A Coalgebraic Semantics for Fischer Servi Logic

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written by

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Abstract

We present a new coalgebraic semantics for the intuitionistic modal logic known as **IK** or *Fischer Servi logic*, providing representations both for its modal spaces and for its image-finite Kripke frames. Our work is based on a recent construction by Almeida [Alm24], which has made coalgebraic analysis of intuitionistic modal logics possible. In particular, it provides a functorial method of turning coalgebras for a positive modal logic into coalgebras for its least intuitionistic extension, as shown by Almeida and Bezhanishvili [AB24]. This does not suffice on its own to treat Fischer Servi logic, as it is not the least intuitionistic extension of a positive modal logic. Thus, we fill this gap in the research by providing a modified approach, which yields coalgebraic completeness for **IK**. As an application of these results, we study bisimulations for Fischer Servi logic, describe the dual spaces of free **IK** algebras, and show how our approach can be used to capture extensions of **IK** with rank-1 axioms.

Contents

Chapter 1

Introduction

Modal logic and intuitionistic logic represent two major and well-established areas of research in logic. On the one hand, modal reasoning allows for qualification over truth – equipped with the expressive power to capture not only what is true, but what is necessarily or possibly true. On the other hand, intuitionistic logic embodies the principle of constructivism – that truth amounts to provability and evidence – providing a foundational framework for constructive mathematics and computation.

Thus, one can easily imagine settings in which the combination of modal and intuitionistic logic would be natural – refining the modal notions of necessity and possibility to be grounded in verification rather than abstract truth. However, it is not entirely obvious what a constructive analogue to classical modal logic should be: in an intuitionistic setting, the modalities $\Box$ and $\Diamond$ must not be defined as each others' duals, and should instead be independent. This results in much more freedom of how an intuitionistic modal logic might be defined. One popular approach is to study intuitionistic modal logic with only the $\Box$ modality (e.g. [Ono77], [BD84], [Doš85], [AB24]), which evades the inherent complications of working with both modalities over an intuitionistic base. However, it is clearly desirable to have an intuitionistic modal logic which is capable of expressing both necessity and possibility. Several bimodal intuitionistic logics have been proposed, but there are good reasons to prefer some over others. In his PhD thesis, Simpson [Sim94] puts forth that a constructive modal logic IML should meet the following conditions:

1. The propositional fragment of IML should yield IPC.
2. IML contains all substitution instances of theorems of IPC and is closed under modus ponens.
3. The addition of the schema $A \vee \neg A$ to IML yields classical modal logic.
4. If $A \vee B$ is a theorem of IML then either A is a theorem of IML or B is.
5. $\Box$ and $\Diamond$ are independent.
6. There is an intuitionistically comprehensible explanation of the meaning of the modalities, relative to which IML is sound and complete.

Simpson shows that, unlike many other intuitionistic modal logics, the logic IK or *Fischer Servi logic* ([Fis84], [Ewa86], [PS86]) meets each one of these conditions. Thus, Fischer Servi logic has strong claims to providing an excellent, bimodal analogue for classical modal logic. In this thesis, we aim to fill a gap in the model theory of Fischer Servi logic, by providing its *coalgebraic semantics*.

The conceptual idea behind the coalgebraic approach to modal logic is to conceive of relational structures in terms of categorically-flavoured transition systems (see [Ven07]). To expand on this, let us briefly recall the definition of a coalgebra and a homomorphism between coalgebras. A coalgebra is a pair $(X, \alpha : X \to F(X))$, where F is an endofunctor on a category $\mathbb{C}$, and X and α are respectively an object and a morphism in $\mathbb{C}$. A map $f : (X, \alpha) \to (Y, \gamma)$ is a homomorphism between F-coalgebras if f is a map in $\mathbb{C}$ such that $\gamma \circ f = Ff \circ \alpha$. These definitions correspond in a very natural way to relational structures and bounded morphisms between them. Generally speaking, a relational structure (X, R) can be seen as a coalgebra $(X, R[-] : X \to F(X))$, where $R[-]$ is the map sending a point in X to its set of successors, the "shape" of which is restricted by the functor F; in the case of a classical Kripke frame, F is the powerset functor. On the other hand, the nature of a bounded morphism corresponds naturally to the commutativity of its corresponding diagram (see [Ven07, Example 9.10]). Under this perspective, providing the coalgebraic semantics for a given modal logic amounts to determining the correct functor. If we have this representation, then studying the properties of the logic amounts to studying the coalgebraic behaviour – which naturally generates notions such as bisimulation and free algebras. Thus, coalgebraic modal logic offers a general and uniform theory, capable of subsuming both Kripke and topological semantics into a category-theoretic framework.

While classical and positive modal logics have well-known coalgebraic semantics (see [VV14], [Pal04a], [BHM23]), the intuitionistic case is much more elusive (as raised by [Lit14] and [DP20]). The issue is that, in a coalgebraic representation, the morphisms *within* the coalgebras and the homomorphisms *between* the coalgebras should belong to the same category. Unlike the classical and positive cases, intuitionistic propositional logic is already modal – with intuitionistic implication treated semantically like a modality. Thus, to preserve truth of intuitionistic implication, bounded morphisms between intuitionistic frames must respect the $\leq$ relation, so the corresponding coalgebra homomorphisms should be p-morphisms (with respect to the order). If we then add another relation R to govern the modalities, $R[-]$ as a function is not necessarily required to be a p-morphism. Thus, representing intuitionistic modal frames coalgebraically requires a way to impose different conditions on the coalgebra morphisms and the homomorphisms between coalgebras.

It was only recently that a technique was introduced [Alm24], [AB24] (generalizing the construction in [Ghi92]) to derive coalgebraic semantics for intuitionistic modal logics, by turning coalgebras for a positive modal logic into coalgebras for its least intuitionistic extension. The essential idea is to first ensure a correspondence on the level of objects, turning frames into coalgebras in a category whose morphisms need only be monotone. Provided a way to ensure one-to-one correspondence on objects, one can then lift these coalgebras to a category whose maps are additionally p-morphisms, thereby removing all unwanted coalgebra homomorphisms.

This method springs from the construction, due to Almeida [Alm24], of two special functors. Namely the right adjoint $\mathcal{V}_G$ to the inclusion of Esakia spaces into Priestley spaces, and its 'discrete' analogue P_G – the right adjoint to the inclusion of image-finite posets with p-morphisms into posets with monotone maps. The former allows for coalgebraic treatment of intuitionistic descriptive general frames (or intuitionistic *modal spaces*), and the latter for *image-finite* intuitionistic Kripke frames. This was used successfully by Almeida and Bezhanishvili [AB24] to provide coalgebraic semantics for the $\Box$-only fragment of intuitionistic modal logic.

However, a central limitation of such an approach lies in the fact that one needs to start with a positive modal logic, and find its least intuitionistic extension. Thus, it is insufficient on its own to treat intuitionistic modal logics which are not the least intuitionistic extension of a positive reduct – as is the case with Fischer Servi logic. In this thesis we show how the approach of [AB24] can be modified to yield coalgebraic completeness for **IK** and other related logics. Schematically, this is achieved by choosing a positive fragment, then modifying it in steps by adding single layers of implications, thereby allowing for the remaining axioms to be imposed. The resulting positive structure can then be extended to capture the full intuitionistic modal logic, following the same methods as in [AB24].

Using our adapted approach, we develop the coalgebraic representation of both modal spaces (Theorem 4.18) and image-finite Kripke frames (Theorem 4.35) for Fischer Servi logic. As an application of our results, we derive a notion of bisimulation for Fischer Servi logic, describe the dual spaces of free **IK**-algebras, and show how our approach can be used to capture extensions of the logic with rank-1 axioms. This thesis presents a uniform approach to Fischer Servi logic, with applications to other related intuitionistic modal logics. Thus, our contribution is to provide a coalgebraic treatment of a class of intuitionistic modal logics that has not previously been addressed, thereby advancing the program of obtaining uniform results for intuitionistic modal logic within the coalgebraic framework for modal logic.

This thesis is structured as follows: in Chapter 2, we introduce the relevant logics together with their topological, algebraic, and Kripke-style semantics. In Chapter 3, we introduce in detail the coalgebraic approach to modal logic and describe the step-by-step constructions of $\mathcal{V}_G$ and P_G due to [Alm24]. In Chapter 4, we present our main results, providing a coalgebraic representation for both image-finite Kripke frames and modal spaces for Fischer Servi logic. In Chapter 5, we present applications of our construction; we derive a notion of bisimulation between Fischer Servi frames, provide an explicit construction of the dual space to the free **IK**-algebra, and exemplify how our approach subsumes rank-1 extensions of Fischer Servi logic.

Chapter 2

Preliminaries

The goal of this chapter is to introduce modal logics over positive and intuitionistic bases, together with their semantics – namely Kripke-style, algebraic, and topological. We note that, while we introduce Fischer Servi logic in this chapter, it will be presented and discussed in much more detail in Chapter 4. This chapter is structured as follows: we begin by introducing the logics and their Kripke-style semantics. We then present their algebraic semantics, as well as several useful notions from universal algebra. The third section is concerned with topology and the order-topological semantics for our logics. We end this chapter by presenting the dualities for the categories of algebras we will work with.

Throughout, we assume familiarity with modal logic and category theory. We point the reader to [BRV01] or [CZ97] for an overview of modal logic, and to [Awo10] for the necessary concepts from category theory.

2.1 Logics

We begin this chapter by introducing the logics we will be working with, together with their Kripke-style semantics. We first present intuitionistic propositional logic (IPL) and positive propositional logic (PPL), then introduce modal logics with a single additional modality over classical, intuitionistic, and positive bases. With this in place, we will introduce the Fischer Servi logic IK, which will be the central focus of this thesis. We assume familiarity with classical propositional and modal logic. We refer the reader to [BRV01], [CZ97] for an overview of modal logic.

2.1.1 Intuitionistic and Positive Propositional Logic

Throughout, let Prop be a set of proposition letters, whose elements are denoted $p, q, r, \ldots$.

Definition 2.1 (Formulas of IPL)**.** *The well-formed formulas* Form *of the intuitionistic propositional language* $\mathcal{L}_{\text{IPL}}$ *are given by the rule*

$$\mathsf{Form} ::= p \ \mid \ \bot \ \mid \ (\phi \vee \psi) \ \mid \ (\phi \to \psi)$$

Negation and conjunction are introduced as abbreviations, with $\neg\phi := \phi \to \bot$*, and* $\phi \wedge \psi := \neg(\neg\phi \vee \neg\psi)$*.*

For an axiomatization of IPL, see for example [Min00]. We recall here that the contrasting factor between intuitionistic logic and classical logic is that the law of excluded middle $\phi \vee \neg\phi$ is not a theorem of intuitionistic logic. That is, adding the schema $\phi \vee \neg\phi$ to IPL yields classical propositional logic.

We now introduce positive propositional logic (PPL) – the implication-free fragment of IPL – as it will prove useful in our work to consider modal logic over a positive base.

Definition 2.2 (Formulas of PPL)**.** *The well-formed formulas* Form *of the positive propositional language* $\mathcal{L}_{\mathrm{PPL}}$ *are given by the rule*

$$\mathsf{Form} ::= p \mid \bot \mid (\phi \wedge \phi) \mid (\phi \vee \phi)$$

Note: In the absence of implication, one cannot introduce negation as in the intuitionistic case – hence the nomenclature 'positive logic'. Thus it is also necessary to add both conjunction and disjunction to the logic if we wish to have both, as they can no longer be defined in terms of each other.

Both IPL and PPL admit Kripke-style semantics over partially odered sets. Thus, we take this opportunity for a brief interlude to introduce the categories of posets we will work with and collect a few useful properties, before introducing the relational semantics for IPL and PPL.

Categories of posets

Definition 2.3. *A partially ordered set or poset is a pair* $(X, \leq)$ *where* X *is a set and* $\leq$ *is a relation that is reflexive* $(\forall a \in X.(a \leq a))$*, transitive* $(\forall a, b, c \in X.(a \leq b \wedge b \leq c \to a \leq c))$*, and antisymmetric* $(\forall a, b \in X.(a \leq b \wedge b \leq a \to a = b))$*.*

Definition 2.4 (Monotone map)**.** *If* P, P' *are posets, we say that a map* $f : P \to P'$ *is* monotone *if, whenever* $x, y \in P$ *and* $x \leq y$*,* $f(x) \leq f(y)$ *in* P'*.*

We denote by **Pos** the category of posets and monotone maps.

Given a set X and $A \subseteq X$, we write $\uparrow A := \{x \in X | x \geq a$ for some $a \in A\}$, and call this the *upset* of A. If $A = \uparrow A$ we say that A itself is an upset. Likewise, we write $\downarrow A := \{x \in X | x \leq a$, and say that A is a downset if $A = \downarrow A$. We will write $\uparrow x$ as a shorthand for $\uparrow\{x\}$.

Definition 2.5 (Image-finite poset)**.** *Given a poset* $\mathbf{P} = (P, \leq)$*, we say that* $\mathbf{P}$ *is* image-finite *if for every* $x \in P$*,* $\uparrow x$ *is finite.*

Definition 2.6 (p-morphism)**.** *Let* P *and* P' *be posets. A map* $f : P \to P'$ *is called a* p-morphism *if*

- f *is monotone (*forth condition*);*
- *If* $f(x) \leq y$ *then there is* $x' \geq x$ *in* X *such that* $f(x') = y$ *(* back condition*).*

We denote by $\mathbf{ImFinPos}_p$ the category of image finite posets and p-morphisms.

