between bounded distributive lattices with the property that for all $a, b \in D_0$ the Heyting implication $i(a) \to i(b)$ exists in D_1. We say that a one-step Heyting algebra (D_0, D_1, i) is *conservative* if $i\colon D_0 \to D_1$ is an embedding of bounded distributive lattices and D_1 is generated (as a bounded distributive lattice) by the set $\{i(a) \to i(b)\colon a, b \in D_0\}$. Finally, we say that (D_0, D_1, i) is *finite* if both D_0 and D_1 are finite.

Remark 3.3. In [12] a *one-step modal algebra* was defined to be a quadruple $(A_0, A_1, i, \Diamond)$ such that $i\colon A_0 \to A_1$ is a Boolean algebras homomorphism between Boolean algebras A_0 and A_1 and $\Diamond\colon A_0 \to A_1$ is a map preserving $0, \vee$. Thus the main conceptual difference between one-step modal algebras and their Heyting algebra counterparts is that since for any finite distributive lattice there is only one choice of a Heyting implication it is not necessary to consider an additional operation $\to\colon D_0^2 \to D_1$, satisfying appropriate equations, as part of the definition. his makes the one-step Heyting algebras somewhat simpler to work with. In particular, the duals of finite one-step Heyting algebras are simpler than the duals of finite one-step modal algebras. Instead of working with a relation between two different sets we may simply work with order-preserving maps between two standard intuitionistic Kripke frames.

Definition 3.4. A *one-step homomorphism* between two one-step Heyting algebras $\mathcal{H} = (D_0, D_1, i)$ and $\mathcal{H}' = (D_0', D_1', i')$ is a pair (g_0, g_1) of bounded lattice homomorphisms $g_0\colon D_0 \to D_0'$ and $g_1\colon D_1 \to D_1'$ making the diagram

$$\begin{array}{ccc} D_0 & \xrightarrow{g_0} & D_0' \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i'} \\ D_1 & \xrightarrow{g_1} & D_1' \end{array}$$

commute, such that for all $a, b \in D_0$

$$g_1(i(a) \to i(b)) = g_1(i(a)) \to g_1(i(b)).$$

A *one-step extension* of a one-step Heyting algebra $\mathcal{H}_0 := (D_0, D_1, i_0)$ is a one-step Heyting algebra $\mathcal{H}_1 := (D_1, D_2, i_1)$ such that $(i_0, i_1)\colon \mathcal{H}_0 \to \mathcal{H}_1$ is a one-step homomorphism with i_1 injective.

Note that if $\mathfrak{A}$ is a Heyting algebra, then $\mathcal{H}_{\mathfrak{A}} = (\mathfrak{A}, \mathfrak{A}, \mathrm{Id})$ is a one-step Heyting algebra. Consequently, we may, given a one-step Heyting algebra $\mathcal{H}$, speak of one-step homomorphism between $\mathfrak{A}$ and $\mathcal{H}$ by way of $\mathcal{H}_{\mathfrak{A}}$.

The above definitions determines a category OSHA of one-step Heyting algebras and one-step homomorphisms between them. This is a non-full subcategory of the arrow category $\mathsf{bDL}^{\to}$. We let OSHA_ω and $\mathsf{OSHA}_\omega^{\text{cons}}$ denote the full subcategories of OSHA consisting of finite one-step Heyting algebras and finite conservative one-step Heyting algebras, respectively.

3.1. **Duality.** Since in the following we are only concerned with finite one-step Heyting algebras the duality is particularly well behaved. We construct categories dually equivalent to the categories OSHA_ω and $\mathsf{OSHA}_\omega^{\mathrm{cons}}$. To this end we need the following proposition.

Proposition 3.5. *Let $f\colon P \to Q$ and $g\colon Q \to R$ be order-preserving maps between finite posets. Then the following are equivalent:*

(1) *The bounded lattice homomorphism $f^*\colon \mathrm{Do}(Q) \to \mathrm{Do}(P)$ preserves all Heyting implications of the form $g^*(U) \to g^*(V)$, for $U, V \in \mathrm{Do}(R)$;*
(2) $\forall a \in P\ \forall b \in Q\ (b \leq f(a) \implies \exists a' \in P\ (a' \leq a \text{ and } g(f(a')) = g(b)))$

Proof. Straightforward. □

Definition 3.6 ([31])**.** Given order-preserving maps $f\colon P \to Q$ and $g\colon Q \to R$ satisfying one (and therefore both) of the conditions of Proposition 3.5 we say that f is *open relative to g* or simply that f is *g-open*.

Definition 3.7. An *intuitionistic one-step frame* is a triple (P_1, P_0, f) such that $f\colon P_1 \to P_0$ is an order-preserving map between posets. We say that an intuitionistic one-step frame $(P_1, P_0.f)$ is *conservative* if $f\colon P_1 \to P_0$ is a surjection satisfying

$$\forall a, b \in P_1\ (f[{\downarrow}a] \subseteq f[{\downarrow}b] \implies a \leq b).$$

Definition 3.8. A one-step map from an intuitionistic one-step frame $\mathcal{F}' = (P_1', P_0', f')$ to an intuitionistic one-step frame $\mathcal{F} = (P_1, P_0, f)$ is a pair (μ_1, μ_0) of order-preserving maps $\mu_1\colon P_1' \to P_1$ and $\mu_0\colon P_0' \to P_0$, where μ_1 is f-open, making the diagram

$$\begin{array}{ccc} P_1' & \xrightarrow{\mu_1} & P_1 \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ P_0' & \xrightarrow{\mu_0} & P_0 \end{array}$$

commute.

A *one-step extension* of an intuitionistic one-step frame $\mathcal{F}_0 = (P_1, P_0, f_0)$ is an intuitionistic one-step frame $\mathcal{F}_1 = (P_2, P_1, f_1)$ such that $(f_1, f_0)\colon \mathcal{F}_1 \to \mathcal{F}_0$ is a one-step map, with f_1 surjective.

It is easy to check that this yields a category IOSFrm of intuitionistic one-step frames and one-step maps. Moreover, the finite and the finite conservative intuitionistic one-step algebras form full subcategories IOSFrm_ω and $\mathsf{IOSFrm}_\omega^{\mathrm{cons}}$ of IOSFrm, respectively.

Note that if $\mathfrak{F}$ is an intuitionistic Kripke frame then $\mathcal{F}_{\mathfrak{F}} = (\mathfrak{F}, \mathfrak{F}, \mathrm{Id})$ will be an intuitionistic one-step frame. Consequently, we may, given an intuitionistic one-step frame $\mathcal{F}$, speak of one-step homomorphism between $\mathfrak{F}$ and $\mathcal{F}$ by way of $\mathcal{F}_{\mathfrak{F}}$.

Proposition 3.9. *The categories OSHA_ω and IOSFrm_ω are dually equivalent. Moreover, this dual equivalence restricts to a dual equivalence between the categories $\mathsf{OSHA}_\omega^{cons}$ and $\mathsf{IOSFrm}_\omega^{cons}$.*

Proof. That the duality between bDL_ω and Pos_ω extends to a duality between the categories OSHA_ω and IOSFrm_ω is straightforward given Proposition 3.5.

To see that the dual equivalence between OSHA_ω and IOSFrm_ω restricts to a dual equivalence between $\mathsf{OSHA}_\omega^{\mathrm{cons}}$ and $\mathsf{IOSFrm}_\omega^{\mathrm{cons}}$ it suffices to note that under the isomorphism between the poset of bounded sublattices of $\mathrm{Do}(P)$ and the poset of compatible quasi-orders on P ([43, Thm. 3.7], [7, Thm. 6.15]) the sublattice generated by the set $\mathscr{U} \subseteq \mathrm{Do}(P)$ corresponds to the compatible quasi-order $\preceq_{\mathscr{U}}$ given by

$$a \preceq_{\mathscr{U}} b \quad \text{iff} \quad \forall U \in \mathscr{U}\ (b \in U \implies a \in U).$$

Thus $\mathscr{U} \subseteq \mathrm{Do}(P)$ generates $\mathrm{Do}(P)$ as a bounded distributive lattice iff the quasi-order $a \preceq_{\mathscr{U}} b$ coincides with the order on P.

From this it is easy to see that (P_1, P_0, f) is a conservative intuitionistic one-step frame if and only if $(\mathrm{Do}(P_0), \mathrm{Do}(P_1), f^*)$ is a conservative one-step Heyting algebra. □

3.2. **One-step semantics.** Recall from Section 2 that for a given set Prop of propositional letters the set $Form_n(\mathsf{Prop})$ is the subset of $Form(\mathsf{Prop})$ consisting of formulas of implicational degree at most n. We show how one-step Heyting algebras may interpret reduced hypersequent rules, i.e., rules using only formulas from $Form_1(\mathsf{Prop})$.

Given two disjoint finite sets Prop_0 and Prop_1 of propositional variables, a *valuation* on a one-step algebra $\mathcal{H} = (D_0, D_1, i)$ is a pair of functions $v = (v_0, v_1)$ such that $v_0 \colon \mathsf{Prop}_0 \to D_0$ and $v_1 \colon \mathsf{Prop}_1 \to D_1$.

Given a one-step algebra $\mathcal{H}$ together with a valuation $v = (v_0, v_1)$ for every formula $\varphi(\vec{p}) \in Form_0(\mathsf{Prop}_0)$ we define an element $\varphi^{\mathrm{v}_0} \in D_0$ as follows:

$$\bot^{\mathrm{v}_0} = 0 \quad \text{and} \quad \top^{\mathrm{v}_0} = 1 \quad \text{and} \quad p_i^{\mathrm{v}_0} = v_0(p_i) \quad \text{for } p_i \in \vec{p},$$

and

$$(\varphi_1 * \varphi_2)^{\mathrm{v}_0} = \varphi_1^{\mathrm{v}_0} * \varphi_2^{\mathrm{v}_0}, \qquad * \in \{\wedge, \vee\}.$$

Moreover, for every formula $\psi(\vec{p}, \vec{q}) \in Form_1(\mathsf{Prop}_0 \cup \mathsf{Prop}_1)$, where the elements of $\vec{q} \subseteq \mathsf{Prop}_1$ do not have any occurrence in the scope of an implication, we define an element $\psi^{\mathrm{v}_1} \in D_1$ as follows:

$$\bot^{\mathrm{v}_1} = 0 \quad \text{and} \quad q^{\mathrm{v}_1} = v_1(q) \quad \text{and} \quad p^{\mathrm{v}_1} = i(v_0(p)) \quad \text{for } q \in \vec{q} \text{ and } p \in \vec{p},$$

and

$$(\psi_1 * \psi_2)^{\mathrm{v}_1} = \psi_1^{\mathrm{v}_1} * \psi_2^{\mathrm{v}_1} \qquad * \in \{\wedge, \vee, \to\}.$$

To see that this is well defined note that if the the main connective of $\psi(\vec{p}, \vec{q})$ is $\to$, say $\psi_1(\vec{p}, \vec{q}) \to \psi_2(\vec{p}, \vec{q})$, then we must have $\vec{q} = \emptyset$, whence $\psi_1, \psi_2 \in Form_0(\mathsf{Prop}_0)$. By the definition of a one-step Heyting algebra the implications of the form $i(a) \to i(a)$ exist in D_1 and so the above is indeed well defined.

Since the function i preserves 0 as well as the connectives $\wedge$ and $\vee$ it is easily seen that $i(\varphi^{\mathrm{v}_0}) = \varphi^{\mathrm{v}_1}$, for all $\varphi \in Form_0(\mathsf{Prop}_0)$.

A valuation $v = (v_0, v_1)$ on a one-step algebra $\mathcal{H}$ is *suitable* for an expression (i.e., for a formula, sequent, or hypersequent) ϵ of degree at most 1 iff the domain of v_0 includes all propositional variables having in ϵ an occurrence located inside an implication; a *0-valuation* is a valuation $v = (v_0, v_1)$ where the domain of v_1 is empty (thus, a 0-evaluation is always suitable for any expression ϵ).

We say that a sequent $\Gamma \Rightarrow \Delta$ of degree at most 1 is *true* in one-step algebra $\mathcal{H}$ *under a suitable valuation* $v = (v_0, v_1)$ if

$$\left(\bigwedge \Gamma\right)^{\mathrm{v}_1} \le \left(\bigvee \Delta\right)^{\mathrm{v}_1},$$

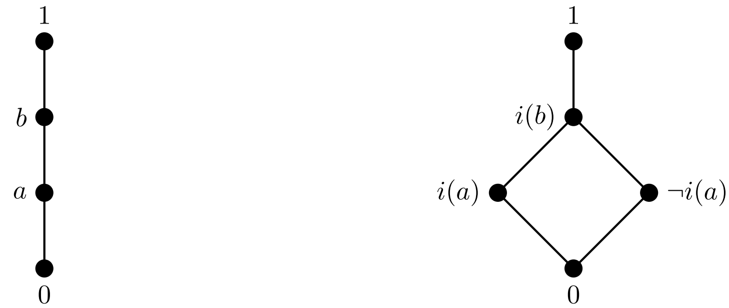
with the convention that $\bigwedge \emptyset = \top$ and $\bigvee \emptyset = \bot$.

A hypersequent S is true in a one-step algebra $\mathcal{H}$ under a suitable valuation v if at least one of the components of S is true in $\mathcal{H}$ under v. We write $(\mathcal{H}, v) \models S$, if this is the case.

Finally, we say that a one-step algebra $\mathcal{H}$ validates a hypersequent S if it is true under all possible suitable valuations v on $\mathcal{H}$, in which case we write $\mathcal{H} \models S$. Moreover, if $(r) = (S_1, \ldots, S_n)/S$ is a hypersequent rule of degree at most 1 write $(\mathcal{H}, v) \models (r)$, for a suitable valuation v, if $(\mathcal{H}, v) \models S$, whenever $(\mathcal{H}, v) \models S_i$, for all $i \in \{1, \ldots, n\}$. We say that $\mathcal{H}$ validates (r) if for all suitable valuations v on $\mathcal{H}$ we have that $(\mathcal{H}, v) \models (r)$.

We say that an intuitionistic one-step frame $\mathcal{F} = (P_1, P_0, f)$ validates a sequent, hypersequent or hypersequent rule if its dual one-step Heyting algebra $\mathcal{F}^* = (\mathrm{Do}(P_0), \mathrm{Do}(P_1), f^*)$ does. The notion of *0-validation* of a sequent, hypersequent or hypersequent rule is defined in the same way, by restricting to 0-valuations.

Example 3.10. Consider the one-step Heyting algebra (D_0, D_1, i) drawn below



D_0 D_1

It is straightforward to verify that this is a finite conservative one-step Heyting algebra. Moreover, (D_0, D_1, i) will validate the formula $(p \to q) \vee (q \to p)$. This is despite the fact that D_1, considered as a Heyting algebra on its own, does not validate this formula. The point of the one-step semantics is thus that it allows us to restrict the kinds of valuations we allow on D_1 to valuations taking values in a certain bounded sublattice D_0 of D_1.

With these definitions we can establish the soundness of the derivability relation with respect to the one-step semantics. There is a subtlety to take care of here, however: A propositional variable p not occurring in a set of hypersequents $\mathscr{S} \cup \{S\}$ under the scope of an implication may still occur inside an implication in a derivation witnessing $\mathscr{S} \vdash^1_{\mathrm{HC}} S$. Thus, if p is in the domain of v_1 when we evaluate S in $\mathcal{H}$, it may happen that we cannot give a meaning to such a derivation inside $\mathcal{H}$. This is why the correct semantics for the relation $\mathscr{S} \vdash^1_{\mathrm{HC}} S$ requires the restriction to 0-valuations for $\mathscr{S} \cup \{S\}$ (but not for the rules of HC, because the variables from the latter can be instantiated indifferently with formulas of degree 0 or 1).

Proposition 3.11. *Let $\mathcal{H}$ be a one-step algebra,* HC *a reduced hypersequent calculus, and $\mathscr{S} \cup \{S\}$ a set of hypersequents of degree at most* 1*. If $\mathscr{S} \vdash^1_{\mathrm{HC}} S$ and $\mathcal{H}$ validates* HC*, then $\mathcal{H}$ 0-validates $\mathscr{S}/S$.*

Proof. By induction on the length of a derivation witnessing $\mathscr{S} \vdash^1_{\mathrm{HC}} S$ (notice that we can assume that in such a derivation only propositional variables occurring in $\mathscr{S} \cup \{S\}$ occur, because extra variables can be replaced by, say, $\top$). □

If $(g_0, g_1)\colon \mathcal{H} \to \mathcal{H}'$ is a one-step homomorphism such that both g_0 and g_1 are injective then we say that (g_0, g_1) is an embedding. The following lemma shows that the embeddings between one-step Heyting algebras preserve validity.

Lemma 3.12. *Let $(g_0, g_1)\colon \mathcal{H} \to \mathcal{H}'$ be an embedding of one-step algebras. If $v = (v_0, v_1)$ and $v' = (v'_0, v'_1)$ are valuations on $\mathcal{H}$ and $\mathcal{H}'$, respectively, such that $v'_0(p) = g_0(v_0(p))$ and $v'_1(q) = g_1(v_1(p))$, for all $p \in \mathsf{Prop}_0$ and $q \in \mathsf{Prop}_1$, then for any hypersequent rule (r) of degree at most 1 we have that $(\mathcal{H}, v) \models (r)$ iff $(\mathcal{H}', v') \models (r)$.*

Proof. It suffices to show that for all formulas $\varphi, \psi \in Form_1(\mathsf{Prop}_0 \cup \mathsf{Prop}_1)$

$$\varphi^{\mathrm{v}_1} \le \psi^{\mathrm{v}_1} \iff \varphi^{\mathrm{v}'_1} \le \psi^{\mathrm{v}'_1}. \tag{3}$$

Since (g_0, g_1) is a map of one-step algebras an easy inductive argument shows that the assumption $v'_0(p) = g(v_0(p))$ and $v'_1(q) = g_1(v_1(p))$ for all $p \in \mathsf{Prop}_0, q \in \mathsf{Prop}_1$ implies that $\varphi^{\mathrm{v}'_1} = g_1(\varphi^{\mathrm{v}_1})$ for all $\varphi \in Form_1(\mathsf{Prop}_0 \cup \mathsf{Prop}_1)$. From this (3) readily follows as any injective lattice homomorphism will necessarily be both order-preserving and order-reflecting. □

In particular, we have that if $\mathcal{H}'$ is a one-step Heyting algebra validating HC and $\mathcal{H}$ embeds into $\mathcal{H}'$ then $\mathcal{H}$ validates HC as well.