Note: We will later introduce classes of frames with both a partial ordering and additional relations. The p-morphism conditions for these frames will vary across the different relations. Thus, wherever ambiguity may arise, we distinguish between the two by calling maps which are p-morphisms with respect to $\leq$ *order p-morphisms.*

Kripke-style semantics for IPL and PPL

The relational semantics for intuitionistic propositional logic are given over posets $(X, \leq)$. One can think of the order as an information ordering, wherein new facts accumulate, but no previously accepted information can be refuted. This is called *persistence*, and is captured by the restriction that valuations be monotone:

Definition 2.7 (Model of IPL)**.** *Let* $(X, \leq)$ *be a poset, and* $V : X \to \mathcal{P}(\mathsf{Prop})$ *a valuation. Then* $(X, \leq, V)$ *is an* intuitionistic Kripke model *provided that* V *is* monotone *with respect to inclusion. That is, for* $x, x' \in X$

$$x \leq x' \implies V(x) \subseteq V(x').$$

It is worth noting here that intuitionistic propositional logic is already modal in nature. This is evident from the semantics of implication, given below.

Definition 2.8. *Let* $M = (X, \leq, V)$ *be a model for IPL, and* $x \in X$*. The satisfaction relation is defined as follows:*

- $M, x \models p$ *iff* $p \in V(x)$*,*
- $M, x \models \bot$ *never,*
- $M, x \models \phi \vee \psi$ *iff* $M, x \models \phi$ *or* $M, x \models \psi$*,*
- $M, x \models \phi \to \psi$ *iff for all* $x' \in X$ *such that* $x \leq x'$*, it holds that if* $M, x' \models \phi$ *then* $M, x' \models \psi$*.*

Note that the satisfaction of $\phi \to \psi$ is defined in terms of the order. Thus, already in the basic propositional case, the semantics are given over Kripke frames, with the relation $\leq$ governing the $\to$ modality. As we will see, this attribute will add a layer of complication when adding modalities to IPL.

As in the case of IPL, the semantics of PPL are given over posets $(X, \leq)$, and valuations are again required to be monotone [1]. The semantics of the positive connectives are standard.

While both IPL and PPL are complete with respect to posets, they require different notions of bounded morphisms between their Kripke frames. For PPL frames, the bounded morphisms need only be monotone maps, while those for IPL frames must additionally be p-morphisms in order to preserve truth of formulas $\phi \to \psi$.

Thus, while the category **Pos** captures positive frames and their bounded morphisms, we will require an appropriate subcategory for IPL frames. For this, we will use $\mathbf{ImFinPos}_p$ – noting that this is justified as IPL has the finite model property so is also complete with respect to image-finite posets.

2.1.2 Modal $\Box$-logics and their relational semantics

We now present the modal logics resulting from adding the $\Box$ operator to classical, intuitionistic, and positive propositional bases (respectively), together with their Kripke-style semantics. In all cases, the following axioms are added to account for the addition of the $\Box$ modality:

[1]The partial order and monotonicity of valuations is not always included in the literature on positive logic (see for example [Dun95]). It is not strictly necessary in the absence of implication. However, it is desirable for several reasons to include it. For a discussion on this, see [CJ97]

$$\Box(\phi \wedge \psi) \leftrightarrow \Box\phi \wedge \Box\psi,$$
$$\Box\top \leftrightarrow \top.$$

We begin with the classical case. Recall that classical modal logic (CML) extends classical propositional logic with the modality $\Box$.

Definition 2.9 (Formulas of CML). *The well-formed formulas* Form *of the language* $\mathcal{L}_{\mathrm{CMC}}$ *are given by the rule*

$$\mathsf{Form} ::= p \ \mid \ \bot \ \mid \ (\phi \vee \psi) \ \mid \ \neg\phi \ \mid \ \Box\phi$$

We recall that a Kripke frame for CML is a pair (X, R) where X is a set and $R \subseteq X \times X$ is a binary relation. Models are constructed by taking *valuations*, i.e. functions $V : \mathrm{Prop} \to \mathcal{P}(W)$, assigning a set of worlds to each proposition letter.

Definition 2.10. *Let $M = (W, R, V)$ be a model for classical modal logic, and $x \in W$. The satisfaction relation is defined as follows:*

- *$M, w \models p$ iff $w \in V(p)$,*
- *$M, w \models \bot$ never,*
- *$M, w \models \neg\phi$ iff $M, w \not\models \phi$,*
- *$M, w \models \phi \vee \psi$ iff $M, w \models \phi$ or $M, w \models \psi$,*
- *$M, w \models \Box\phi$ iff for all $v \in W$ such that wRv, $M, v \models \phi$.*

Maps between bounded Kripke frames are required to be p-morphisms (as in Definition 2.6) with respect to R.

Recall that in the classical case, the $\Diamond$ modality can be defined as the *dual* of $\Box$, i.e. $\Diamond\phi := \neg\Box\neg\phi$, with the satisfaction of a formula $\Diamond\phi$ given by

$$M, w \models \Diamond\phi \text{ iff there is } v \in W \text{ such } wRv \text{ and } M, v \models \phi.$$

Thus, one may give an equivalent definition by adding the $\Diamond$ operator instead, and defining $\Box := \neg\Diamond\neg$. In either case, adding one of the operators automatically gives you the other, and the resulting logics are the same. As we will see shortly, this is not the case for modal logics with intuitionistic or positive bases.

We now turn to the intuitionistic case. Denote by $\mathrm{IML}_\Box$ the modal logic resulting from adding the $\Box$-operator to IPL.

Definition 2.11 (Formulas of $\mathrm{IML}_\Box$). *The well-formed formulas* Form *of the language* $\mathcal{L}_{\mathrm{IML}_\Box}$ *are given by the rule*

$$\mathsf{Form} ::= p \ \mid \ \bot \ \mid \ (\phi \vee \psi) \ \mid \ \phi \to \psi \ \mid \ \Box\phi$$

In the absence of the law of excluded middle, it is not possible to define $\Diamond$ and $\Box$ as each others' duals. Thus, adding the $\Box$ modality to IPL results in a different logic than if we were to add the $\Diamond$ modality.

Definition 2.12 (Intuitionistic $\Box$-Kripke frame)**.** *Let $F = (X, \leq, R)$ where $(X, \leq)$ is a poset and $R \subseteq X \times X$ is a binary relation. We say that F is a* Intuitionistic $\Box$-Kripke frame *if $\Box$ and $\leq$ satisfy the following* mix law*:*

$$R \ = \leq \circ R \circ \leq .$$

Given two intuitionistic $\Box$-Kripke $(X, \leq, R)$ and $(Y, \leq, S)$ frames, we say that a map $f : X \to Y$ is a $\Box$-intuitionistic p-morphism if

1. *f is a* p-morphism *with respect to $\leq$;*
2. *xRy implies $f(X)Rf(y)$;*
3. *$f(x)Sy$ implies there is $z \in X$ such that xRz and $f(z) \leq y$* [2]*.*

Models are again constructed by taking monotone valuations. The semantics of the intuitionistic connectives remains the same as in the propositional case, and the clause for $M, w \models \Box\phi$ is the same as that in Definition 2.10 for the classical case.

For the purposes of this thesis, we will work with intuitionistic Kripke frames over image-finite posets (Definition 2.5), rather than arbitrary posets. We note that completeness with respect to image-finite Kripke frames follows from the finite model property. We denote by $\mathbf{IKF}^{\Box}_{ImFin}$ the category of image-finite $\Box$-Kripke frames and $\Box$-intuitionistic p-morphisms.

We now introduce the positive modal logic $\mathrm{PML}_{\Box}$ resulting from adding the $\Box$ modality to PPL.

Definition 2.13 (Formulas of $\mathrm{PML}_{\Box}$)**.** *The well-formed formulas* Form *of the language $\mathcal{L}_{\mathrm{PML}_{\Box}}$ are given by the rule*

$$\mathsf{Form} ::= p \ \mid\ \bot \ \mid\ (\phi \vee \psi) \ \mid\ (\phi \wedge \psi) \ \mid\ \Box\phi$$

Kripke frames for $\mathrm{PML}_{\Box}$ are defined as in the $\Box$-intuitionistic case (Definition 2.12), though we will consider them over arbitrary posets. Models are again constructed by taking monotone valuations. The semantics of the positive connectives remain the same, and the clause for $\Box$ is again defined as in the classical case (Definition 2.10). The notion of p-morphisms between positive $\Box$-frames is similar to the intuitionistic case, with the exception that f is only required to be monotone (rather than a p-morphism) with respect to the order. We denote by $\mathbf{PKF}^{\Box}_{Pos}$ the category of positive $\Box$-Kripke frames over posets and $\Box$-positive p-morphisms.

Here, in the absence of negation, it is again not possible to define the modal operators as each others' duals. Thus, the logics resulting from adding either $\Diamond$ or $\Box$ to PPL are different.

2.1.3 Modal $\Diamond$-logics and their relational semantics

We denote by $\mathrm{IML}_{\Diamond}$ and $\mathrm{PML}_{\Diamond}$ the logics resulting from adding the diamond operator to IPL and PPL, respectively. Here the additional axioms will instead be

$$\Diamond(a \vee b) \leftrightarrow \Diamond a \vee \Diamond b,$$
$$\Diamond\bot \leftrightarrow \bot.$$

In both the positive and intuitionistic case, adding the $\Diamond$ modality in place of $\Box$ is almost the same,

[2]We note here that this reduces to the standard back condition, given that f is an order p-morphism. We discuss this in more detail later on, and define it like this here given its symmetry with the other cases in this chapter.

Definition 2.14. *Kripke frames for $IML_\Diamond$ and $PML_\Diamond$ are triples $(X, \leq, R)$, where $(X, \leq)$ is a poset and R a binary relation, just as in the case for $\Box$. The mix law for positive and intuitionistic $\Diamond$-frames is instead*

$$R \; = \geq \circ R \circ \geq .$$

We note that these mix laws are required in order to ensure persistence of formulas of the form $\Box\phi$ and $\Diamond\phi$ in their respective settings. The semantics of $\Diamond$ also result in different notions of p-morphisms than for positive and intuitionistic $\Box$-Kripke frames.

Definition 2.15 (p-morphisms for positive and intuitionistic $\Diamond$-Kripke frames)**.** *Given two positive $\Diamond$-Kripke frames $(X, \leq, R)$ and $(Y, \leq, S)$ frames, we say that a map $f : X \to Y$ is a* $\Diamond$-positive p-morphism *if*

- *f is* monotone *with respect to $\leq$;*
- *xRy implies $f(x)Rf(y)$;*
- *$f(x)Sy$ implies there is $z \in X$ such that xRz and $f(z) \geq y$.*

We say that f is a $\Diamond$-intuitionistic p-morphism *if f is additionally a p-morphism with respect to $\leq$.*

We define the following categories:

- $\mathbf{IKF}^{\Diamond}_{ImFinPos_p}$ is the category of intuitionistic $\Diamond$-Kripke frames over image-finite posets and their p-morphisms;
- $\mathbf{PKF}^{\Diamond}_{Pos}$ the category of positive $\Diamond$-Kripke frames over posets and their p-morphisms.

We remark that completeness is known for these $\Box$- and $\Diamond$- modal logics, regarding their respective semantics (see for example [CZ97], [Pal04a], [CJ97], [BHM23]).

The logics induced by adding only one operator to an intuitionistic or positive base are comparatively simple. The case becomes more complicated if we wish to equip these logics with both operators. While the modalities should be independent, they should still satisfy certain compatibilities. The question arises of whether there should be two separate relations governing each modality, and in what way they should interact with each other.

Dunn [Dun95] puts forth a positive modal logic with both $\Box$ and $\Diamond$, imposing the interaction postulates $\Box(a \vee b) \leq \Box a \vee \Diamond b$ and $\Box a \wedge \Diamond b \leq \Diamond(a \wedge b)$, which allow for the semantics of both modalities to be governed by the same relation. Dunn-style positive modal logic is well-studied, and has known coalgebraic semantics (see [BHM23], [Pal04a]). However, as discussed in the introduction, the intuitionistic case is more complicated.

2.1.4 Fischer Servi Intuitionistic Modal logic (IK)

In this thesis, we will study the intuitionistic modal logic referred to as **IK** or *Fischer Servi* logic, defined by Fischer Servi in [Fis77] [Fis80] and axiomatized in [Fis84]. We note that **IK** was also introduced independently by Ewald [Ewa86] and Plotkin and Stirling [PS86]. In the remainder of this section, we introduce this logic together with its relational semantics.

Definition 2.16. *The well-formed formulas* Form *of the language* $\mathcal{L}_{\mathrm{IK}}$ *are given by the rule*

$$\mathsf{Form} ::= p \mid \bot \mid (\phi \vee \psi) \mid \phi \to \psi \mid \Box\phi \mid \Diamond\phi$$

Definition 2.17 (Axiomatization of **IK**)**.** *The axiom system of* ***IK*** *is given as follows* [3]*:*

Axioms:

0. All substitution instances of theorems of IPL
1. $\Box(A \to B) \to (\Box A \to \Box B)$
2. $\Box(A \to B) \to (\Diamond A \to \Diamond B)$
3. $\neg\Diamond\bot$
4. $\Diamond(A \vee B) \to (\Diamond A \vee \Diamond B)$
5. $(\Diamond A \to \Box B) \to \Box(A \to B)$

Rules:

(MP) From A and $A \to B$, deduce B.
(Nec) From A, deduce $\Box A$.

Figure 2.1: Axiomatization of **IK**

Definition 2.18 (**IK** frame)**.** *Let* $F = (X, \leq, R)$ *where* $(X, \leq)$ *is a poset and* R *a binary relation. We say that* F *is an* ***IK*** *frame provided the conditions* ***F1*** *and* ***F2*** *are satisfied, listed below and illustrated diagrammatically in Figure 2.2.*

F1 $x' \geq xRy \implies \exists y'.x'Ry' \geq y$*;*

F2 $xRy \leq y' \implies \exists x'.x \leq x'Ry'$*.*

Models for ***IK****, sometimes referred to as* birelation models*, are again constructed by taking monotone valuations.*

$$\begin{tikzcd} x' \arrow[r, dotted, "R"] & y' \\ x \arrow[u, "\leq"] \arrow[r, "R"'] & y \arrow[u, dotted, "\leq"'] \end{tikzcd} \qquad \begin{tikzcd} x' \arrow[r, dotted, "R"] & y' \\ x \arrow[u, dotted, "\leq"] \arrow[r, "R"'] & y \arrow[u, "\leq"'] \end{tikzcd}$$

F1 **F2**

Figure 2.2: **IK** frame conditions

[3]This axiomatization is due to [PS86]

Definition 2.19 (Kripke-style semantics of **IK**)**.** *Let $(X, \leq, R)$ be an **IK** frame. The semantics of the intuitionistic connectives are kept the same, and satisfaction for the modal connectives is given as follows:*

- $w \models \Box A \iff \forall w' \geq w. \forall v'. w' R v' \implies v' \models A$
- $w \models \Diamond A \iff \exists v. w R v \& v \models A$

A proof of completeness for **IK** with respect to these semantics can be found in [Fis84].