We wish to establish the algebraic completeness of the derivability relation $\vdash^1$ with respect to one-step Heyting algebras.

Proposition 3.13. *Let $\mathscr{S} \cup \{S\}$ be a finite set of hypersequents of implicational degree at most* 1, *and let* HC *be a (reduced) hypersequent calculus. If all one-step Heyting algebras validating* HC *0-validate the hypersequent rule $\mathscr{S}/S$ then $\mathscr{S} \vdash^1_{\mathrm{HC}} S$.*

Proof. Let $\mathscr{S} \cup \{S\}$ be a finite set of hypersequents of degree at most 1 such that $\mathscr{S} \nvdash^1_{\mathrm{HC}} S$. We then construct a one-step Heyting algebra $\mathcal{LT}_{\mathrm{HC}}(\mathscr{S}, S)$ validating HC and a 0-valuation on $\mathcal{LT}_{\mathrm{HC}}(\mathscr{S}, S)$ under which all hypersequents in $\mathscr{S}$ are true but S is false. This is completely similar to the construction found in the proof of Proposition 2.10. As before we let Prop be the set of propositional letters occurring in $\mathscr{S} \cup \{S\}$ and let $\widetilde{\mathscr{S}}$ be a maximal set of hypersequents, based on $Form(\mathsf{Prop})$, containing $\mathscr{S}$ such that $\widetilde{\mathscr{S}} \nvdash^1_{\mathrm{HC}} S$. We then have that if $s_1, \dots, s_n$ are sequents of degree at most 1

$$\widetilde{\mathscr{S}} \vdash^1_{\mathrm{HC}} \; s_1 \mid \dots \mid s_n \mid S \implies \widetilde{\mathscr{S}} \vdash^1_{\mathrm{HC}} s_i \text{ for some } i \leq n. \tag{4}$$

Letting D_k be the set of equivalence classes of formulas of degree at most k, for $k \in \{0, 1\}$, of the equivalence relation

$$\varphi \approx \psi \text{ iff } \widetilde{\mathscr{S}} \vdash_{\mathrm{HC}} \Rightarrow \varphi \leftrightarrow \psi,$$

we obtain a (finite conservative) one-step Heyting algebra $\mathcal{LT}_{\mathrm{HC}}(\mathscr{S}, S) := (D_0, D_1, i)$ where $i \colon D_0 \to D_1$ is the evident inclusion. From (4) we see that $\mathcal{LT}_{\mathrm{HC}}(\mathscr{S}, S)$ validates HC and moreover that under the valuation v on $\mathcal{LT}_{\mathrm{HC}}(\mathscr{S}, S)$ determined by sending propositional variables to the corresponding equivalence classes in D_0,[8] we have that $(\mathcal{LT}_{\mathrm{HC}}(\mathscr{S}, S), v) \models \mathscr{S}$ but $(\mathcal{LT}_{\mathrm{HC}}(\mathscr{S}, S), v) \not\models S$. ☐

Note that since there are only finitely many formulas of degree at most 1 when Prop is finite, the one-step algebra $\mathcal{LT}_{\mathrm{HC}}(\mathscr{S}, S)$ obtained in the proof of Proposition 3.13 is in fact a finite conservative one-step Heyting algebra.

4. Characterising the bounded proof property

Given a finite conservative one-step Heyting algebra $\mathcal{H} = (D_0, D_1, i)$ we will define the *diagram* associated with $\mathcal{H}$. This construction is analogous to the diagrams of a finite conservative one-step modal algebra from [12]. In fact they are a two-sorted version of the diagrams known from model theory, see e.g., [19].

We introduce a set of propositional variables $\mathsf{Prop}_0^{\mathcal{H}} = \{p_a \colon a \in D_0\}$. Then by the conservativity of $\mathcal{H}$ it follows that for each $a \in D_1$ there exists a formula $\theta_a \in Form_1(\mathsf{Prop}_0^{\mathcal{H}})$ such that $\theta_b^{\mathsf{v}_1} = b$, where v is the *natural* 0-valuation on $\mathcal{H}$ given by $v_0(p_a) = a$. In particular, we have that $\theta_{i(a)} = p_a$ for all $a \in D_0$.

Now let

$$\begin{aligned} \mathscr{S}^0_{\mathcal{H}} :=& \{p_{a\wedge b} \Rightarrow p_a \wedge p_b,\ p_a \wedge p_b \Rightarrow p_{a\wedge b} \colon a, b \in D_0\} \\ & \cup \{p_{a\vee b} \Rightarrow p_a \vee p_b,\ p_a \vee p_b \Rightarrow p_{a\vee b} \colon a, b \in D_0\} \\ & \cup \{p_0 \Rightarrow \bot\}, \cup \{\top \Rightarrow p_1\}, \end{aligned}$$

and

$$\begin{aligned} \mathscr{S}^1_{\mathcal{H}} :=& \{\theta_{a\wedge b} \Rightarrow \theta_a \wedge \theta_b,\ \theta_a \wedge \theta_b \Rightarrow \theta_{a\wedge b} \colon a, b \in D_1\} \\ & \cup \{\theta_{a\vee b} \Rightarrow \theta_a \vee \theta_b,\ \theta_a \vee \theta_b \Rightarrow \theta_{a\vee b} \colon a, b \in D_1\} \\ & \cup \{\theta_{i(a)\to i(b)} \Rightarrow \theta_{i(a)} \to \theta_{i(b)},\ \theta_{i(a)} \to \theta_{i(b)} \Rightarrow \theta_{i(a)\to i(b)} \colon a, b \in D_0\}. \end{aligned}$$

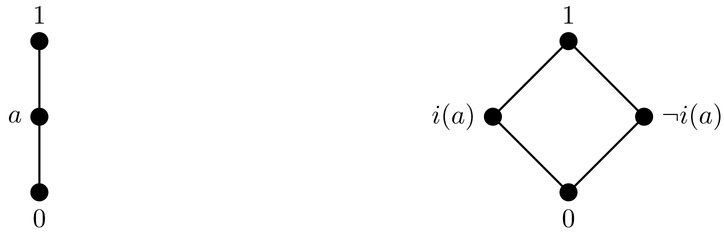
We then define the *positive diagram of* $\mathcal{H}$ to be $\mathscr{S}_{\mathcal{H}} := \mathscr{S}^0_{\mathcal{H}} \cup \mathscr{S}^1_{\mathcal{H}}$. For each $a, b \in D_1$ we let s_{ab} be the sequent $\theta_a \Rightarrow \theta_b$ if $a \nleq b$ and the empty sequent if $a \leq b$. We then define the *negative diagram of* $\mathcal{H}$ to be the hypersequent

$$S_{\mathcal{H}} := \{s_{ab} \colon a, b \in D_1\}.$$

Definition 4.1. By the *diagram* of a finite conservative one-step Heyting algebra we will understand the hypersequent rule $\mathscr{S}_{\mathcal{H}}/S_{\mathcal{H}}$.

[8] Notice that this is a 0-valuation.

Note that writing down the diagram of even relatively simple one-step Heyting algebra quickly becomes rather involved. Of course, when considering concrete cases, a lot of clauses in the diagram will be redundant and may therefore be eliminated. For example, the one-step diagram of the finite conservative one-step Heyting algebra drawn below



D_0 D_1

will be (equivalent to) the hypersequent rule.

$$\frac{p_0 \Rightarrow \bot \qquad \top \Rightarrow p_1 \qquad \neg p_a \Rightarrow p_{\neg a} \qquad p_{\neg a} \Rightarrow \neg p_a \qquad p_1 \Rightarrow p_{\neg a} \vee p_a}{p_a \Rightarrow p_0 \mid p_1 \Rightarrow p_a}$$

We say that a one-step Heyting algebra $\mathcal{H}'$ *refutes* a diagram $\mathscr{S}_\mathcal{H}/S_\mathcal{H}$ under a 0-valuation v if $(\mathcal{H}', v) \models \mathscr{S}_\mathcal{H}$ but $(\mathcal{H}', v) \not\models S_\mathcal{H}$.

The following proposition shows why we are interested in diagrams.

Proposition 4.2. *Let $\mathcal{H} = (D_0, D_1, i)$ and $\mathcal{H}' = (D_0', D_1', i')$ be one-step Heyting algebras with $\mathcal{H}$ finite and conservative. Then the following are equivalent:*

(1) *There exists a one-step embedding from $\mathcal{H}$ into $\mathcal{H}'$;*
(2) *There exists a 0-valuation v on $\mathcal{H}'$ such that $(\mathcal{H}', v')$ refutes the diagram of $\mathcal{H}$.*

Proof. First assume that there exists a one-step embedding $(g_0, g_1)\colon \mathcal{H} \to \mathcal{H}'$. We then define a 0-valuation $v' = (v_0', v_1')$ on $\mathcal{H}'$ by $v_0(p_a) = g_0(a)$. Then as $\mathcal{H}$ evidently refutes its own diagram under the natural valuation $v_0(p_a) = a$ it immediately follows from Lemma 3.12 that $(\mathcal{H}', v')$ refutes $\mathscr{S}_\mathcal{H}/S_\mathcal{H}$ as well.

Conversely if there exists a 0-valuation $v' = (v_0', v_1')$ on $\mathcal{H}'$ such that $(\mathcal{H}', v')$ refutes the diagram of $\mathcal{H}$, then we claim that defining $(g_0, g_1)\colon \mathcal{H} \to \mathcal{H}'$ by

$$g_0(a) = v_0'(p_a) \quad \text{and} \quad g_1(b) = \theta_b^{\mathrm{v}_1'},$$

yields an embedding of one-step algebras.

First of all since $\mathcal{H}$ is conservative the function g_1 is well defined, and because i' is an injection and $(\mathcal{H}', v') \models \mathscr{S}^0_\mathcal{H}$ we see that g_0 must be a bounded lattice homomorphism. Since $(\mathcal{H}', v')$ also validates $\mathscr{S}^1_\mathcal{H}$ we see that g_1 is a bounded lattice homomorphism as well.

To see that $i' \circ g_0 = g_1 \circ i$ we simply observe that for all $a \in D_0$

$$i(g_0(a)) = i(v_0'(p_a)) = p_a^{\mathrm{v}_1'} = \theta_{i(a)}^{\mathrm{v}_1'} = g_1(i(a)).$$

From the assumption that $(\mathcal{H}', v')$ does not validate any of the sequents $\theta_a \Rightarrow \theta_b$ when $a \not\leq b$ it immediately follows that g_1 is an injection. So as i is an injection, we must have that g_0, being the first component of the injection $g_1 \circ i$, is an injection as well.

Finally because $(\mathcal{H}', v')$ makes all sequents of the form $\theta_{i(a) \to i(b)} \Rightarrow \theta_{i(a)} \to \theta_{i(b)}$ and $\theta_{i(a)} \to \theta_{i(b)} \Rightarrow \theta_{i(a) \to i(b)}$ true, we have that

$$g_1(i(a) \to i(a)) = i'(g_0(a)) \to i'(g_0(b)),$$

and so we can conclude that (g_0, g_1) is indeed an embedding of one-step algebras. □

Definition 4.3. A class K of one-step Heyting algebras (or intuitionistic one-step frames) has the extension property if all members of K have a one-step extension also belonging to K.

Lemma 4.4. *Let* HC *be a hypersequent calculus and let* $Con_\omega^{Alg}(\mathrm{HC})$ *be the class of finite conservative one-step Heyting algebras validating* HC*. If every* $\mathcal{H} \in Con_\omega^{Alg}(\mathrm{HC})$ *embeds into some standard Heyting algebra validating* HC *then the class* $Con_\omega^{Alg}(\mathrm{HC})$ *has the extension property.*

Proof. Let $\mathcal{H} = (D_0, D_1, i)$ be a finite (conservative) one-step Heyting algebra and suppose that there exists an embedding $(g_0, g_1)\colon \mathcal{H} \to \mathfrak{A}$ into some Heyting algebra $\mathfrak{A}$ validating HC. Letting A be the bounded lattice reduct of $\mathfrak{A}$, we see that $\mathcal{H}'' = (D_1, A, g_1)$ is a one-step algebra validating HC and extending $\mathcal{H}$.

To obtain a finite conservative one-step Heyting algebra validating HC and extending $\mathcal{H}$ let D_2 be the bounded distributive sublattice of A generated by the set $\{g_1(a) \to g_1(b) \colon a, b \in D_1\}$. As the variety of bounded distributive lattices is locally finite D_2 is finite. Moreover, we have $g_1[D_1] \subseteq D_2$. Therefore, $\mathcal{H}' = (D_1, D_2, g_1)$ will be a finite conservative one-step algebra validating HC and extending $\mathcal{H}$. ☐

Theorem 4.5. *Let* HC *be a (reduced) hypersequent calculus. Then the following are equivalent:*

(1) *The calculus* HC *has the bounded proof property;*
(2) *The class of finite conservative one-step Heyting algebras validating* HC *has the extension property;*
(3) *The class of finite conservative intuitionistic one-step frames validating* HC *has the extension property.*

Proof. That items 2 and 3 are equivalent is an immediate consequence of the dual equivalence between the categories $\mathsf{OSHA}_\omega^{\mathrm{cons}}$ and $\mathsf{IOSFrm}_\omega^{\mathrm{cons}}$.

To see that item 1 implies item 2 let $\mathcal{H}$ be a finite conservative one-step Heyting algebra validating HC. Since $\mathcal{H}$ refutes its own diagram $\mathscr{S}_\mathcal{H}/S_\mathcal{H}$ we obtain from Proposition 3.11 that $\mathscr{S}_\mathcal{H} \not\vdash^1_{\mathrm{HC}} S_\mathcal{H}$. Therefore, if HC enjoys the bounded proof property it follows that $\mathscr{S}_\mathcal{H} \not\vdash_{\mathrm{HC}} S_\mathcal{H}$. By algebraic completeness we must have a Heyting algebra $\mathfrak{A}$ validating HC and refuting $\mathscr{S}_\mathcal{H}/S_\mathcal{H}$. But then by Proposition 4.2 there exists embedding $(g_0, g_1)\colon \mathcal{H} \to \mathfrak{A}$ and so by Lemma 4.4 we may conclude that the class of finite conservative Heyting algebras validating HC has the extension property.

Finally, to see that item 2 implies item 1 let $\mathscr{S} \cup \{S\}$ be a finite set of hypersequents of implicational degree at most 1 such that $\mathscr{S} \not\vdash^1_{\mathrm{HC}} S$. By Proposition 2.14 it then suffices to show that $\mathscr{S} \not\vdash_{\mathrm{HC}} S$.

Let $\mathcal{H}_0 = (D_0, D_1, i_0)$ be the finite conservative one-step Heyting algebra $\mathcal{LT}_{\mathrm{HC}}(\mathscr{S}, S)$ constructed in the proof of Proposition 3.13. Moreover, let v^0 be a 0-valuation on $\mathcal{H}_0$ such that $(\mathcal{H}_0, v^0) \models \mathscr{S}$ but $(\mathcal{H}_0, v^0) \not\models S$. If the class of finite conservative one-step algebras validating HC has the extension property then we have a one-step extension in form of a finite conservative one-step Heyting algebra $\mathcal{H}_1 = (D_1, D_2, i_1)$ validating HC. Moreover, i_0, i_1 induce a 0-valuation v^1 on $\mathcal{H}_1$ under which $\mathscr{S}$ is true but S it not. In this way we obtain a chain

$$D_0 \xhookrightarrow{i_0} D_1 \xhookrightarrow{i_1} \dots \xhookrightarrow{i_{n-1}} D_{n-1} \xhookrightarrow{i_n} D_n \xhookrightarrow{i_{n+1}} \dots$$

of Heyting algebras in the category bDL_ω, with the property that

$$i_{n+1}(i_n(a) \to_{n+1} i_n(b)) = i_{n+1}(i_n(a)) \to_{n+2} i_{n+1}(i_n(b)).$$

Consequently, taking the colimit of the above diagram, in the category bDL_ω, we obtain a Heyting algebra $\mathfrak{A}$ with Heyting implication

$$[a] \to [b] := [i_{n,k+1}(a) \to_{k+2} i_{m,k+1}(b)], \quad k = \max\{n, m\},$$

for $a \in D_n$ and $b \in D_m$ and $i_{n,k}\colon D_n \to D_k$ the evident map for $n \leq k$.

It is then easy to see that $\mathfrak{A}$ must validate HC and moreover that the 0-valuations v^n on $\mathcal{H}_n$ induces a valuation v on $\mathfrak{A}$ which by the injectivity of the i_n's is such that $(\mathfrak{A}, v) \models \mathscr{S}$ and $(\mathfrak{A}, v) \not\models S$. We may therefore conclude that $\mathscr{S} \not\vdash_{\mathrm{HC}} S$. ☐

In concrete cases it is not so easy to work with one-step extensions of frames. However, assuming the finite model property we obtain a version of Theorem 4.5 which avoids the concept of one-step extensions altogether.

Definition 4.6. We say that a hypersequent calculus HC has the *(global) finite model property* if for each set $\mathscr{S} \cup \{S\}$ of hypersequents, $\mathscr{S} \not\vdash_{\mathrm{HC}} S$ iff there exists a finite Heyting algebra $\mathfrak{A}$ validating HC and a valuation v on $\mathfrak{A}$ such that $(\mathfrak{A}, v) \models \mathscr{S}$ and $(\mathfrak{A}, v) \not\models S$.

Proposition 4.7. *A hypersequent calculus* HC *has the finite model property iff for each set* $\mathscr{S} \cup \{S\}$ *of hypersequents,* $\mathscr{S} \nvdash_{\mathrm{HC}} S$ *iff there exists a finite intuitionistic Kripke frame* $\mathfrak{F}$ *validating* HC *and a valuation* v *on* $\mathfrak{F}$ *such that* $(\mathfrak{F}, v) \models \mathscr{S}$ *and* $(\mathfrak{F}, v) \not\models S$.

Proof. Immediate by the duality between finite Heyting algebras and finite intuitionistic Kripke frames. □

Lemma 4.8. *Let* HC *be a hypersequent calculus. Then* HC *has the finite model property iff if for each set* $\mathscr{S} \cup \{S\}$ *of hypersequents of degree at most* 1*,* $\mathscr{S} \nvdash_{\mathrm{HC}} S$ *iff there exists a finite Heyting algebra* $\mathfrak{A}$ *validating* HC *and a valuation* v *on* $\mathfrak{A}$ *such that* $(\mathfrak{A}, v) \models \mathscr{S}$ *and* $(\mathfrak{A}, v) \not\models S$.