2.2 Algebra

In this section, we introduce the algebraic semantics for the logics we have defined, and take the opportunity to fix the algebraic notions we will need. We begin by defining the categories of distributive lattices and Heyting algebras, which provide the algebraic semantics for our PPL and IPL respectively. We then present the modal algebras corresponding to the $\Box$- and $\Diamond$- modal logics over intuitionistic and positive bases, and finally present the algebraic semantics for Fischer Servi logic. We assume familiarity with universal algebra, and refer the reader to [BS81], for example, for a comprehensive overview.

2.2.1 General concepts

Recall that an algebra over a type $\Sigma = \{f_i | i \in I\}$ is a structure $\mathbb{A} = (A, (f_i)_{i \in I})$ where A is a non-empty set and $(f_i)_{i \in I}$ is a family of operations $f_i : A^n \to A$ whose arity is specified by Σ.

A homomorphism between algebras of the same signature is defined as follows:

Definition 2.20 (Algebra homomorphism)**.** *Let $\mathbb{A}$ and $\mathbb{B}$ be two algebras over the same signature Σ. A map $\phi : A \to B$ is a* homomorphism *from A to B if for every $f \in \Sigma$ and all $x_1, ..., x_n \in A$,*

$$\phi(f(x_1, ..., x_n)) = f(\phi(x_1), ...\phi(x_n)).$$

Recall that a variety $\mathcal{K}$ of algebras is said to provide the algebraic semantics of a logic L if for any formula ϕ in the language $\mathcal{L}_{\mathsf{L}}$ of L, ϕ is valid in L if and only if for each algebra $A \in \mathcal{K}$, $A \Vdash \phi = 1$, where this means that, model theoretically, $A \vDash \forall \overline{x}(\phi(\overline{x}) = 1)$.

Definition 2.21 (Free algebra)**.** *Let $\mathbf{K}$ be a class of algebras of type $\mathcal{F}$, and X a set of variables. We say that an algebra $\mathcal{F}(X)$ is* free over *X in $\mathbf{K}$ if there is a map $i : X \to \mathcal{F}(X)$, and for any $A \in \mathbf{K}$ and any function $f : X \to A$ there is a unique homomorphism $\overline{f} : \mathcal{F}(X) \to A$ such that $\overline{f} \circ i = f$. This is illustrated in Figure 2.3.*

$$\begin{tikzcd}
X \arrow[rr, "i"] \arrow[dr, "f"'] & & \mathcal{F}(X) \arrow[dl, dotted, "\overline{f}"] \\
& A &
\end{tikzcd}$$

Figure 2.3: Commuting Diagram for Free Algebras

Given a type Σ, the free Σ-algebra over a set X of generators is given by freely closing X under the operations in Σ, and quotienting over the axioms governing the operations. That is, add $f_i^n(x_0, ..., x_n)$

for all $\{x_0, ..., x_n\} \subseteq X$, quotient, and repeat, until we have all terms built from the elements of X and the operations in Σ.

2.2.2 Algebraic semantics for positive and intuitionistic logic

Recall the definition of a poset from Definition 2.3.

Definition 2.22 (Lattice)**.** *A* lattice *is a poset $(L, \leq)$ such that for each $a, b \in L$, $a \wedge b$ and $a \vee b$ both exist.*

Where $a \wedge b$ is the *infimum* of $\{a, b\}$ (the greatest element that is less than both a and b) and $a \vee b$ is the *supremum* of $\{a, b\}$ (the least element that is greater than both a and b).

We say that a lattice L is *bounded* if there are elements $\bot, \top \in L$ such that $\bot \leq a \leq \top$ for all $a \in L$. Henceforth, all lattices will be assumed to be bounded, so we will not specify this.

We begin by defining the algebraic semantics of PPL, which is given by the variety of distributive lattices.

Definition 2.23 (Distributive lattice)**.** *A lattice $(L, \leq)$ is* distributive *if for all $a, b, c \in L$:*

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$
$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$

A lattice $(L, \leq)$ can equivalently be given as a triple $(L, \wedge, \vee)$, where L is a set and $\wedge, \vee : L^2 \to L$ are binary operations satisfying additional conditions. Thus, bounded distributive lattices are algebras over the signature $\{\bot, \top, \wedge, \vee\}$, and provide the algebraic semantics for positive propositional logic.

Theorem 2.24. *PPL is complete with respect to the variety of distributive lattices.*

Definition 2.25 (Bounded lattice homomorphism)**.** *Let L and L' be bounded lattices. A map $h : L \to L'$ is called a* bounded lattice homomorphism *if it preserves the operations. That is,*

$$h(a \wedge b) = h(a) \wedge h(b);$$
$$h(a \vee b) = h(a) \vee h(b);$$
$$h(0) = 0;$$
$$h(1) = 1.$$

We denote by **DL** the category of distributive lattices and bounded lattice homomorphisms.

We now present Heyting algebras, which provide the algebraic semantics for IPL.

Definition 2.26 (Heyting algebra)**.** *A lattice H is called a* Heyting algebra *if for all $a, b \in H$ there is an element $a \to b \in H$, called the* relative pseudocomplement *(of a relative to b), such that for any $c \in H$*

$$c \leq a \to b \textit{ iff } a \wedge c \leq b.$$

Thus, $a \to b = \bigvee\{c \in H | c \wedge a \leq b\}$.

Definition 2.27 (Heyting algebra homomorphism)**.** *If H and H' are Heyting algebras, then a map $h : H \to H'$ is a* Heyting algebra homomorphism *if h is a lattice homomorphism such that $h(a \to b) = h(a) \to h(b)$.*

We denote by **HA** the category of Heyting algebras and Heyting algebra homomorphisms.

Note: Every Heyting algebra is a distributive lattice, and as every Heyting algebra homomorphism is a bounded lattice homomorphism, **HA** is a subcategory of **DL**.

Theorem 2.28. *IPL is complete with respect to the variety of Heyting algebras.*

We will now define the category of *profinite* Heyting algebras. This might seem strange or out of place at first sight, but it is both well-motivated and will prove very useful later on. Indeed, the variety of all Heyting algebras is generated by the profinite ones, which means that IPL is complete with respect to its profinite algebras. Moreover, all finite Heyting algebras are profinite, so this is includes many interesting algebras.

Profinite Heyting algebras are characterized by being *isomorphic to the projective limit of a projective system of finite Heyting algebras.* Let us first define what this means.

Definition 2.29 (Projective system)**.** *A projective system of finite Heyting algebras is a pair $((A_i)_{i\in I}, (\pi_{ji})_{i\leq j\in I})$ where*

- *$(A_i)_{i\in I}$ is a family of finite Heyting algebras;*
- *$(\pi_{ji})_{i\leq j\in I}$ is a family of Heyting algebra morphisms, with $\pi_{ji} : A_j \to A_i$;*
- *If $i \leq j \leq k$, then $\pi_{ij}\pi_{jk} = \pi_{ik}$;*
- *$\pi_{ii} = id_{A_i}$.*

Definition 2.30 (Projective limit of finite Heyting algebras)**.** *Let $S = ((A_i)_{i\in I}, (f_{i,j})_{i\leq j\in I}$ be a projective system of finite Heyting algebras. The projective limit is given by $S_\infty = (A_\infty, (\mathbf{p}_i)_{i\in I})$ where*

$$A_\infty = \{\overrightarrow{a} \in \prod_{i\in I} A_i | \forall i \leq j.a_i = f_{ij}(a_j)\},$$

and $\mathbf{p}_i : A_\infty \to A_i$ are the projections inherited from the product.

Definition 2.31 (Profinite Heyting algebra)**.** *A Heyting algebra A is called* profinite *if A is isomorphic to the inverse limit of an inverse system of finite Heyting algebras.*

Let $\mathbf{HA}_{pf}$ denote the category of profinite Heyting algebras and Heyting algebra homomorphisms.

2.2.3 Modal algebras

Having presentend the algebraic semantics for our propositional languages, we now introduce the algebraic semantics for their modal extensions.

The modal operations $\Box$ and $\Diamond$ can be interpreted on *modal algebras.* If they are present, the modal operators will be required to satisfy their respective *normality axioms*:

The following definitions can be found, for example, in [BHM23].

Normality axioms for $\Box$	Normality axioms for $\Diamond$
1. $\Box\top = \top$	3. $\Diamond\bot = \bot$
2. $\Box(a \wedge b) = \Box a \wedge \Box b$	4. $\Diamond(a \vee b) = \Diamond a \vee \Diamond b$

Table 2.1: Normality axioms for $\Box$ and $\Diamond$

Definition 2.32 (positive $\Box$-algebra)**.** *A* positive $\Box$-algebra *is a pair $(D, \Box)$ such that $L \in$ **DL** and $\Box$ is a unary function on D satisfying the normality axioms for $\Box$. Let $\mathbf{PMA}^{\Box}$ be the category of positive $\Box$-algebras and bounded lattice homomorphisms preserving $\Box$.*

Definition 2.33 (positive $\Diamond$-algebra)**.** *A* positive $\Diamond$-algebra *is a pair $(D, \Diamond)$ such that $D \in DL$ and $\Diamond$ is a unary function on D satisfying the normality axioms for $\Diamond$. Let $\mathbf{PMA}^{\Diamond}$ be the category of positive $\Diamond$-algebras and bounded lattice homomorphisms preserving $\Diamond$.*

Definition 2.34 (Intuitionistic $\Box$-algebra)**.** *An* intuitionistic $\Box$-algebra *is a pair $(H, \Box)$ such that $H \in$ **HA** and $\Box$ is a unary function on H satisfying the normality axioms for $\Box$. Let $\mathbf{IMA}^{\Box}$ be the category of intuitionistic $\Box$-algebras and bounded lattice homomorphisms preserving $\Box$.*

Definition 2.35 (Intuitionistic $\Diamond$-algebra)**.** *An* intuitionistic $\Diamond$-algebra *is a pair $(H, \Diamond)$ such that $H \in$ **HA** and $\Diamond$ is a unary function on H satisfying the normality axioms for $\Diamond$. Let $\mathbf{IMA}^{\Diamond}$ be the category of intuitionistic $\Diamond$-algebras and bounded lattice homomorphisms preserving $\Diamond$.*

Note that the definitions of $\Box$- and $\Diamond$-algebras in both the intuitionistic and positive case are almost the same, differing only in the requirement that they be distributive lattices in the positive case, and Heyting algebras in the intuitionistic case. It is important to highlight that the homomorphisms for the intuitionistic cases must be Heyting homomorphisms, which is not required in the positive cases.

We end this section by presenting the algebraic semantics for Fischer Servi logic, given by **IK**-algebras. These can be found in [Pal04b], for instance.

Definition 2.36. *An algebra $(H, \wedge, \vee, \to, \Box, \Diamond, \top, \bot)$ is called an* **IK***-algebra if $(H, \wedge, \vee, \to, \top, \bot)$ is a Heyting algebra wich additionally satisfies the normality axioms for $\Box$ and $\Diamond$, as well as the following interaction axioms:*

$$\textbf{\textit{A}}.\ \Diamond(a \to b) \leq \Box a \to \Diamond b,$$
$$\textbf{\textit{B}}.\ \Diamond a \to \Box b \leq \Box(a \to b).$$

We denote by **IKA** the category of **IK**-algebras and Heyting algebra homomorphisms preserving $\Box$ and $\Diamond$.

Theorem 2.37. *The logic **IK** is complete with respect to **IK** algebras.*

2.3 Topology

In this section, we present the spaces which will provide the topological semantics for our logics. Not only are modal spaces dual to modal algebras (which we cover in the following section), but also correspond to descriptive general frames [Ven07]. Essentially, they are Kripke frames with additional topological structure, where the basis determines the admissible subsets. Thus, modal spaces provide a relational alternative to Kripke frames which, through their added structure, avoid the problem of Kripke-incompleteness (see [BRV01, Chapter 4]).

We will begin with Priestley and Esakia spaces, modal spaces with a single modality, and the **IK**-spaces for Fischer Servi logic. We will then collect the basic properties of these spaces that will be useful throughout this thesis. We assume basic familiarity with topology, and point the reader to [Eng77] for a comprehensive overview.

2.3.1 The categories of Stone, Priestley, and Esakia spaces

Notation: Given a map $f : X \to Y$ and a subset $A \subseteq X$, we use square brackets $f[A]$ to denote the *direct image* of A under f, i.e. $f[A] = \{f(a) | a \in A\}$.

Definition 2.38. *A* Stone space *is a compact, Hausdorff space with a basis of clopens.*

We denote by **Stone** the category of Stone spaces and continuous maps.

Definition 2.39 (Priestley space)**.** *A Priestley space is a triple $(X, \leq, \tau)$ such that $(X, \leq)$ is a poset, $(X, \leq, \tau)$ is* compact*, and $(X, \leq, \tau)$ satisfies the* Priestley Separation axiom*:*

$$\textbf{\textit{(PSA)}} \quad \forall x, y \in X. x \not\leq y \implies \exists U \in ClopUp(X). x \in U \text{ and } y \notin U$$

We will denote a Priestley space $(X, \leq, \tau)$ by X wherever no ambiguity arises. We will also not make notational distinctions between the orders of two different Priestley Spaces.