Proof. The statement follows from the fact that given $\mathscr{S} \cup \{S\}$ it is possible to produce at set of hypersequents $\mathscr{S}' \cup \{S'\}$ having degree at most 1, such that for every Heyting algebra $\mathfrak{A}$ validating HC (finite or not) we have that $\mathfrak{A}$ validates $\mathscr{S}/S$ iff it validates $\mathscr{S}'/S'$ (thus, in particular, $\mathscr{S} \vdash_{\mathrm{HC}} S$ iff $\mathscr{S}' \vdash_{\mathrm{HC}} S'$ by Proposition 2.12). In order to construct $\mathscr{S}' \cup \{S'\}$ from $\mathscr{S} \cup \{S\}$, we just need to abstract out implicative subformulas with fresh propositional variables (we have already applied this procedure e.g., in the proof of Propositions 2.14 and 2.15). □

Theorem 4.9. *Let* HC *be a (reduced) hypersequent calculus. Then the following are equivalent:*

(1) *The calculus* HC *has the bounded proof property and the finite model property;*
(2) *Each finite conservative one-step algebra validating* HC *embeds into some finite Heyting algebra validating* HC*;*
(3) *Each finite conservative intuitionistic one-step frame validating* HC *is the relative open image of some finite intuitionistic Kripke frame validating* HC.

Proof. As in the proof of Theorem 4.5 it is immediate that items 2 and 3 are equivalent.

To see that item 1 implies item 2 we observe that if $\mathcal{H}$ is a finite conservative one-step Heyting algebra validating HC then as $\mathcal{H}$ refutes its diagram $\mathscr{S}_{\mathcal{H}}/S_{\mathcal{H}}$ we must have that $\mathscr{S}_{\mathcal{H}} \nvdash^1_{\mathrm{HC}} S_{\mathcal{H}}$ by Proposition 3.11. Consequently, it follows from the assumption that HC has the bounded proof property that $\mathscr{S}_{\mathcal{H}} \nvdash_{\mathrm{HC}} S_{\mathcal{H}}$ and therefore as HC has the finite model property we obtain a finite Heyting algebra $\mathfrak{A}$ which validates HC and refutes the diagram $\mathscr{S}_{\mathcal{H}}/S_{\mathcal{H}}$. By Proposition 4.2 it then follows that $\mathcal{H}$ embeds into $\mathfrak{A}$.

Conversely, to see that item 2 implies item 1 we first note that by Theorem 4.5, item 2 implies that HC enjoys the bounded proof property. To see that it also enjoys the finite model property it suffices, by Lemma 4.8, to consider finite set of hypersequents $\mathscr{S} \cup \{S\}$ of degree at most 1. Given such a set $\mathscr{S} \cup \{S\}$ with the property that $\mathscr{S} \nvdash_{\mathrm{HC}} S$ let $\mathcal{H}$ be the finite conservative one-step algebra $\mathcal{LT}_{\mathrm{HC}}(\mathscr{S}, S)$ as constructed in the proof of Proposition 3.13. Then by assumption we have a finite Heyting algebra $\mathfrak{A}$ validating HC such that $\mathcal{H}$ embeds into $\mathfrak{A}$, and this embedding induces a valuation on $\mathfrak{A}$ under which $\mathscr{S}$ is true but S is not. □

It is easy to see that if a rule (r) is of degree 0 then an intuitionistic one-step Heyting algebra (D_0, D_1, i) validates (r) if and only if D_1, considered as a standard Heyting algebra, validates (r). This observation together with Theorem 4.9 yields the following corollary.

Corollary 4.10. *If* $\mathcal{R}$ *is a set of rules of degree* 0*, then the calculus* $\mathbf{HLJ}' + \mathcal{R}$ *enjoys the bounded proof property as well as the finite model property.*

Remark 4.11. Single-succedent (hyper)sequent rules of degree 0 are equivalent to what is know as *structural* (hyper)sequent rules [22]. Any structural single-succedent hypersequent rule (r) may (effectively) be transformed into an equivalent structural single-succedent hypersequent rule (r') such $\mathbf{HLJ} + (r')$ enjoys cut-elimination [22, Thm. 7.1(b), Cor. 8.6]. Whether or not this is also the case for multi-succedent hypersequent rules is not known. In any case Corollary 4.10 guarantees the bound proof property also in the multi-succedent case.

Remark 4.12. Recall [6, 9] that an intermediate logic L is *stable* if for all subdirectly irreducible Heyting algebras $\mathfrak{A}$ and $\mathfrak{B}$ such that $\mathfrak{A}$ is isomorphic to a bounded sublattice of $\mathfrak{B}$ we have that $\mathfrak{B} \models L$ implies that $\mathfrak{A} \models L$. By [9, Thm. 5.3] these are precisely the intermediate logics L axiomatised by multi-conclusion rules corresponding to hypersequent rules of degree 0. Furthermore, if a stable intermediate logic is finitely axiomatisable then it is also axiomatisable by finitely many multi-conclusion rules (hypersequent rules) [9, Rem. 5.7]. Consequently, by the correspondence between multi-conclusion consequence relations and hypersequent calculi outlined in section 2.2 we obtain that the intermediate logics which admit a hypersequent calculus of the form $\mathbf{HLJ}' + \mathcal{R}$, where $\mathcal{R}$ is a set of rules of degree 0 are precisely the stable intermediate logics—of which there are continuumly many [6, Thm. 6.13]. Moreover, if we restrict to $\mathcal{R}$ a finite set of degree 0 rules we obtain precisely the finitely axiomatisable stable intermediate logics. In particular all finitely axiomatisable stable intermediate logics must be decidable. Stable modal logics were defined in [8]. In [13, Thm. 5.3] it was proven that (finitely axiomatisable) stable modal logics have (finite) multi-conclusion axiomatisations with the bounded proof property. That this is also the case for stable intermediate logics is then an easy consequence of Corollary 4.10. The latter, of course, is not as surprising as in the modal case.

5. Examples

In this section we provide a number of examples showing how to use the methods developed above to determine whether or not a given sequent or hypersequent calculus enjoys the bounded proof property.

We warn the reader that as we base the duality between one-step Heyting algebras and intuitionistic one-step frames on the downset functor $\mathrm{Do}\colon \mathsf{Pos}_\omega^{open} \to \mathsf{HA}_\omega$ the partial order on Kripke frames may be the opposite of what the reader is familiar with.

It is possible to adapt the algorithmic correspondence theory for intuitionistic logic (see e.g., [25]) to the framework of one-step semantics for hypersequent rules. However, as the examples we will be considering here are rather simple we will derive the correspondence results we need manually.

Finally, we would like to mention the following result[9] due to Ciabattoni, Galatos and Terui:

Theorem 5.1 ([22])**.** *There is an effective procedure which given an axiom φ belonging to the level $\mathcal{P}_3$ of the substructural hierarchy produces a finite set of structural hypersequent rules $\mathcal{R}_\varphi$ such that when added to* **HLJ** *yields a hypersequent calculus for* $\mathbf{IPC} + \varphi$ *enjoying cut-elimination and the subformula property.*

The hypersequent calculi obtained by this procedure evidently have the bounded proof property. Thus in order to obtain truly novel results of a positive nature using Theorem 4.5 and 4.9 it will be necessary to consider axioms at the level $\mathcal{N}_3$ of the substructrual hierarchy [22]. Since all intermediate logics are axiomatizable by canonical formulas [47] which belong to the level $\mathcal{N}_3$ over **IPC** the substructural hierarchy collapses at this level[10].

Note that using the normal form representation given in [22] it is easy to see that each formula appearing at level $\mathcal{P}_3$ of the substructural hierarchy is provably equivalent (over **IPC**) to an ONNILLI-formula [10]. Consequently, all formulas in the class $\mathcal{P}_3$ axiomatise stable intermediate logics [10, Thm. 5].[11] Therefore, non-trivial examples will require that we consider non-stable logics which tend to have rather complicated axiomatisations, i.e., of degree at least 3. However, for sequent calculi the situation may be different.

5.1. Calculi for LC.

The intermediate logic **LC**, known as the Gödel-Dummett logic, is obtained by adding the axiom $(p \to q) \vee (q \to p)$ to a Hilbert-style presentation of **IPC**. Using our methods we show that the sequent calculus obtained by adding the rule

$$\frac{}{\Rightarrow (p \to q), (q \to p)}\ (r_{LC})$$

does not enjoy the bounded proof property.

Proposition 5.2. *A intuitionistic one-step frame (P_1, P_0, f) validates the rule (r_{LC}) iff*

$$\forall a_0, a_1, a_2 \in P_1\ (a_1 \le a_0 \text{ and } a_2 \le a_0 \implies (f(a_1) \le f(a_2) \text{ or } f(a_2) \le f(a_1))).$$

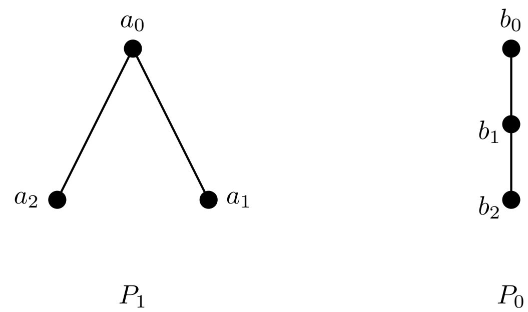
Proof. Straightforward. □

[9] In fact it holds more generally for Full Lambek calculus with exchange and weakening $\mathbf{FL}_{ew}$.

[10] Somewhat surprisingly this is also the case for $\mathbf{FL}_e$ as recently established in [40].