Definition 2.40 (Priestley morphism)**.** *Let X and Y be Priestley spaces. A map $f : X \to Y$ is a* Priestley morphism *if it is continuous and* monotone*, i.e. $x \leq y \implies f(x) \leq f(y)$.*

We let **Pris** denote the category of Priestley spaces and continuous monotone maps.

Definition 2.41 (Esakia space)**.** *An Esakia space is a Priestley space such that whenever $U \subseteq X$ is clopen, then $\downarrow U$ is clopen.*

We denote by **Esa** the category of Esakia spaces and continuous p-morphisms.

2.3.2 Modal spaces

Notation: We recall at this point that if X is a set and $R \subseteq X \times X$ is a binary relation, and $U \subseteq X$, we write $\Box_R(U) := \{x \in X | R[x] \subseteq U\}$ and $\Diamond_R(U) := R^{-1}[U]$.

Definition 2.42 (Modal Stone space)**.** *A modal space is a pair (X, R) where X is a Stone space and $R \subseteq X \times X$ such that*

1. *$R[x]$ is closed for every $x \in X$,*
2. *If $U \in Clop(X)$, then $\Box_R U \in Clop(X)$.*

We now define the categories of $\Box$-only and $\Diamond$-only modal Priestley spaces. These definitions can be found, for example, in [BHM23].

Definition 2.43 ($\Box$-Priestley space)**.** *A $\Box$-Priestley space is a pair $(X, R_\Box)$ where X is a Priestley space and $R_\Box$ is a binary relation on X such that*

1. $R_\Box(x)$ *is a* closed upset *for each* $x \in X$.

2. $R_\Box U$ *is a clopen upset for each clopen upset* U *of* X.

Definition 2.44 ($\Box$-Priestley morphism)**.** *Let* $(X, R_\Box)$ *and* $(X', R'_\Box)$ *be two modal* $\Box$*-Priestley spaces. A Priestley map* $f : X \to X'$ *is a* $\Box$-Priestley morphism *provided*

1. f *is monotone;*

2. $xR_\Box z$ *implies* $f(x)R'_\Box f(z)$*;*

3. $f(x)R'_\Box y$ *implies there is* $z \in X$ *such that* $xR_\Box z$ *and* $f(z) \leq y$.

Note that (2) is the standard forth condition, while (3) is weaker than the usual back condition in that it does not impose equality of $f(z)$ and y.

We denote by $\mathbf{MPS}^\Box$ be the category of $\Box$-Priestley spaces and morphisms.

Definition 2.45 ($\Diamond$-Priestley space)**.** *A* $\Diamond$*-Priestley space is a pair* $(X, R_\Diamond)$ *where* X *is a Priestley space and* $R_\Diamond$ *is a binary relation on* X *such that*

1. $R_\Diamond(x)$ *is a closed* downset *for each* $x \in X$.

2. $\Diamond_R U$ *is a clopen upset for each clopen upset* U *of* X.

Definition 2.46 ($\Diamond$-Priestley morphism)**.** *Let* $(X, R_\Diamond)$ *and* $(X', R'_\Diamond)$ *be two modal* $\Diamond$*-Priestley spaces. A map* $f : X \to X'$ *is a p-morphism provided*

1. f *is monotone;*

2. $xR_\Diamond z$ *implies* $f(x)R'_\Diamond f(z)$*;*

3. $f(x)R'_\Diamond y$ *implies there exists* $z \in X$ *such that* $xR_\Diamond z$ *and* $y \leq f(z)$.

Note that, once again, (3) is weaker than the standard back condition in that it does not impose equality of $f(z)$ and y.

We denote by $\mathbf{MPS}^\Diamond$ the category of $\Diamond$-Priestley spaces and morphisms.

$\Box$- and $\Diamond$-Esakia spaces are defined as $\Box$- and $\Diamond$-Priestley spaces which are additionally required to meet the Esakia condition (see [Pal04b]). The morphisms are also defined much the same, but are additionally required to be p-morphisms with respect to the order (i.e. Esakia morphisms). We denote these categories by $\mathbf{MES}^\Box$ and $\mathbf{MES}^\Diamond$, respectively.

Note that in the case of $\Box$-Esakia morphisms, the condition

$$f(x)R'_\Box y \text{ implies there is } z \in X \text{ such that } xR_\Box z \text{ and } f(z) \leq y$$

is equivalent to the standard back condition with $f(z) = y$, as f is required to be an order p-morphism and thus preserves upsets. No such simplification is possible in the case of $\Diamond$-Esakia morphisms, as these concern downsets.

Here, as in the algebra section, the correspondence between the intuitionistic and positive semantics is quite evident. Indeed, the only difference in these logics is the presence of implications; the intuitionistic cases have the same operations and axioms as the positive ones, with only the addition of implications (and the axioms governing them). In other words, $\mathrm{IML}_\Box$ and $\mathrm{IML}_\Diamond$ are the *smallest intuitionistic extensions* of $\mathrm{PML}_\Box$ and $\mathrm{PML}_\Diamond$, respectively.

We finish this section by presenting the modal spaces for Fischer Servi logic.

Definition 2.47 (**IK**-space)**.** *A **IK**-space is a modal Esakia space* (X, R) *such that the following conditions hold:*

(T1) $R[x]$ *is closed*

(T2) $R[\uparrow x]$ *is a closed upset*

(T3) If U *is a clopen upset, then* $\Diamond_R U$ *and* $\Box_{(\leq \circ R)} U$ *are clopen upsets*

(T4) $R[x] = R[\uparrow x] \cap \downarrow R[x]$

Definition 2.48 (**IK**-space p-morphisms)**.** *Let* (X, R) *and* (Y, S) *be **IK**-spaces. A map* $f : X \to Y$ *is a p-morphism iff for every* $x, x', y \in X, z \in Y$,

1. *if* $x \leq_X y$ *then* $f(x) \leq_Y f(y)$.
2. *If* $f(x) \leq_Y z$ *then* $f(x') = z$ *for some* $x' \in \uparrow x$.
3. *For every* $A \in ClopUp(Y)$, $f^{-1}[A] \in ClopUp(X)$.
4. *If* xRy *then* $f(x)Sf(y)$.
5. *If* $f(x)Sz$ *then* $z \leq_Y f(x')$ *for some* $x' \in R[x]$.
6. *If* $f(x)(\leq_Y \circ S)z$ *then* $f(x') \leq_Y z$ *for some* $x' \in R[\uparrow x]$.

We denote the category of **IK**-spaces and p-morphisms by **IKS**.

2.3.3 Properties of Priestley and Esakia spaces

Throughout this thesis, we will need some basic properties of Priestley and Esakia spaces, so we collect them here.

Lemma 2.49. *Let* $(X, \leq)$ *be a Priestley space. Then if* D *is a closed upset of* X, $D = \bigcap\{U \in ClopUp(X) | D \subseteq U\}$. *Likewise, if* C *is a closed downset of* X, *then* $C = \bigcap\{U \in ClopDown(X) | C \subseteq U\}$.

Proposition 2.50 (Terminal object in **Pris**)**.** *The terminal object in the category **Pris** is the Priestley space* $(\{*\}, \leq)$ *where* $\{*\}$ *is a singleton and* $\leq = \{(\{*\}, \{*\})\}$. *The terminal map* t_X *from any Priestley space* X *is defined by* $t_X(x) = *$.

Proof. It is simple to check that t_X is a Priestley morphism, and uniqueness follows from the fact that Priestley morphisms must be total functions. $\square$

Proposition 2.51. *Let* X *be a Priestley space. Then for any* $x \in X$, *the set* $\uparrow x$ *is a closed subset of* X.

Proof. Let X be a Priestley space with $x \in X$. Then $X - \uparrow x = \{y \in X | x \not\leq y\}$. By the PSA, for each $y \in X - \uparrow x$, there is $U_y \in ClopUp(X)$ such that $x \in U_y$ and $y \notin U_y$.

Then $\bigcap\{U_y | y \notin X\}$ is a closed upset (as it is the intersection of clopen upsets) containing x, so $\uparrow x \subseteq \bigcap\{U_y | y \notin X\}$, and $\forall y \not\geq x. y \notin \bigcap\{U_y | y \notin X\}$. Thus, $\uparrow x = \bigcap\{U_y | y \notin X\}$ is closed. $\square$

Lemma 2.52. *Let X be a Priestley space and $U \subseteq X$ a closed upset. Then for any $z \in X$ such that $z \notin U$, there is $U' \in ClopUp(X)$ such that $U \subseteq U'$ and $z \notin U$.*

Proof. As $z \notin U$ and U is an upset, it must be that $\forall w \in U.(w \not\leq z)$. Then by the PSA (Definition 2.39), there exists for each $w \in U$ a clopen upset V_w such that $w \in V_w$ and $z \notin V_w$. Then $\bigcup\limits_{w \in U} V_w$ is an open cover of U. Recall that as U is a closed subset of a compact space, U is itself compact. Then there is a finite subcover such that $U \subseteq V_{w_1} \cup ... \cup V_{w_n}$. Let $U' := V_{w_1} \cup ... \cup V_{w_n}$. Then as U' is a finite union of clopen upsets, it is itself a clopen upset, and $z \notin U'$ by construction. This concludes the proof. □

Lemma 2.53. *Let X be a Priestley space $C \subseteq X$ a closed downset. Then for any $z \in X$ such that $z \notin C$, there is a clopen upset U such that $z \in U$ and $C \cap U = \emptyset$.*

Proof. Fix a closed downset $C \subseteq X$. Then $\forall y \in C.(z \not\leq y)$. By the PSA, for each $y \in C$ there is $U_y \in ClopUp(X)$ such that $z \in U_y$ and $y \notin U_y$. For each U_y, let $V_y := X - U_y$. Then $\bigcup_{y \in C} V_y$ forms an open cover of C, and as C is compact there is a finite subcover $V = V_{y_1} \cup ... \cup V_{y_n}$ which is a clopen downset. Furthermore, $z \in X - V$, which is a clopen upset. Then $z \in X - V$ and $C \cap (X - V) = \emptyset$, as desired. □

In Chapters 4 and 5, we will make use of inverse limits of Priestley spaces. Thus, we provide the specific definition of projective systems and limits of Priestley spaces, and prove a few additional properties. For a detailed discussion on projective limits of topological spaces, see for example [Eng77, Chapter 2.5].

Definition 2.54 (Projective system of Priestley spaces)**.** *An projective system of spaces is a family $((X_i)_{i \in I}, (\pi_{ij})_{i \leq j \in I})$ where:*

- *$((X_i)_{i \in I}$ is a collection of topological spaces indexed by positive integers;*
- *For any $i \leq j$, $\pi_{ji} : X_j \to X_i$ is a Priestley map.*
- *If $i \leq j \leq k$, then $\pi_{ji}\pi_{kj} = \pi_{ki}$*
- *$\pi_{ii} = id_{X_i}$*

Definition 2.55 (Projective limit of Priestley spaces)**.** *Let $S = ((X_i)_{i \in I}, (\pi_{ij})_{i \leq j \in I})$ be a projective system of spaces; an element $\{x_i\}$ of the Cartesian product $\Pi_{i \in \Sigma} X_i$ is called a thread of S if $\pi_{ji}(x_j) = x_i$ for any $i \leq j$ in I.*

The projective limit *$(X_\infty, (\mathbf{p}_i)_{i \in I})$ of S is the subspace of $\Pi_{i \in I} X_i$ consisiting of all threads of S, together with the projections inherited from the product. In other words,*

- *$X_\infty := \{\vec{a} \in \prod\limits_{i \in I} X_i | \forall i \leq j. x_i = f_{ij}(x_j)\}$, the subspace of $\Pi_{i \in I} X_i$ consisiting of all threads of S, equipped with the subspace topology inherited from the product space.*
- *The family $(\mathbf{p}_i : X_\infty \to X_i)_{i \in I}$ consists of the projections inherited from the product.*

Furthermore, it holds that $\mathbf{p}_i = \pi_{ji} \circ \mathbf{p}_j$ for all $i \leq j$.

We will in general omit explicit mention of the projections $\mathbf{p}_i$ and just write X_∞, wherever this does not lead to ambiguity.

Lemma 2.56. *Let X_∞ be an inverse limit of Priestley spaces. The topology of X_∞ is given by the following base:*

$$\bigcup_{i \in I} \{\mathbf{p}_i^{-1}[U] | U \in ClopUp(X_i)\}.$$

Lemma 2.57. *$U \in ClopUp(X_\infty)$ if and only if $U = \mathbf{p}_i^{-1}[V]$ for some $V \in ClopUp(X_i)$.*

As an intuition for the algebraically minded reader, recall that an inverse limit of Priestley spaces is dual to a direct limit of distributive lattices. Thus, any clopen upset in X_∞ must have arrived from some stage X_i.

Proposition 2.58. *If $U \in ClopUp(X_\infty)$ then $\mathbf{p}_i[U] \in ClopUp(X_i)$.*

Proof. This follows from the surjectivity of the projections, since if $U = \mathbf{p}_i^{-1}[V]$, then $\mathbf{p}_i[U] = \mathbf{p}_i[\mathbf{p}_i^{-1}[V]]$, and as $\mathbf{p}_i$ is surjective, $\mathbf{p}_i[\mathbf{p}_i^{-1}[V]] = V \in ClopUp(X_i)$.

□

2.4 Duality

In this section, we introduce the dualities for the algebraic categories we have presented. We begin by recalling Priestley duality for distributive lattices and Esakia duality for Heyting algebras, and the duality between image-finite posets and profinite Heyting algebras.

2.4.1 Priestley, Esakia, and profinite dualities

Recall that a prime filter of an algebra A is a non-empty, proper subset F of A such that $F = \uparrow F$, F is closed under meets, and if $a \vee b \in F$ then $a \in F$ or $b \in F$.

Given a Priestley space $(X, \leq)$, let $ClopUp(X)$ be the set of clopen *upsets* of X. Then the structure $(ClopUp(X), \subseteq, \cap, \cup, X, \emptyset)$ is a bounded distributive lattice.

Conversely, given a distributive lattice D, let X_D denote the set of prime filters of D. Define the map

$$\phi(a) = \{F \in X_D | a \in F\}$$

We generate a topology τ on X_D by the basis $\{\phi(a)|a \in D\} \cup \{X - \phi(a)|a \in D\}$. Then $(X_D, \subseteq, \tau)$ is a Priestley space.

We then have that if D is a distributive lattice, then $D \approxeq ClopUp(X_D)$. Conversely, if X is a Priestley space, $X \approxeq X_{ClopUp(X)}$.

Furthermore, if $f : A \to B$ is a homomorphism between distributive lattices, then $f^{-1}[-] : X_B \to X_A$ is a bounded lattice homomorphism, and conversely if $g : X \to Y$ is Priestley morphism, then $g^{-1}[-] : ClopUp(Y) \to ClopUp(X)$ is a bounded lattice homomorphism. This leads to the following theorem:

Theorem 2.59 (Priestley duality [Pri70])**.** *The categories **DL** and **Pris** are dually equivalent.*

The duality between **Esa** and **HA** is essentially a special case of Priestley duality. Indeed, we have the following:

- The Priestley dual of a Heyting algebra is an Esakia space;
- The Priestley dual of a Heyting algebra homomorphism is an *Esakia morphism.*

The converse direction is also given much the same, but here we must also consider implication. That is, if X is an Esakia space, then $(ClopUp(X), \subseteq, \cap, \cup, \to, X, \emptyset)$ is a Heyting algebra where for $U, V \in ClopUp(X)$,

$$U \to V := -{\downarrow}(U - V).$$

Remark 2.60. *Recall that the Priestley dual of a bounded lattice homomorphism is a Priestley map. It is worth emphasizing here that the Priestley dual of a Heyting homomorphism is not only continuous and monotone, but also a p-morphism.*

Theorem 2.61 (Esakia duality [Esa74])**.** *The categories* $\boldsymbol{HA}$ *and* $\boldsymbol{Esa}$ *are dually equivalent.*

We now present the duality between $\mathbf{ImFinPos}_p$ and $\mathbf{HA}_{pf}$, established in [BB08]. Together with the algebraic semantics covered earlier, this duality will give us completeness of IPL and PPL with respect to profinite Heyting algebras. This will later allow us to treat **IK** Kripke frames over image-finite posets by considering their dual Heyting algebras.

It is well-known that if X is a poset, the structure $(Up(X), \supseteq)$ is a distributive lattice. Specifically, it is a complete and completely join-prime generated distributive lattice. Furthermore, a distributive lattice D is complete and completely join-prime generated iff D is isomorphic to the distributive lattice $(Up(X), \supseteq)$ of upsets of some poset X (see [Bez99, Section 7]). This leads us to the following:

Theorem 2.62. *The category* $\boldsymbol{Pos}$ *is dually equivalent to* $\mathbf{DL}^+$ *of complete and completely join-prime generated distributive lattices and complete homomorphisms.*

If X is image-finite, then $Up(X)$ defines a profinite Heyting algebra. Indeed, the duality in Theorem 2.62 restricts to the following result, due to [BB08]:

Theorem 2.63 (Profinite duality)**.** *The categories* $\mathbf{HA}_{pf}$ *and* $\mathbf{ImFinPos}_p$ *are dually equivalent.*

For full details, we point the reader to [BB08].

2.4.2 Dualities between modal spaces and modal algebras

We now turn to the dualities for modal spaces and modal algebras. These will be special cases of Priestley and Esakia duality, with the added correspondence between the relation in modal spaces, and the modal operator in modal algebras.

The following dualities for positive spaces and algebras were established by Goldblatt [Gol89]:

Theorem 2.64.

- *The categories* $\mathbf{PMA}^\Box$ *and* $\mathbf{MPS}^\Box$ *are dually equivalent.*
- *The categories* $\mathbf{PMA}^\Diamond$ *and* $\mathbf{MPS}^\Diamond$ *are dually equivalent.*

Finally, let us move on to the intuitionistic case. The following is due to Palmigiano [Pal04b]:

Theorem 2.65.

- *The categories* $\mathbf{IMA}^{\Box}$ *and* $\mathbf{MES}^{\Box}$ *are dually equivalent.*
- *The categories* $\mathbf{IMA}^{\Diamond}$ *and* $\mathbf{MES}^{\Diamond}$ *are dually equivalent.*
- *The categories* $\boldsymbol{IKS}$ *and* $\mathbf{IKA}$ *are dually equivalent.*

With this, we now have completeness for these logics with respect to their Kripke, algebraic, and topological semantics.

Chapter 3

Coalgebraic modal logic

We begin this chapter by introducing the theory of coalgebras and the coalgebraic approach to modal logic, followed by established coalgebraic semantics for positive modal logics. For an overview of the theory of coalgebras and its application to modal logic, we point the reader to [Kur01] and [Ven07]. We then present the step-by-step construction, due to [Alm24], which will enable us to turn coalgebras for positive modal logic into coalgebras for its smallest intuitionistic extension.

3.1 Coalgebras and coalgebraic modal logic

This section introduces the theory of coalgebras and the well-known coalgebraic representations of classical Kripke frames and modal Stone spaces. This will serve as a foundation and allow us to fix terminology for when we present our coalgebraic approach to Fischer Servi frames in Chapter 4.

Definition 3.1 (Coalgebra)**.** *Let $\mathbb{C}$ be some category, and $F : \mathbb{C} \to \mathbb{C}$ an endofunctor. An F-coalgebra is a pair $(X, \alpha : X \to FX)$ for X an object in $\mathbb{C}$ and α a morphism in $\mathbb{C}$.*

Definition 3.2 (Coalgebra homomorphism)**.** *Let $(X, \alpha : X \to FX)$ and $(Y, \beta : Y \to FY)$ be two F-coalgebras. A* coalgebra homomorphism *from (X, α) to (Y, β) is an arrow $h : X \to Y$ in $\mathbb{C}$ such that the diagram in Figure 3.1 commutes, i.e. $\beta \circ h = Fh \circ \alpha$.*

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "\alpha"] & Y \arrow[d, "\beta"] \\ FX \arrow[r, "Fh"] & FY \end{tikzcd}$$

Figure 3.1: Coalgebra homomorphism

In general, a frame for a modal logic consists of (1) a carrier set (possibly with additional structure, as with modal spaces), (2) a relation, and (3) the way in which the relation behaves. Intuitively, to turn a frame into a coalgebra, (1) is captured by the object, (2) by the morphism, and (3) by the functor. Thus, the theory of coalgebras provides a very natural way of interpreting the semantics of modal logics (see for example [Ven07]). This is represented most intuitively in the classical case. Thus, we present the coalgebraic representation of classical Kripke frames and models in Example 3.3 below, for which we refer to [Ven07, Example 9.4].

Example 3.3. *Let (X, R) be a classical Kripke frame, where X is a set and $R \subseteq X \times X$ is a binary relation. We want to turn this into a coalgebra for an endofunctor on* ***Set*** *– the category of sets and functions. The critical observation is that the relation R can be viewed instead as a function $R[-]$, taking a point $x \in X$ to its set of successors. In the case of Kripke frames, there is no restriction on the admissible sets of successors, so $R[x]$ can be any subset of X. Thus, $R[-] : X \to \mathcal{P}(X)$. Furthermore, $\mathcal{P}$ defines a functor, by taking a set to its powerset and a function f to its direct image $f[-]$. Then clearly $(X, R : X \to \mathcal{P}(X))$ is a coalgebra on the powerset endofunctor on* ***Sets****. Thus, (X, R) may be represented as the $\mathcal{P}$-coalgebra $(X, R[-] : X \to \mathcal{P}X)$.*

Likewise, given a $\mathcal{P}$-coalgebra (X, α), one can define a Kripke frame (X, R_α), where R_α is given by

$$xR_\alpha y \iff y \in \alpha(x)$$

Furthermore, the homomorphisms for $\mathcal{P}$-coalgebras coincide with p-morphisms between Kripke frames. To see this, let (X, R) and (Y, S) be two Kripke frames, and consider their respective coalgebraic representations $(X, R[-])$ and $(Y, S[-])$. Now let $f : X \to Y$ be a function. It is straightforward to show that

- *f satisfies the forth condition iff $(\mathcal{P}f) \circ R[x] \subseteq S[-] \circ f(x)$ for all $x \in X$,*
- *f satisfies the back condition iff $(\mathcal{P}f) \circ R[x] \subseteq S[-] \circ f(x)$ for all $x \in X$.*

Thus, f is a p-morphism from X to Y if and only if it is a coalgebra homomorphism from $(X, R[-])$ to $(Y, S[-])$.

One can also see a Kripke model *as a coalgebra. Letting* Prop *be the set of proposition variables, a valuation $V : \mathsf{Prop} \to \mathcal{P}(X)$ on a frame (X, R) can be seen as a map $V^{-1}[x] = \{p \in \mathsf{Prop} | x \in V(p)\}$. Thus, a model (X, R, V) corresponds to the coalgebra $(X, (V^{-1}[-], \alpha) : X \to \mathcal{P}(\mathsf{Prop}) \times \mathcal{P}(X))$ for the functor $\mathcal{P}(\mathsf{Prop}) \times \mathcal{P}(-)$.*

The idea of viewing a relation as a map is what underpins giving a coalgebraic semantics of any modal logic. The question is finding the correct endofunctor for the desired correspondence both on objects and on morphisms.

This also works for modal spaces. If we take a modal Stone space (X, R) (see Definition 2.42), the corresponding coalgebra should be over an endofunctor on **Stone**. Furthermore, in this case, there is the additional restriction that for any x, $R[x]$ must be closed. Thus, we use a functor that takes a Stone space to its hyperspace of closed subsets – the Vietoris functor (see e.g. [Eng77, page 2.7.20]), which is essentially a topological analogue of the powerset functor.

Definition 3.4. *Let X be a Stone space, and $K(X)$ be the set of closed subsets of X. The Vietoris functor $\mathcal{V}$ sends X to the set $K(X)$, equipped with the* Vietoris topology *(or* hit-or-miss topology*), generated by a subbasis consisting of sets of the form*

$$\langle U \rangle = \{F \in K(X) | F \cap U \neq \emptyset\} \text{ and } [U] = \{F \in K(X) | F \subseteq U\},$$

for $U, V \in Clop(X)$. The resulting space is called the Vietoris hyperspace *of X* [1].

[1] Note that the given subbasis is not the usual one of the Vietoris topology as found for example in [Eng77], but yields the same topology.

Given a continuous map $f : X \to Y$ between Stone spaces, $\mathcal{V}(f) : \mathcal{V}(X) \to \mathcal{V}(Y)$ is defined by $\mathcal{V}(f)(A) = f[A]$, where $f[A] = \{f(x) | x \in A\}$ is the direct image [2].

We then have the following (see e.g. [VV14] [KKV04]):

Proposition 3.5. *$\mathcal{V}$ is an endofunctor on the category **Stone**. Furthermore, the category $Coalg(\mathcal{V})$ is isomorphic to category **MSS** of modal stone spaces.*

The coalgebraic representation of modal Stone spaces sprung from the work of Esakia [Esa74], who first introduced topological kripke frames using the Vietoris topology. We point the reader to [VV14] for a discussion on the Vietoris functor and its role in coalgebraic modal logic.

3.1.1 Coalgebra Bisimulations

Recall that bisimulations in modal logic serve the purpose of capturing modal equivalence. Thus, bisimulations tend to be defined on a case-by-case basis, depending on the modal logic in question. In this section, we introduce the coalgebraic notion of bisimulation, which generalizes the definition. Given a coalgebraic representation for a class of frames, this formulation allows one to find the natural notion of bisimulation for them – as will be exemplified in Chapter 5.

To elaborate on what this means, suppose we have a coalgebraic representation for a given class of frames. Then there exists a frame-level bisimulation between F and F' with corresponding coalgebras (X, α) and (X', α') if and only if there is a coalgebra bisimulation between (X, α) and (X', α'). Thus, the notion of bisimulations for frames of this class can be determined by analysing coalgebraic behaviour, rather than the specific properties of the relational semantics.

This exemplifies one of the primary advantages of having a coalgebraic semantics for a given logic: many notions will spring naturally from analyzing the coalgebra morphisms, eliminating the need to define them on a case-by-case basis. The following definition, as well as further discussion on coalgebra bisimulations, can be found in [Ven07, Chapter 11]:

Definition 3.6 (Coalgebra bisimulation)**.** *Let (X, α) and (Y, γ) be coalgebras for a functor F. A relation $B \subseteq X \times Y$ is a* bisimulation *between (X, α) and (Y, γ) if we can endow it with a coalgebra map $\beta : B \to F(B)$ such that the two projections $\pi_X : B \to X$ and $\pi_Y : B \to Y$ (inherited from the product) are coalgebra homomorphisms from (B, β) to (X, α) and (Y, β), respectively, i.e. the diagram in Figure 5.1 commutes.*

$$\begin{tikzcd}
X \arrow[d, "\alpha"] & B \arrow[l, "\pi_X"'] \arrow[r, "\pi_Y"] \arrow[d, dotted, "\beta"] & Y \arrow[d, "\gamma"] \\
FX & FB \arrow[l, "F\pi_X"] \arrow[r, "F\pi_Y"'] & FY
\end{tikzcd}$$

Figure 3.2: Bisimulation between coalgebras

As an example, let us show that bisimulations between $\mathcal{P}$-coalgebras correspond to bisimulations between their corresponding Kripke frames. We recall first the definition.

[2] Note that this is well defined because maps between Stone spaces are closed maps, hence the direct image of a closed set is closed.