[11] Incidentally, each formula appearing at level $\mathcal{P}_2$ is provably equivalent to a NNIL-formula and thus $\mathcal{P}_2$-formulas axiomatise subframe logics [10].

To see that adding the rule (r_{LC}) does not yield a sequent calculus with the bounded proof property, consider the one-step frame $\mathcal{F} = (P_1, P_0, f)$ presented as:



The function f is the obvious map given by $a_i \mapsto b_i$ for $i \in \{0,1,2\}$. This is easily seen to be a finite conservative one-step frame validating the rule (r_{LC}). Now suppose towards a contradiction that $\mathcal{F}$ has a one-step extension, say $\mathcal{F}' = (P_2, P_1, g)$. As f is bijective it follows from the assumption that g is f-open that g must be an open map. Therefore, we must have $a_0', a_1', a_2' \in P_2$ with $a_2', a_1' \leq a_0'$, such $g(a_i') = a_i$ for $i \in \{0,1,2\}$. But this shows that $\mathcal{F}'$ fails to validate the rule (r_{LC}) and consequently that $\mathcal{F}$ does not have any one-step extension validating (r_{LC}).

Thus, by Theorem 4.5, we see that the hypersequent calculus obtained by adding the rule (r_{LC}) does not have the bounded proof property.

However, we know from [5] that adding the so-called communication rule

$$\frac{G \mid \Gamma_1, \Gamma_1' \Rightarrow \Pi \qquad G \mid \Gamma_2, \Gamma_2' \Rightarrow \Pi'}{G \mid \Gamma_1, \Gamma_2 \Rightarrow \Pi \mid \Gamma_1', \Gamma_2' \Rightarrow \Pi'}\ (com)$$

to the hypersequent calculus **HLJ** yields a hypersequent calculus for the logic **LC** which preserves cut-eliminability.

Since this rule is structural it follows from Corollary 4.10 that the rule (com) enjoys the bounded proof property and the finite model property.

In fact, it is easy to see that an intuitionistic one-step frame (P_1, P_0, f) validates the rule (com) iff P_1 is a linear order.

Finally, we consider a (variant of the) calculus for **LC** due to Sonobe [45], see also [26], given by adding the following rules to the standard multi-succedent sequent calculus **LJ**$'$:

$$\frac{p_0, p_1 \Rightarrow q_1, p_2 \to q_2, \ldots, p_n \to q_n \qquad \ldots \qquad p_0, p_n \Rightarrow q_n, p_1 \to q_1, \ldots, p_{n-1} \to q_{n-1}}{p_0 \Rightarrow p_1 \to q_1, \ldots, p_n \to q_n}\ (LR_n)$$

for each $n \geq 1$.

Note that (LR_1) is just the rule $(r \to)$. As the original calculus has cut-elimination we should expect to be able to establish the bounded proof property, as well as the finite model property, using the methods developed here. In the following we show that this is indeed the case.

Proposition 5.3. *Let $\mathcal{F} = (P_1, P_0, f)$ be a finite intuitionistic one-step frame then $\mathcal{F} \models (LR_n)$ for all $n \geq 1$ iff*

$$\forall a \in P_1\ \forall T \subseteq {\downarrow} a\ \exists a_1 \in T\ \exists a_2 \leq a\ (f(a_1) = f(a_2) \text{ and } f[T] \subseteq f[{\downarrow} a_2]).$$

Proof. Let $n \geq 1$ be given. The rule (LR_n) fails on $\mathcal{F}$ iff there exists $U_1, \ldots, U_n, V_1, \ldots, V_n \in \mathrm{Do}(P_0)$ and $W \in \mathrm{Do}(P_1)$ such that

$$\mathsf{AND}_{i=1}^n \left(W \cap f^{-1}(U_i) \subseteq f^{-1}(V_i) \cup \bigcup_{j \neq i} P_1 \backslash {\uparrow} f^{-1}(U_j \backslash V_j) \right) \text{ and } W \not\subseteq \bigcup_{i=0}^n P_1 \backslash {\uparrow}(U_i \backslash V_i).$$

This happens precisely when

$$\exists a_0 \in W \exists a_1 \ldots, a_n \leq a_0\ \mathsf{AND}_{i=1}^n \left(W \cap f^{-1}(U_i) \subseteq f^{-1}(V_i) \cup \bigcup_{j \neq i} P_1 \backslash {\uparrow} f^{-1}(U_j \backslash V_j) \text{ and } f(a_i) \in U_i \text{ and } f(a_i) \notin V_i \right).$$

By using Ackermann's Lemma [2] with $W := {\downarrow}a_0$ and $U_i := {\downarrow}f(a_i)$ and $V_i := ({\uparrow}f(a_i))^{\mathrm{c}}$, we obtain that this is equivalent to

$$\exists a_0 \in P_1 \exists a_1 \ldots, a_n \leq a_0 \ \mathsf{AND}_{i=1}^{n} \left({\downarrow}a_0 \cap f^{-1}({\downarrow}f(a_i)) \subseteq f^{-1}(({\uparrow}f(a_i))^{\mathrm{c}}) \cup \bigcup_{j \neq i} P_1 \backslash {\uparrow} f^{-1}({\downarrow}f(a_j) \backslash ({\uparrow}f(a_j))^{\mathrm{c}}) \right).$$

Negating this we obtain the following first-order condition

$$\forall a_0, \ldots, a_n \ \left(\{a_1, \ldots, a_n\} \subseteq {\downarrow}a \Longrightarrow \mathsf{OR}_{i=1}^{n} ({\downarrow}a_0 \cap f^{-1}({\downarrow}f(a_i)) \not\subseteq f^{-1}(({\uparrow}f(a_i))^{\mathrm{c}}) \cup \bigcup_{j \neq i} P_1 \backslash {\uparrow} f^{-1}({\downarrow}f(a_j) \backslash ({\uparrow}f(a_j)^{\mathrm{c}})) \right).$$

This first-order condition may in turn be rewritten as

$$\forall a_0, \ldots, a_n \ (\{a_1, \ldots, a_n\} \subseteq {\downarrow}a_0 \Longrightarrow \exists a' \in \{a_1, \ldots, a_n\} \exists a'' \leq a_0 \ (f(a') = f(a'') \text{ and } f[\{a_1, \ldots, a_n\} \backslash \{a'\}] \subseteq f[{\downarrow}a''])).$$

Finally, by the assumption that $\mathcal{F}$ is finite it follows that $\mathcal{F} \models (LR_n)$ for all $n \geq 1$, iff

$$\forall a \in P_1 \ \forall T \subseteq {\downarrow}a \ \exists a_1 \in T \ \exists a_2 \leq a(f(a_1) = f(a_2) \text{ and } f[T] \subseteq f[{\downarrow}a_2]),$$

where T is required to be non-empty. □

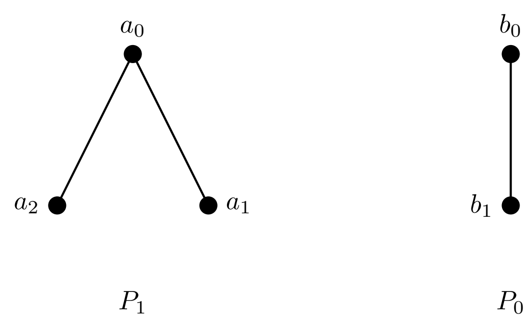
Remark 5.4. It is worth noting that the first-order condition obtained in Proposition 5.3 is somewhat similar to the condition on modal one-step frames corresponding to the calculus of **S4.3** considered in [12, Sec. 8.2.1].

Remark 5.5. Note that as an immediate consequence of Proposition 5.3 we have that if (P_1, P_0, f) is a finite intuitionistic one-step frame validating the rules $(LR_n)_{n \geq 1}$ then we have that

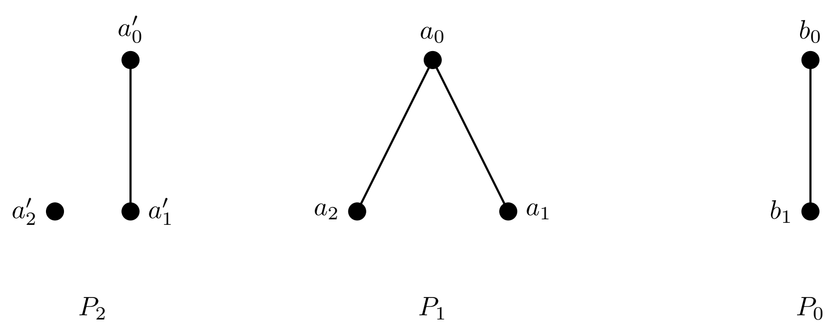
$$\forall a_0 \in P_1 \forall a_1, a_2 \leq a_0 \ (f(a_1) \leq f(a_2) \text{ or } f(a_2) \leq f(a_1)).$$

In particular ${\downarrow}f(a)$ is linearly ordered for all $a \in P_1$.

Example 5.6. Given this it is easy to verify that the intuitionistic one-step frame (P_1, P_0, f) given by



with $f(a_2) = f(a_0) = b_0$ and $f(a_1) = b_1$ is a conservative intuitionistic one-step frame validating (LR_n) for all $n \geq 1$. Evidently, considered by itself P_1 is not a frame for **LC**. However, one may easily verify that we have an extension in terms of



For an intuitionistic one-step frame $\mathcal{F} \coloneqq (P_1, P_0, f)$ we define a relation $<_f$ on (the underlying set of) P_1 as follows

$$a <_f a' \quad \text{iff} \quad a \leq_1 a' \text{ and } f(a) \neq f(a'),$$

and let $\leq_f$ be the reflexive closure of $<_f$. It is easy to verify that $\leq_f$ is a partial order. A $\leq_f$*-path* will then by a (finite) sequences $\mathfrak{p} = a_1, \ldots, a_n$ such that $a_i <_f a_j$ for all indices $i < j$. We say that an $\leq_f$-path $\mathfrak{p}$ is *good* if for all $a \leq \max \mathfrak{p}$ there exists an initial segment $\mathfrak{p}_a$ of $\mathfrak{p}$ such that $f(a) = f(\max \mathfrak{p}_a)$.

Lemma 5.7. *Any finite (conservative) one-step frame $\mathcal{F} \coloneqq (P_1, P_0, f)$ validating the rules $(LR_n)_{n\geq 1}$ has enough good $\leq_f$-paths in the sense that for every $a \in P_1$ there exist a good $\leq_f$-path $\mathfrak{p}$ in P_1 with* $\max \mathfrak{p} = a$*.*

Proof. We proceed by induction on the cardinality of ${\downarrow} f(a)$. If $|{\downarrow} f(a)| = 1$, then a itself will be a good path and clearly $\max a = a$. Suppose that for all a' with $|{\downarrow} f(a')| < |{\downarrow} f(a)|$ there is a good path $\mathfrak{p}_{a'}$ such that $\max \mathfrak{p}_{a'} = a'$. Then to find a good path for a let $T_a = \{a' \leq a \colon f(a) \neq f(a')\}$. Since by assumption $\mathcal{F}$ validates all the rules $(LR_n)_{n\geq 1}$ we obtain from Proposition 5.3 that there exists $a'' \leq a$ such that $f[T_a] \subseteq f[{\downarrow} a'']$ and $f(a'') = f(a')$ for some $a' \in T_a$. From the latter it follows that $|{\downarrow} f(a'')| = |{\downarrow} f(a')| < |{\downarrow} f(a)|$ and so by induction hypothesis we have that there is a good path $\mathfrak{p}_{a''}$ for a''. We claim that $\mathfrak{p}_a = \mathfrak{p}_{a''}, a$ is a good path for a. To see this we note that if $a' \leq \max \mathfrak{p}_a$ then either $f(a') = f(a)$ or $a' \in T_a$. In the former case we may take $\mathfrak{p}_a$ itself as initial segment of $\mathfrak{p}_a$ witnessing that $\mathfrak{p}_a$ is a good path. In the latter case we have that $f(a') = f(b)$ for some $b \leq a'' = \max \mathfrak{p}_{a''}$ and so as $\mathfrak{p}_{a''}$ is a good path for a'' there must be an initial segment $\mathfrak{p}_b$ of $\mathfrak{p}_{a''}$ such that $f(\max \mathfrak{p}_b) = f(b) = f(a')$. Since $\mathfrak{p}_b$ is also an initial segment of $\mathfrak{p}_a$ this shows that $\mathfrak{p}_a$ is indeed a good path for a. □

As an immediate corollary of Lemma 5.7 we obtain that for any finite conservative intuitionistic one-step frame (P_1, P_0, f) validating the rules $(LR_n)_{n\geq 1}$ the poset P_2 of good $\leq_f$-paths is a finite (standard) **LC**-frame extending (P_1, P_0, f) via the map $\mathfrak{p} \mapsto \max \mathfrak{p}$. Consequently, we obtain that the calculus determined by adding the rules $\{(LR_n)\}_{n\geq 1}$ to $\mathbf{LJ'}$ enjoys the bounded proof property as well as the finite model property.

Remark 5.8. Note that the poset of good $\leq_f$-paths is a generated subframe of the poset of all $\leq_f$-paths in P_1, in fact even of all $\leq_1$-paths in P_1. It follows that the Heyting algebra $\mathrm{Do}(P_2)$ is a homomorphic image, i.e., a quotient, of the Heyting algebra dual to the poset $\mathsf{Path}(P_1)$ of all $\leq_1$-paths in P_1. It is known [3, Thm. 1] (see also [34, Sec. 2]) that $\mathrm{Do}(\mathsf{Path}(P_1)) \simeq \mathcal{F}_{\mathsf{LC}}(D_1)$ the free **LC** Heyting algebra over the distributive lattice $D_1 = \mathrm{Do}(P_1)$. Thus, from the algebraic point of view we obtain a standard extension D_2 of a one-step Heyting algebra (D_0, D_1, i) validating the rules $(LR)_{n\geq 1}$ by taking a quotient of $\mathcal{F}_{LC}(D_1)$.

The above example illustrates how the bounded proof property for a (collection of) degree 1 rule(s) may be established in an almost mechanical manner. The only part of the above which requires a bit of ingenuity is to find the appropriate one-step extensions.

5.2. **Calculi for KC.** Recall that the logic **KC** is obtained by adding the axiom $\neg p \vee \neg\neg p$ to **IPC**. It is well known that this is the logic of (finite) directed frames.

Now consider the rule

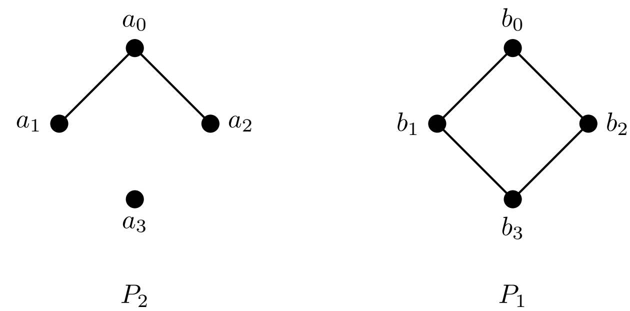
$$\frac{p, q \Rightarrow \bot}{\Rightarrow \neg q, \neg p}\,(r_{KC})$$

Proposition 5.9. *A step frame (P_1, P_0, f) validates the rule (r_{KC}) iff*

$$\forall a_0, a_1, a_2 \in P_1\ (a_1 \leq a_0 \text{ and } a_2 \leq a_0 \implies \exists b \in P_0\ (b \leq f(a_1) \text{ and } b \leq f(a_2)).$$

Proof. Straightforward. □

Consider the one-step frame $\mathcal{F} = (P_1, P_0, f)$ presented as



with f given by $a_i \mapsto b_i$, for $i \in \{0, \dots, 3\}$. Then $\mathcal{F}$ is a finite conservative one-step frame validating the rule (r_{KC}). If P_2 is a finite poset and $g \colon P_2 \to P_1$ is an f-open surjection, then as f is a bijection the f-openness condition on g implies that g is an open surjection and therefore, that for $a_0' \in g^{-1}(a_0)$ we have $a_1', a_2' \leq a_0'$ such that $g(a_1') = a_1$ and $g(a_2') = a_2$. But as $\downarrow a_1$ and $\downarrow a_2$ are disjoint we see that (P_2, P_1, g) does not validate the rule (r_{KC}), and thus $\mathcal{F}$ does not have any one-step extension validating (r_{KC}).

By Theorem 4.5, it then immediately follows that the calculus obtained by adding the rule (r_{KC}) does not have the bounded proof property.

However, we know from [21] that adding the rule

$$\frac{G \mid \Gamma, \Gamma' \Rightarrow}{G \mid \Gamma \Rightarrow \mid \Gamma' \Rightarrow}\ (lq)$$

to the hypersequent calculus **HLJ** yields a hypersequent calculus for the logic **KC**, which enjoys cut-elimination.

Again from Corollary 4.10 it follows that the rule (lq) enjoys the bounded proof property and the finite model property.

Remark 5.10. The two examples above show that unlike what usually happens in the modal case, adding a rule of the form $/\!\Rightarrow \varphi$ for φ a formula of degree 1 does not necessarily yield a calculus for $\mathbf{IPC} + \varphi$ with the bounded proof property. Moreover, we see that reducing even a simple rule may not necessarily ensure the bounded proof property. This indicates that even though the bounded proof property only place a seemingly weak requirements on derivations it is nevertheless not so easy to obtain sequent calculi for even relatively simple intermediate logics with this property. This is to some extent to be expected from the modal case: A calculus for $\mathbf{K} + \varphi$ with the bounded proof property often does not always lift to a calculus for $\mathbf{S4} + \varphi$ with the bounded proof property.

5.3. **Calculi for $\mathbf{BD}_2$.** The logic $\mathbf{BD}_2$, consisting of formulas valid precisely on frames of depth at most 2, is axiomatized by the axiom $p_2 \vee (p_2 \to (\neg p_1 \vee p_1))$ which belongs to the class $\mathcal{P}_4$, in fact by [40, Thm. 3.2] it is equivalent (over **IPC**) to a $\mathcal{N}_3$-formula, and so Theorem 5.1 does not apply.

Consider the rule

$$\frac{G \mid \Gamma', \Gamma \Rightarrow \Delta' \qquad G \mid \Gamma, p \Rightarrow q, \Delta}{G \mid \Gamma' \Rightarrow \Delta' \mid \Gamma \Rightarrow p \to q, \Delta}\ (bd_2)^*$$

In [24] it is shown that this rule determines a calculus for $\mathbf{BD}_2$ which enjoys cut-elimination. In fact, the resulting calculus $\mathbf{HBD_2} \coloneqq \mathbf{HLJ'} + (bd_2)^*$ will enjoy the subformula property and consequently also the bounded proof property. Therefore, we should be able to establish this using our methods.