Definition 3.7 (Bisimulation). *Let $F = (W, R)$ and $F' = (W', R')$ be Kripke frames. A non-empty binary relation $\sim \subseteq W \times W'$ is called a* bisimulation *between F and F' if it satisfies the following conditions:*

(i) If $w \sim w'$ and wRv, then there exists $v' \in M'$ such that $v \sim v'$ and $w'R'v'$ (the forth condition*);*

(ii) if$w \sim w'$ and $w'R'v'$, then there exists $v \in M$ such that $v \sim v'$ and wRv (the back condition*).*

If $M = (W, R, V)$ and $M' = (W', R', V')$ are Kripke models, then $\sim \subseteq W \times W'$ is a bisimulation if it satisfies the additional condition that:

(iii) If $w \sim w'$ then w and w' satisfy the same proposition letters.

Proof. Let (X, α) and (Y, γ) be $\mathcal{P}$-coalgebras, and the coalgebra (B, h) yield a commuting diagram (where $B \subseteq X \times Y$). To show the forth condition, assume $(x, y) \in B$ and $x' \in \alpha(\pi_X(x, y))$. As $x' \in \alpha(\pi_X(x, y))$ and (the left side of) the diagram commutes, we have that $x' \in \pi_X[h(x, y)]$, so there is some $(x', y') \in h(x, y)$ and therefore $y' \in \pi_Y[h(x, y)]$. Then as (the right side) of the diagram commutes, $y' \in \gamma(\pi_Y(x, y))$. The proof for the back condition is analogous.

Conversely, let (X, R) and (Y, S) be Kripke frames, and B a bisimulation between them. Define a map $h : B \to \mathcal{P}(B)$ sending a pair (x, y) to the set $((R[x] \times R[y]) \cap B) \in \mathcal{P}(B)$. Then for $(x, y) \in B$, $y' \in R[y]$ there must be $(x', y') \in B$ by the back condition, and thus $y' \in \mathcal{P}(\pi_Y)((R[x] \times R[y]) \cap B)$. The case for the left side of the diagram is similar, using the forth condition. $\square$

3.1.2 Coalgebras for Positive Modal Logic

We move one step closer to the intuitionistic case by introducing coalgebraic representations for modal Priestley spaces.

Definition 3.8 (Upper Vietoris functor). *Let $(X, \leq, \tau)$ be a Priestley space, and let $\mathsf{ClUp}(X)$ be the set of* closed upsets *of X. The functor $\mathcal{V}^{\uparrow}$ (called the* upper Vietoris functor*) is defined as follows:*

- *On **Objects:** $\mathcal{V}^{\uparrow}(X, \leq, \tau) = (\mathsf{ClUp}(X), \supseteq, \tau^*)$, with the topology τ^* given by sets of the form $[U]$ and $\langle X - V \rangle$ for U, V clopen upsets of X. We call this the* upper Vietoris space *of X.*
- *On **morphisms:** $\mathcal{V}^{\uparrow}$ sends a Priestley morphism $f : X \to Y$ to the map $V^{\uparrow}(f)$ defined by $V^{\uparrow}(f)(A) = \uparrow f[A]$*

Definition 3.9 (Lower Vietoris functor). *Let $(X, \leq, \tau)$ be a Priestley space, and* $\mathrm{ClDown}(X)$ *be the set of closed downsets of X. The functor $\mathcal{V}^{\downarrow}$ (called the* lower Vietoris functor*) is defined as follows:*

- *On **objects:** $\mathcal{V}^{\downarrow}(X, \leq, \tau) = (\mathrm{ClDown}(X), \subseteq, \tau_*)$, with the topology τ_* given by sets of the form $[U]$ and $\langle X - V \rangle$ for U, V clopen downsets of X. We call this the* lower Vietoris space *of X.*
- *On **morphisms:** $\mathcal{V}^{\downarrow}$ sends a Priestley morphism $f : X \to Y$ to the map $\mathcal{V}^{\downarrow}(f) : \mathcal{V}^{\downarrow}(X) \to \mathcal{V}^{\downarrow}(Y)$ defined by $V^{\downarrow}(f)(A) = \downarrow f[A]$.*

Recall from the previous chapter the category $\mathbf{MPS}^{\Box}$ of $\Box$-Priestley spaces and morphisms (Definitions 2.43, 2.44), and the category $\mathbf{MPS}^{\Diamond}$ of $\Diamond$-Priestley spaces and morphisms (Definitions 2.45 and 2.46). The following theorem is well-known (see for example [BHM23]):

Theorem 3.10. *The category* $\mathbf{MPS}^{\Box}$ *is equivalent to* $\mathbf{Coalg}(\mathcal{V}^{\uparrow})$*, and the category* $\mathbf{MPS}^{\Diamond}$ *is equivalent to* $\mathbf{Coalg}(\mathcal{V}^{\downarrow})$*.*

Finding coalgebras for these positive modal logics is relatively straight-forward. Given a Kripke frame (X, R), the corresponding coalgebra map $R[-]$ is only required to be monotone [3]. Likewise, in the absence of implication, bounded morphisms between positive frames need only be monotone. Thus, the standard recipe (see Example 3.3) works, using an appropriate **Pos** endofunctor; both the coalgebra morphisms and the homomorphisms between coalgebras will be monotone maps, and thus morphisms in **Pos** as required.

However, in the intuitionistic case, maps between Kripke frames are required to be order p-morphisms, while the maps $R[-]$ are still only required to be monotone. Thus, the usual recipe will not work, as the coalgebra morphisms and homomorphisms between coalgebras will not belong to the same category. Working in a category with monotone maps would thus result in more coalgebra homomorphisms than frame p-morphisms, and restricting to a category with p-morphisms would result in more frames than coalgebras. Thus, securing a correspondence both on objects and morphisms is not straight-forward. The solution is this: deal first with the objects through a **Pos** (or **Pris**) endofunctor F, then turn F into an $\mathbf{ImFinPos}_p$ (or **Esa**) endofunctor.

The key to ensuring a unique lifting lies in the universal property of free algebras (see Figure 2.3) – or rather its dual notion. Thus, the solution is provided by functors which dually correspond to the free functor from **DL** to **HA** and the free functor from **Pos** to $\mathbf{ImFinPos}_p$. The construction of these functors is the topic of the next section.

3.2 The step-by-step construction

In this section, we describe the step-by-step construction, due to Almeida [Alm24], of the endofunctors $\mathcal{V}_G$ on **Pris**, and P_G on **Pos**. $\mathcal{V}_G$ is the right adjoint to the inclusion of **Esa** into **Pris**, and P_G is the right adjoint to the inclusion of $\mathbf{ImFinPos}_p$ into **Pos**. As shown in [AB24], these functors will allow us to turn coalgebras for PML into coalgebras for IML, using $\mathcal{V}_G$ in the case of modal spaces, and P_G in the case of image-finite Kripke frames.

As a general intuition about the construction, say we have a distributive lattice D, with X_D its dual Priestley space (or poset in the discrete case). If we want to generate the free Heyting algebra over D, then we must begin by adding relative pseudocomplements $a \to b$ for all $a, b \in D$. This will be dually achieved in the construction by taking the rooted subsets of X_D. The resulting distributive lattice D_1 will have added elements $a \to b$ for all $a, b \in D$, but the new elements do not necessarily have implications between themselves. Thus, we must repeat this process for D_1, and so on, until we reach the free Heyting algebra over D. However, each step will add new implications to every element in the previous lattice, which need not agree with those that were previously added. If this happens infinitely often, it could be in the end that no element is the relative pseudocomplement of a and b. Thus, we must ensure at each step that previously-added relative pseudocomplements are preserved. This quotient is achieved by dually taking the rooted subsets that are additionally *g-open* – a notion which will be introduced shortly.

[3] We prove this explicitly in Chapter 4, Theorem 4.3

3.2.1 The topological case

We begin with the functor $\mathcal{V}_G$. Recall the Vietoris endofunctor $\mathcal{V}$ on **Stone** from Definition 3.4. $\mathcal{V}$ also acts as an endofunctor on **Pris**, sending a Priestley space X to the space $(\mathcal{V}(X), \supseteq)$ (see [Alm24, Lemma 9]). From this point onwards, we will treat $\mathcal{V}$ as an endofunctor on **Pris**.

Definition 3.11. *The functor $\mathcal{V}_r$ is defined as follows:*

- ***On objects:*** *Let X be a Priestley space. Then $\mathcal{V}_r(X) = \{A \subseteq X | A \text{ is closed and rooted } \}$, equipped with the subspace topology inherited from $\mathcal{V}(X)$ and ordered by reverse inclusion.*
- ***On morphisms:*** *Given a Priestley map $f : X \to Y$, $\mathcal{V}_r(f) : \mathcal{V}_r(X) \to \mathcal{V}_r(Y)$ is defined by $\mathcal{V}_r(f)(A) = f[A]$.*

Lemma 3.12. *The functor $\mathcal{V}_r$ is an endofunctor on **Pris** [4].*

Definition 3.13 (g-open map)**.** *Let X, Y, and Z be Priestley spaces, and let $f : X \to Y$ and $g : Y \to Z$ be Priestley morphisms. We say f is* g-open, *or* open relative to g, *if f^{-1} preserves relative pseudocomplements of the form $g^{-1}[U] \to g^{-1}[V]$ for $U, V \in ClopUp(Z)$. That is,*

$$f^{-1}(g^{-1}[U] \to g^{-1}[V]) = f^{-1}(g^{-1}[U]) \to f^{-1}(g^{-1}[V])$$

Lemma 3.14. *A Priestley map $f : X \to Y$ is g-open if and only if the following condition holds:*

$$(*) \qquad \forall a \in X, \forall b \in Y, (f(a) \leq b \implies \exists a' \in X, (a \leq a' \;\&\; g(f(a')) = g(b))),$$

illustrated in Figure 3.3.

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

$$\begin{tikzcd}
a' \arrow[r, dotted, "f"] & f(a') \arrow[r, dotted, "g"] & g(f(a')) \\
 & b \arrow[r, "g"] & g(b) \arrow[u, "\leq"'] \\
a \arrow[uu, dotted, "\leq"'] \arrow[r, "f"'] & f(a) \arrow[u, dotted, "\leq"'] &
\end{tikzcd}$$

Figure 3.3: Condition $(*)$ for g-openness

The proof of this lemma can be found in [Alm24, Lemma 12].

Definition 3.15 (g-open subset)**.** *We say that a closed subset $S \subseteq X$ is g-open (seen as a Priestley space with the induced order and topology) if the inclusion $\iota : S \hookrightarrow X$ is g-open. By Lemma 3.14, this is equivalent to the following condition:*

$$\forall s \in S, \forall b \in X (s \leq b \implies \exists s' \in S (s \leq s' \& g(s') = g(b))).$$

Definition 3.16. *Let $g : X \to Y$ be a Priestley morphism. Define $\mathcal{V}_g(X) \subseteq \mathcal{V}_r(X)$ as the Priestley space*

[4] See [Alm24, Lemma 13] for a proof of this.

$$V_g(X) := \{C \subseteq X | C \text{ is closed, rooted, and } g\text{-open}\}$$

with the subspace topology and order inherited from $\mathcal{V}_r(X)$ [5].

Additionally, $V_g(X)$ is equipped with the g-open, surjective Priestley morphism $r_g : V_g(X) \to X$ sending each rooted subset to its root ([Alm24, Lemma 15]). We will often drop the subscript and refer to r_g as r, wherever no ambiguity arises.

Lemma 3.17. *Let X, Y, Z be Priestley spaces, $g : X \to Y$ be a Priestley morphism, and $h : Z \to X$ be a g-open Priestley map. Then the map h' defined by $h'(x) = h[\uparrow x]$ is the unique r_g-open Priestley map $r \circ h' = h$, i.e. the diagram in Figure 3.4 commutes.*

$$\begin{tikzcd} Z \arrow[rr, "h'"] \arrow[dr, "h"'] & & V_g(X) \arrow[dl, "r_g"] \\ & X \arrow[d, "g"] & \\ & Y & \end{tikzcd}$$

Figure 3.4: Commuting Triangle of Priestley spaces

Lemma 3.18. *Let $f : X \to Y$ be a Priestley map, and t the terminal map to the one-element poset $\{*\}$. Then f is t-open.*

Proof. Let $a \in X$ and $b \in Y$ such that $f(a) \leq b$. Then trivially $a \leq a$ and clearly $t(f(a)) = * = t(b)$, as the terminal map maps everything to $*$. $\square$

Note that this also means that any subset $S \subseteq Y$ is t-open.

Lemma 3.19. *Let $g : X \to Y$ be a Priestley map. Then for any $S \subseteq X$ such that S is an upset, S is g-open.*

Proof. Let $s \in S$ and $b \in X$ such that $s \leq b$. As S is an upset, $b \in S$. Then clearly $s \leq b \in S$ and $g(b) = g(b)$. $\square$

Lemma 3.20. *Let $p : X \to Y$ be a Priestley morphism, and $g_X : X \to Z$ and $g_Y : Y \to Z$ be two maps such that $g_Y \circ p = g_X$, and the relative pseudocomplements indexed by g_X and g_Y exist. Then $p[-] : \mathcal{V}_{g_X}(X) \to \mathcal{V}_{g_Y}(Y)$ is a unique Priestley morphism such that $p \circ r_X = r_Y \circ p[-]$.*

Note that this is a special case of Lemma 3.17, where $p[-] = r_X p[\uparrow -]$ (by monotonicity), and is thus the unique g_Y-open extension of $r_X \circ p$. For the full proof of this lemma, see [Alm24, Lemma 19].

Definition 3.21. *Let $g : X \to Y$ be a Priestley morphism. The **g-Vietoris complex** $(V^g_\bullet(X), \leq_\bullet)$ over X is a sequence*

$$(V_0(X), V_1(X), ..., V_n(X), ...)$$

Connected by morphisms $r_{i+1} : V_{i+1}(X) \to V_i(X)$ such that:

[5] The proof that $(V_g(X), \supseteq)$ is a Priestley space can be found in [Alm24, Lemma 14].

- $V_0(X) = X$
- $r_0 = g$
- *For* $i \geq 0$, $V_{i+1}(X) := \mathcal{V}_{r_i}(V_i(X))$
- $r_{i+1} = r_{r_i} : V_{i+1}(X) \to V_i(X)$ *is the root map.*

The projective limit *of this family is called* $\mathcal{V}_G^g(X)$*. When* g *is the terminal map to the one-element poset, we will omit it and simply write* $\mathcal{V}_G$

As $\mathcal{V}_G(X)$ is a projective limit, it comes equipped with projections $\lambda_i : \mathcal{V}_G(X) \to V_i(X)$. Furthermore, the projection $\lambda_0 : \mathcal{V}_G(X) \to X$ is surjective, as for any $x \in X$, $(x, \uparrow x, \uparrow(\uparrow x), ...)$ is an element of $\mathcal{V}_G(X)$ mapping to x. To see that $(x, \uparrow x, \uparrow(\uparrow x), ...) \in \mathcal{V}_G(X)$, recall that upsets of points are always rooted, closed in Priestley spaces (Lemma 2.51) and g-open for any appropriate g (Lemma 3.19).

Proposition 3.22. *Let* $g : Y \to Z$ *be a Priestley map such that* g*-indexed relative pseudocomplements exist. Let* X *be an Esakia space,* Y *a Priestley space, and* $f : X \to Y$ *a* g*-open Priestley morphism. Then there is a unique Esakia morphism* $\overline{f} : X \to \mathcal{V}_G^g(Y)$ *extending* f*, given by*

$$\overline{f}(x) = (f_0(x), f_1(x), ...)$$

for the family $f_n : X \to V_n(Y)$ *given by*

- $f_0 = f$
- $f_{n+1}(x) = f_n[\uparrow x]$

The full proof of this proposition can be found in [Alm24, Lemma 20]. We will, however, recall the proof that $\overline{f}$ is a p-morphism, as it is informative for the next section.

Proof. Assume that for $x \in X$ and $\overline{f}(x) \leq y$. Then consider the set $\uparrow x \cap \bigcap\{f_n^{-1}[y(n)] | n \in \omega\}$. As this set consists of closed spaces, and since X is compact, this intersection will be empty if and only if a finite intersection is empty. So let $\uparrow x \cap f_0^{-1}[y(0)] \cap ... \cap f_n^{-1}[y(n)] = \emptyset$.

As $\overline{f}(x) \leq y$, it follows that $f_{n+1} \leq y(n+1)$; so there is some $k \geq x$ such that $f_n(k) = y(n)$. Then $k \in \uparrow x \cap f_0^{-1}[y(0)] \cap ... \cap f_n^{-1}[y(n)] \neq \emptyset$, a contradiction. Thus, there is $x \leq k$ such that $h(k) = y$, as desired.

□

Note that compactness plays a crucial role in in the previous proof. In the absence of compactness (or the addition of some other requirement), there is no clear way to show that $\overline{f}$ is a p-morphism. This is one reason why the restriction to image-finiteness is made in the next section: some form of discreteness is required to ensure that the the analogous proposition holds.

Lemma 3.23. *Let* $g_X : X \to Z$*,* $g_Y : Y \to Z$*, and and* $p : X \to Y$ *be Priestley maps such that* p *is* g_Y*-open and* $g_Y \circ p = g_X$*, and the relative pseudocomplements indexed by* g_X *and* g_Y *exist. Then there is a unique p-morphism* $\overline{p} : \mathcal{V}_G^{g_X}(X) \to \mathcal{V}_G^{g_Y}(Y)$ *such that* $\lambda_0^Y \overline{p} = p\lambda_0^X$*, illustrated in Figure 3.5.* $\overline{p}$ *is defined by* $\overline{p}(x) = (p_0(\lambda_0(x)), p_1(\lambda_1(x)), p_2(\lambda_2(x)), ...)$*.*

$\mathcal{V}_G$: **Pris** $\to$ **Esa** can now be defined as a functor, sending a Priestley space to the Esakia space $\mathcal{V}_G(X)$ and a Priestley map to the Esakia morphism $\overline{p}$. The following is due to [Alm24, Theorem 22]:

$$\begin{tikzcd}
\mathcal{V}_G^{g_X}(X) \arrow[rr, "\overline{p}"] \arrow[d, "\lambda_0^X"] & & \mathcal{V}_G^{g_Y}(Y) \arrow[d, "\lambda_0^Y"] \\
X \arrow[rr, "p"] \arrow[dr, "g_X"'] & & Y \arrow[dl, "g_Y"] \\
& Z &
\end{tikzcd}$$

Figure 3.5: Commuting Diagram for $\overline{p}$

Theorem 3.24. *$\mathcal{V}_G$ is an endofunctor on **Pris**, and is the* right adjoint *to the inclusion of **Esa** into **Pris**.*

We note that, as stated in [Alm24, Theorem 23], dualizing the above construction yields the free functor from **DL** to **HA**.

3.2.2 The discrete setting

We now present the **Pos** endofunctor P_G, which we will use to achieve a coalgebraic representation for image-finite intuitionistic Kripke frames. The construction of P_G very closely mirrors that of $\mathcal{V}_G$, but is concerned with the duality between image-finite posets and profinite Heyting algebras (see Theorem 2.63) in place of Esakia duality [6]. Here, the categories **Pos** and $\mathbf{ImFinPos}_p$play analogous roles to those of **Pris** and **Esa** in the topological case. Furthermore, wherever we require that a subset be closed in the topological setting (such as the definition of $V_g(X)$), we will instead require that it be finite. The reader may find intuition in seeing that compactness or image-finiteness are seemingly necessary, in order to ensure that the lifted maps will be p-morphisms. Thus, we do not expect to have an extension to any larger category than $\mathbf{ImFinPos}_p$ through these methods. There are several other reasons for why this restriction is justified, for which we point the reader to [AB24, Remark 3.12].

Definition 3.25. *The functor P_r is defined as follows:*

- ***On objects:*** *Let X be a poset. Then $P_r(X) = \{A \subseteq X | A$ is* finite *and rooted$\}$, ordered by reverse inclusion.*
- ***On morphisms:*** *Let $f : X \to Y$ be a monotone map between posets. Then $P_r(f) : P_r(X) \to P_r(Y)$ is defined by $P_r(f)(A) = f[A]$.*

The notion of a g-open map in **Pos** is the same as in Definition 3.13, and is also equivalent to condition $(*)$ of Lemma 3.14.

Definition 3.26. *Let $g : X \to Y$ be a monotone map between posets. Define $P_g(X) \subseteq P_r(X)$ as the poset*

$$P_g(X) = \{C \subseteq X | C \text{ is finite, rooted, and } g\text{-open}\}, \text{ ordered by reverse inclusion.}$$

Lemma 3.27. *Let $g : X \to Y$ be a monotone map between posets; $h : Z \to X$ be a monotone and g-open map, where Z is image-finite. Then $h' : Z \to P_g(X)$ defined by $h' = h[\uparrow -]$ is the unique monotone and r_g-open map such that the triangle in Figure 3.6 commutes.*

[6] The role of Priestley duality is taken by the duality between **Pos** and $\mathbf{DL}^+$ (Theorem 2.62).

$$\begin{tikzcd}
Z \arrow[rr, "h'"] \arrow[dr, "h"'] & & P_g(X) \arrow[dl, "r_g"] \\
& X \arrow[d, "g"] & \\
& Y &
\end{tikzcd}$$

Figure 3.6: Commuting Triangle of Posets

Note that $h[\uparrow x]$ will be finite as Z is image-finite.

Lemma 3.28. *Let $p : X \to Y$ be a monotone map between posets, and $g_X : X \to Z$ and $g_Y : Y \to Z$ be two maps such that $g_Y \circ p = g_X$, and the relative pseudocomplements indexed by g_X and g_Y exist. Then $p[-] : P_{g_X}(X) \to P_{g_Y}(Y)$ is a unique monotone map such that $p \circ r_X = r_Y \circ p[-]$.*

Definition 3.29. *Let $g : X \to Y$ be a monotone map between image-finite posets. The* g-discrete complex $(P^g_\bullet(X), \leq_\bullet)$ *over X is a sequence*

$$(P_0(X), P_1(X), ..., P_n(X), ...)$$

connected by morphisms $r_{i+1} : P_{i+1(X) \to P_i(X)}$ such that

- *$P_0(X) = X$;*
- *$r_0 = g$;*
- *$P_{i+1}(X) := \mathcal{P}_{r_i}(P_i(X))$*
- *$r_{i+1} := r_{r_i} : P_{i+1}(X) \to P_i(X)$ is the root map.*

We denote the image-finite part *of the projective limit of this family by $P^g_G(X)$. We will again write P_G when g is the terminal map to the one element poset.*

The following proposition is analogous to Proposition 3.22 for the topological case:

Proposition 3.30. *Let X be an image-finite poset, Y be a poset, and $f : X \to Y$ a monotone map. Then there is a unique p-morphism $\overline{f} : X \to P_G(Y)$ extending f, given by $\overline{f}(x) = (f_0(x), f_1(x), ...)$, with $f_0 = f$ and $f_{n+1}(x) = f_n[\uparrow x]$.*

We again recall the proof that $\overline{f}$ is a p-morphism. For a full proof, see [Alm24, Proposition 58], to illustrate where image-finiteness is used.

Proof. Let $x \in X$, and suppose that $\overline{f}(x) \leq y$. Denote $y = (y_0, y_1, ...) \in \mathcal{P}_G(Y)$.

Let n be arbitrary, and consider $y_{n+1} \subseteq f_{n+1}[\uparrow x]$. Then there is some y' such that $x \leq y'$ and $\overline{f}(y')$ agrees with y up to the n'th position by Lemma 3.27. Since $x \in X$ and x is *image-finite*, there must be a successor $x \leq z$, such that $\overline{f}(z)$ agrees with y on arbitrarily many positions, i.e., $\overline{f}(z) = y$, and therefore $\overline{f}$ is a p-morphism. □

Note that as X is image-finite, this also shows that $\overline{f}$ will factor through the image-finite part of the projective limit, and is thus well-defined.

Lemma 3.31. *Let $g_X : X \to Z$, $g_Y : Y \to Z$, and and $f : X \to Y$ be monotone maps such that p is g_Y-open and $g_Y \circ f = g_X$, and the relative pseudocomplements indexed by g_X and g_Y exist. Then by repeated application of Lemma 3.30, the map $\overline{f}$ defined by $\overline{f}(x) = (f_0(\lambda_0(x)), f_1(\lambda_1(x)), f_2(\lambda_2(x)), ...)$ is the unique p-morphism $\overline{f} : P_G^{g_X}(X) \to P_G^{g_Y}(Y)$ such that $\lambda_0^Y \overline{f} = f\lambda_0^X$.*

Theorem 3.32. *The assignment P_G is an endofunctor on the category **Pos**, and is the right adjoint to the inclusion of **ImFinPos**$_p$ (of image-finite posets and p-morphisms) into **Pos**.*

3.3 Pushing coalgebras for other functors

The functors $\mathcal{V}_G$ and P_G allow for coalgebras for a positive modal logic to be turned into coalgebras for its least intuitionistic extension. We present here the results of Almeida and Bezhanishvili [Alm24], who derived a coalgebraic semantics for □-IML by pushing coalgebras for □-PML.

Recall from Theorem 3.10 that the category $\mathbf{MPS}^\Box$ of □-Priestley spaces is equivalent to $\mathbf{Coalg}(\mathcal{V}^\uparrow)$. The following theorem corresponds to [AB24, Theorem 4.2]:

Theorem 3.33. *The category $\mathbf{MES}^\Box$ of □-Esakia spaces and p-morphisms is equivalent to $\mathbf{Coalg}(\mathcal{V}_G(\mathcal{V}^\uparrow(-)))$.*

Just as $\mathcal{V}_G$ can push coalgebras for positive spaces into coalgebras for intuitionistic spaces, P_G can be applied to turn coalgebras for image-finite positive Kripke frames into coalgebras for image-finite intuitionistic Kripke frames. Given that the category $\mathbf{PKF}^\Box_{Pos}$ of □-positive Kripke frames is equivalent to the category of coalgebras for the functor $Up(-)$ [7], the following equivalence is shown [AB24, Theorem 4.3]:

Theorem 3.34. *The category $\mathbf{IKF}^\Box_{ImFin}$ of intuitionistic □-Kripke frames and p-morphisms is equivalent to $\mathbf{Coalg}(P_G(Up(-)))$.*

In general, coalgebras for a positive modal logic can be lifted to coalgebras for its smallest intuitionistic extension. That is, given an endofunctor **Pris**, one can simply lift it to Esakia spaces through composition with $\mathcal{V}_G$. Likewise, an endofunctor on **Pos** can be lifted to an endofunctor on **ImFinPos**$_p$ by composition with P_G. This provides a clean method for giving coalgebraic semantics for many intuitionistic modal logics, by beginning with a **Pris** or **Pos** endofunctor. Ideally, we would like to use this method for Fisher-Servi logic: simply determine a category of positive coalebras, whose lifting via $\mathcal{V}_G$ or P_G then yields a coalgebraic semantics for **IK**. Then the question is only determining the appropriate positive endofunctor.

However, doing so will not be quite as straight-forward in the case of Fischer Servi logic: the modal axioms for **IK**-algebras (see Definition 4.1) already require implications, so there is no way to ensure these hold on a positive structure – we need at least *some* implications in order to talk about the axioms. This said, it will still be possible – with a little extra work – to achieve a coalgebraic semantics for Fischer Servi logic by lifting coalgebras. This is the topic of the next chapter.

[7] We present this functor later in Definition 4.26. It is entirely analogous to $\mathcal{V}^\uparrow(-)$, dropping the condition that subsets be closed.

Chapter 4

Main results: Coalgebras for Fischer Servi logic

In this chapter, we present our coalgebraic semantics for Fischer Servi logic. Our construction is based on the work by Almeida [Alm24] and Almeida and Bezhanishvili [AB24] presented in the previous chapter. The first section will provide an algebraically-flavoured intuition of our construction. We then present the topological setting, beginning by recalling the necessary notions, followed by our coalgebraic representation for **IK**-spaces. We then do the same for the discrete setting, extending our method to image-finite **IK** Kripke frames.