Proposition 5.11. *A finite one-step frame-validates the rule $(bd_2)^*$ iff*

$$\forall a_2, a_1, a_0 \in P_1\ ((a_2 \leq a_1 \text{ and } a_0 \not\leq a_1) \implies f(a_2) = f(a_1)).$$

Proof. We have that the rule $(bd_2)^*$ fails on a finite one-step frame (P_1, P_0, f) iff there exist $U_1, U_2, V_1, V_2 \in \mathrm{Do}(P_1)$ and there exist $W_1, W_2 \in \mathrm{Do}(P_0)$ such that

$$U_1 \cap U_2 \subseteq V_2 \text{ and } U_1 \cap f^{-1}(W_1) \subseteq f^{-1}(W_2) \cup V_1 \text{ and } U_2 \not\subseteq V_2 \text{ and } U_1 \not\subseteq P_1 \backslash {\uparrow} f^{-1}(W_1 \backslash W_2) \cup V_1$$

This is easily seen to be equivalent to there also existing $a_2, a_1, a_0 \in P_1$ with $a_2 \leq a_1$ such that $U_1 \cap U_2 \subseteq V_2$ and $U_1 \cap f^{-1}(W_1) \subseteq f^{-1}(W_2) \cup V_1$ and $a_0 \in U_2 \cap V_2^{\mathrm{c}}$ and $a_1 \in U_1 \cap V_1^{\mathrm{c}}$ and $f(a_2) \in W_1 \cap W_2^{\mathrm{c}}$. Using Ackermann's Lemma the second-order quantifiers may be eliminated to yield

$$\exists a_1, a_2, a_3 \in P_1 \ (a_2 \leq a_1 \text{ and } {\downarrow} a_0 \cap {\downarrow} a_1 \subseteq ({\uparrow} a_0)^{\mathrm{c}} \text{ and } {\downarrow} a_1 \cap f^{-1}({\downarrow} f(a_2)) \subseteq f^{-1}([{\uparrow} f(a_2)]^{\mathrm{c}}) \cup ({\uparrow} a_1)^{\mathrm{c}}).$$

Consequently, negating this formula we obtain the first-order condition

$$\forall a_2, a_1, a_0 \in P_1(a_2 \leq a_1 \implies ({\downarrow} a_0 \cap {\downarrow} a_1 \not\subseteq ({\uparrow} a_0)^{\mathrm{c}} \text{ or } {\downarrow} a_1 \cap f^{-1}({\downarrow} f(a_2)) \not\subseteq f^{-1}([{\uparrow} f(a_2)]^{\mathrm{c}}) \cup ({\uparrow} a_1)^{\mathrm{c}})),$$

which again may be rewritten to

$$\forall a_2, a_1, a_0 \in P_1((a_2 \leq a_1 \text{ and } a_0 \not\leq a_1) \implies ({\downarrow} a_1 \cap f^{-1}({\downarrow} f(a_2)) \not\subseteq f^{-1}([{\uparrow} f(a_2)]^{\mathrm{c}}) \cup ({\uparrow} a_1)^{\mathrm{c}}))).$$

Further rewriting yields

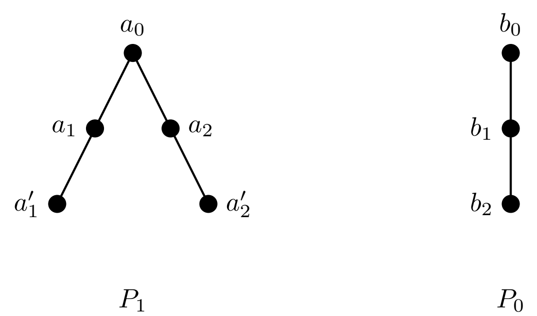
$$\forall a_2, a_1, a_0 \in P_1((a_2 \leq a_1 \text{ and } \leq a_0 \not\leq a_1) \implies \exists a \leq a_1 \ (f(a) \leq f(a_2) \text{ and } f(a_2) \leq f(a) \text{ and } a_1 \leq a)).$$

That is

$$\forall a_2, a_1, a_0 \in P_1((a_2 \leq a_1 \text{ and } a_0 \not\leq a_1) \implies f(a_2) = f(a_1)),$$

as desired. □

Example 5.12. Given Proposition 5.11 it is easy to see that the following is a finite intuitionistic one-step frame validating the rule $(bd_2)^*$



where $f(a_0) = b_0$ and $f(a_i) = f(a_i') = b_i$ for $i \in \{1, 2\}$.

This shows that there exist finite intuitionistic one-step frames (P_1, P_0, f) validating the rule $(bd_2)^*$ such that, considered on their own, neither P_1 nor P_0 are frames for the logic $\mathbf{BD}_2$. Nevertheless, we will still show that all such (not necessarily conservative) one-step frames admit an extension in terms of a standard frame of $\mathbf{BD}_2$.

Proposition 5.13. *Every finite one-step frame validating the rule $(bd_2)^*$ can be extended to a finite standard frame validating $\mathbf{BD}_2$.*

Proof. Let $\mathcal{F} := (P_1, P_0, f_1)$ be a finite one-step frame validating $(bd_2)^*$. Let $\leq_{f_1}$ be the partial order on (the underlying set of) P_1 defined in Section 5.1. The identity on (the underlying set of) P_1 will then be an order-preserving map since $\leq_{f_1} \subseteq \leq_1$. Moreover this map will also be open map relative to f_1, (that this is so, is in fact always the case). It thus remains to be shown that there cannot be any $\leq_{f_1}$-chains of length strictly greater than 2. This, however, is straightforward. For suppose that $a_2 \leq_{f_1} a_1 <_{f_1} a_0$ then $a_2 \leq a_1 < a_0$ and either $f_1(a_2) < f_1(a_1)$ or $a_1 = a_2$. As $\mathcal{F} \models (bd_2)^*$ we have by Proposition 5.11 that $f(a_2) = f(a_1)$ from which we may conclude that $a_1 = a_2$, showing that there are no $\leq_{f_1}$-chains of length strictly greater than 2. □

Remark 5.14. Note that the step-condition for the rule $(bd_2)^*$ in conjunction with conservativity entails that every finite conservative intuitionistic one-step frame $\mathcal{F} = (P_1, P_0, f)$ validating the rule $(bd_2)^*$ is such that P_1 will be of depth at most 2 and thus validate the logic $\mathbf{BD}_2$. That this is indeed the case can be see as follows: If $a_1 < a_0$ in P_1 then for all $a_2 \leq a_1$ we have that $f(a_2) = f(a_1)$. Consequently, $f[{\downarrow} a_1] = \{f(a_1)\}$ and so since $\mathcal{F}$ is conservative this entails that ${\downarrow} a_1$ is a singleton. Therefore, no chain in P_1 is of length greater than 2.

Of course the fact that the calculus $\mathbf{HBD_2}$ obtained by adding the rule $(bd_2)^*$ to $\mathbf{HLJ}'$ enjoys the bounded proof property is an immediate consequence of the fact that the calculus $\mathbf{HBD_2}$ enjoys cut-elimination and the subformula property as shown in [24]. However, the point is that with our method establishing the—admittedly weaker—bounded proof property can be done very easily—and in fact, as we have just seen, almost completely mechanically.

6. Conclusion and future work

We have shown how to transfer the techniques and results of [12, 13] from the setting of modal logic to the setting of intermediate logics. That is, we have established semantic criteria determining when a given hypersequent calculi for an intermediate logic enjoys a certain weakly analytic subformula property; namely the bounded proof property. Analogously to the modal case these criteria are based on extension properties of structures interpreting the degree 1 fragment of the language of **IPC**. Furthermore, we have tested these criteria on a number of examples.

The results obtained in this paper suggest that the methodology introduced in [12] is fairly modular and that it may successfully be applied to obtain similar results for other non-classical logics. For instance, we expect that in the case of intermediate logics it would also be possible to characterise (hyper)sequent calculi for which the maximal number of $\vee$-nestings can be bounded. Moreover, we find it worth investigating if similar results can be obtained for substructural logics. That is, given a connective $*$ and a substructural logic L such that the $*$-free reduct is locally tabular over L can extension properties of appropriate one-step structures characterising the bounded proof property with respect to $*$ of (hyper)sequent calculi for extension of L?

Showing that a given calculus has the bounded proof property and the finite model property via the semantic characterisation of Theorem 4.9 looks an automatisable task: One applies some version of algorithmic correspondence theory and then looks for the appropriate pattern in order to transform one-step frames into Kripke frames. Experience shows that such patterns are classifiable, so that the relevant meta-theory of these logics should effectively be handled with the help of a proof assistant.

Computational complexity issues are still to be investigated: Although the mere invocation of bounded proof property yields heavy (usually non-optimal) complexity bounds, there is still the possibility that semantic constructions employed in this paper could give useful search bounds for sufficient classes of 'one-step' countermodels.

Finally, we point out yet another open question: is it possible to find a class $\mathcal{Q}$ of formulas, extending the class $\mathcal{P}_3$, and an effective procedure, similar to the one found in [22], yielding for each $\varphi \in \mathcal{Q}$ a finite set of (logical) hypersequent rules $\mathcal{R}_\varphi$ which determine a hypersequent calculus for $\mathbf{IPC} + \varphi$ with the bounded proof property?

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