4.1 The algebraic perspective

We begin by recalling the class of algebras corresponding to the logic **IK**, and provide an intuition for what will be happening algebraically.

Definition 4.1. *An algebra* $(H, \wedge, \vee, \rightarrow, \Box, \Diamond, \top, \bot)$ *is called an* **IK**-*algebra if* $(H, \wedge, \vee, \rightarrow, \top, \bot)$ *is a Heyting algebra which additionally satisfies the following modal axioms* [1]*:*

1. $\Box\top = \top$	3. $\Diamond\bot = \bot$
2. $\Box(a \wedge b) = \Box a \wedge \Box b$	4. $\Diamond(a \vee b) = \Diamond a \vee \Diamond b$
A. $\Diamond(a \rightarrow b) \leq \Box a \rightarrow \Diamond b$	**B**. $\Diamond a \rightarrow \Box b \leq \Box(a \rightarrow b)$

Table 4.1: Axioms for **IK**-Algebra

We denote by **IKA** the category of **IK**-algebras and Heyting algebra homomorphisms preserving $\Box$ and $\Diamond$.

Given the interaction between modalities and implications in axioms **A** and **B**, there is no ostensible positive reduct for which **IK** is the smallest intuitionistic extension. Thus, we begin with the fragment consisting of axioms 1-4, which can all be expressed in positive modal logic, and then build upon this.

We will start by looking at spaces that dually satisfy only the normality axioms (1-4), which will allow us to exploit the existing methods discussed in Chapter 3. We will then impose additional

[1] These can be found in [Pal04b], for instance.

restrictions to yield spaces that dually satisfy axioms **A** and **B**. Once this is in place, we apply the $\mathcal{V}_G^g$ construction (Definition 3.21) in order to turn this into an Esakia space.

As an intuition for the algebraically-minded, let us begin by sketching what will happen dually at each step.

1. We begin with an Esakia space X, denoting its dual Heyting algebra by D_X.
2. Denote by $D_{\Box\Diamond}$ the dual distributive lattice of the product $\mathcal{V}^{\uparrow}(X) \times \mathcal{V}^{\downarrow}(X)$, where
   - $\mathcal{V}^{\uparrow}(X)$ (Definition 3.8) dually corresponds to generating the free distributive lattice over $\{\Box a | a \in D_X\}$ and quotienting over the normality axioms for $\Box$, and
   - $\mathcal{V}^{\downarrow}(X)$ (Definition 3.9) dually generates the free distributive lattice over $\{\Diamond a | a \in D_X\}$, quotienting over the normality axioms for $\Diamond$.

   Thus, $D_{\Box\Diamond}$ has all elements of the form $\Box a, \Diamond b$ for all $a, b \in D$, and these behave appropriately.
3. As D_X was a Heyting algebra, we have all relative pseudocomplements $a \to b \in D_X$ for $a, b \in D_X$, so $D_{\Box\Diamond}$ has all elements $\Box(a \to b)$ and $\Diamond(a \to b)$. Then as axiom **A** can be residuated to $\Diamond(a \to b) \wedge \Box a \leq \Diamond b$, we can quotient $D_{\Box\Diamond}$ over **A**, yielding the distributive lattice $D_{\Box\Diamond}^{\mathbf{A}}$.
4. Axiom **B** on the other hand involves implications between modal formulas, so we must freely add relative pseudocomlements of the form $\{a \to b | a, b \in D_{\Box\Diamond}^{\mathbf{A}}\}$ before quotienting. This is dually achieved by applying the functor $\mathcal{V}_r$ (see Definition 3.11). With this one layer of implications added, we may quotient over axiom **B** to yield the distributive lattice $D_{\Box\Diamond}^{\mathbf{AB}}$, which now satisfies all of the **IK** axioms.
5. Finally, completing the $\mathcal{V}_G$ construction, we (dually) freely add relative pseudocomplements to $D_{\Box\Diamond}^{\mathbf{AB}}$, yielding the desired Heyting algebra.

4.2 The topological setting: IK-spaces

In this section, we develop a coalgebraic representation for **IK**-spaces – the descriptive general frames for Fischer Servi logic.

We follow the recipe outlined in Chapter 3. We first introduce a class of relational order-topological spaces which, by duality, yield completeness for Fischer Servi logic. Next, we construct coalgebras corresponding to these spaces, using the methods outlined in Chapter 3. We then prove that this category of coalgebras is equivalent to the category yielded by our spaces.

4.2.1 IK-spaces for Fischer Servi logic

In this subsection, we define the topological categories we will work with, and prove additional properties that will be useful for our construction. Some of these definitions have already been given in the preliminaries, but we choose to recall them here given their importance within this chapter.

As discussed in Chapter 2, while $\Box$ and $\Diamond$ are not interdefinable, they should satisfy the compatibilities imposed by axioms **A** and **B**. These will dually correspond to conditions on the relations governing the modalities.

However, we will first guarantee that the $\Box$ and $\Diamond$ operators behave well, and only after ensure that they interact appropriately. Thus, we begin by looking at a class of frames where each modality is governed by its own relation, and the relations do not interact in any way.

Definition 4.2. *A $\Box\Diamond$-frame is a triple $(X, R_\Box, R_\Diamond)$ such that X is an Esakia space, and the following conditions hold:*

- *$R_\Box[x]$ is a closed upset;*
- *$R_\Diamond[x]$ is a closed downset;*
- *If U is a clopen upset, then $\Diamond_{R_\Diamond} U$ and $\Box_{R_\Box} U$ are clopen upsets.*

It is important to our construction that the relations, when seen as maps, are monotone. This is a known result, but we include it here nevertheless.

Lemma 4.3. *Let $(X, R_\Box, R_\Diamond)$ be a $\Box\Diamond$-frame. Then we have the following:*

1. *$R_\Box$ is monotone w.r.t. reverse inclusion, i.e. if $x \leq y$ then $R_\Box[x] \supseteq R_\Box[y]$;*
2. *$R_\Diamond$ is monotone w.r.t. inclusion, i.e. if $x \leq y$ then $R_\Diamond[x] \subseteq R_\Diamond[y]$.*

Proof. For (1), suppose $x \leq yR_\Box z$. Now assume for contradiction that $z \notin R_\Box[x]$. Since $R_\Box[x]$ is a closed upset, by Lemma 2.52, there is $U \in ClopUp(X)$ such that $R_\Box[x] \subseteq U$ and $z \notin U$. Then $x \in \Box_{R_\Box} U = \{x \in X | R_\Box[x] \subseteq U\}$. Since $x \leq y$ and $\Box_{R_\Box} U$ is an upset, it follows that $y \in \Box_{R_\Box} U$, and thus $R_\Box[y] \subseteq U$. But then $z \in R_\Box[y] \subseteq U$, a contradiction. Thus it follows that $xR_\Box z$. Thus, we have shown that if $x \leq y$, $R_\Box[x] \supseteq R_\Box[y]$.

For (2), suppose that $x \geq yR_\Diamond z$. Assume towards contradiction that $z \notin R_\Diamond[x]$. By Lemma 2.53, there is a clopen upset U such that $z \in U$ and $R_\Diamond[x] \cap U = \emptyset$. As $z \in R_\Diamond[y] \cap U \neq \emptyset$, we have that $y \in \Diamond_{R_\Diamond} U$. But $y \leq x$ and $\Diamond_{R_\Diamond} U$ is an upset, so $x \in \Diamond_{R_\Diamond} U$, a contradiction. Thus, $xR_\Diamond y$, and we have shown that if $y \leq x$, $R_\Box[y] \subseteq R_\Box[x]$.

$\Box$

In fact, these conditions are equivalent to conditions that, on the surface, seem stronger.

Lemma 4.4. *Let $(X, R_\Box, R_\Diamond)$ be a frame such that for all $x \in X$, $R_\Box[x]$ is an upset and $R_\Diamond[x]$ is a downset. Then we have the following:*

(i) $R_\Box = \leq \circ R_\Box \circ \leq$ *iff* 1. *$R_\Box[-]$ is monotone w.r.t $\supseteq$;*

(ii) $R_\Diamond = \geq \circ R_\Diamond \circ \geq$ *iff* 2. *$R_\Diamond[-]$ is monotone w.r.t $\subseteq$.*

We call (i) and (ii) the mix laws*.*

Proof. Assume $R_\Box[-]$ and $R_\Diamond[-]$ are monotone w.r.t $\supseteq$ and $\subseteq$, respectively. The left-to-right inclusions follow immediately by reflexivity of $\leq$. For the right-to-left of (i), suppose $x \leq yR_\Box v \leq z$. Then $yR_\Box z$ as $R_\Box[y]$ is an upset and $z \geq v \in R_\Box[y]$. So $x \leq yR_\Box z$. By Lemma 4.3, we have that $R_\Box[y] \subseteq R_\Box[x]$, and thus $xR_\Box z$. For the right-to-left inclusion of (ii), let $x \geq yR_\Diamond v \geq z$. Then $yR_\Diamond z$ as $R_\Diamond[y]$ is a downset, so $x \geq yR_\Diamond z$. By Lemma 4.3, we have that $R_\Diamond[y] \subseteq R_\Diamond[x]$, and thus $xR_\Diamond z$.

Conversely, monotonicity of $R_\Box[-]$ and $R_\Diamond[-]$ follows trivially by reflexivity of $\leq$. $\Box$

Hence, a $\Box\Diamond$-frame $(X, R_\Box, R_\Diamond)$ is such that for all $x \in X$, $R_\Box[x]$ is an upset and $R_\Diamond[x]$ is a downset, it follows from Lemma 4.3 that the mix laws (i) and (ii) will always be satisfied.

Definition 4.5 ($\Box\Diamond$ p-morphism)**.** *Let $(X, R^X_\Box, R^X_\Diamond)$ and $(Y, R^Y_\Box, R^Y_\Diamond)$ be two $\Box\Diamond$-frames. A* continuous *map $f : X \to Y$ is a $\Box\Diamond$ p-morphism provided*

1. *f is monotone;*
2. *f is a p-morphism w.r.t $\leq$, i.e. $f(x) \leq y \implies \exists z \in X.(z \geq x \& f(z) = y)$.*
3. *$xR^X_\Diamond z$ implies $f(x)R^Y_\Diamond f(z)$;*
4. *$xR^X_\Box z$ implies $f(x)R^Y_\Box f(z)$;*
5. *$f(x)R^Y_\Diamond y$ implies there exists $z \in X$ such that $xR^X_\Diamond z$ and $y \leq f(z)$.*
6. *$f(x)R^Y_\Box y$ implies there exists $z \in X$ such that $xR^X_\Box z$ and$f(z) = y$.*

Note that (5) for $R_\Diamond$ is weaker than the standard back condition for p-morphisms. In contrast, condition (6) for $R_\Box$ is the standard back condition. This follows from the fact that f is an order p-morphism [2]. This underlies a general theme throughout: p-morphisms respect upsets, but not downsets, and thus $R_\Box$ is generally well-behaved, while $R_\Diamond$ is not.

We denote the category of $\Box\Diamond$-frames and $\Box\Diamond$ p-morphisms by $\mathbf{Fr}_{\Box\Diamond}$.

Theorem 4.6. *The category $\mathbf{Fr}_{\Box\Diamond}$ is dually equivalent to the category $\mathbf{Alg}_{\Box\Diamond}$ of Heyting algebras satisfying the normality axioms for $\Box$ and $\Diamond$ and Heyting homomorphisms preserving $\Box$ and $\Diamond$.*

The proof of this theorem is a direct consequence of [Pal04b, Theorem 6.1.11].

Definition 4.7 (**IK**-space)**.** *A **IK**-space is a modal Esakia space [3] (X, R) such that the following conditions hold:*

(T1) $R[x]$ is closed;

(T2) $R[\uparrow x]$ is a closed upset;

(T3) If U is a clopen upset, then $\Diamond_R U$ and $\Box_{(\leq \circ R)} U$ are clopen upsets;

(T4) $R[x] = R[\uparrow x] \cap \downarrow R[x]$.

Palmigiano [Pal04b] gives the following definition of p-morphisms between the category of frames that meet conditions (T1)-(T3), but need not satisfy (T4). As this forms a full subcategory of **IK**, the definition is the same. However, as we will see shortly, adding (T4) will allow for the definition to be simplified.

Definition 4.8 (**IKS** p-morphism)**.** *Let (X, R) and (Y, S) be **IK**-spaces. A map $f : X \to Y$ is a p-morphism iff for every $x, x', y \in X, z \in Y$,*

1. *if $x \leq_X y$ then $f(x) \leq_Y f(y)$.*

[2] In the positive case (see [BHM23]), where f is only required to be monotone, condition (6) only imposes that $f(z) \leq y$

[3] Equivalently, an intuitionistic general frame

2. *If $f(x) \leq_Y z$ then $f(x') = z$ for some $x' \in \uparrow x$.*

3. *For every $A \in ClopUp(Y)$, $f^{-1}[A] \in ClopUp(X)$.*

4. *If xRy then $f(x)Sf(y)$.*

5. *If $f(x)Sz$ then $z \leq_Y f(x')$ for some $x' \in R[x]$.*

6. *If $f(x)(\leq_Y \circ S)z$ then $f(x') \leq_Y z$ for some $x' \in R[\uparrow x]$.*

As stated in Chapter 2 (Theorem 2.65), the category **IKS** of **IK**-spaces and p-morphisms is dually equivalent to **IKA** of **IK** algebras and morphisms.

We will use the fact that **IK**-spaces can be seen as $\Box\Diamond$-frames which satisfy additional conditions, presented in the lemma below (see for example [Cel01]).

Lemma 4.9. *A **IK**-space is a $\Box\Diamond$-frame where $R := R_\Box \cap R_\Diamond$ and the following conditions hold:*

(I) $R_\Diamond = \downarrow(R_\Box \cap R_\Diamond)$ and

(II) $R_\Box = \leq \circ (R_\Box \cap R_\Diamond)$

Or, in other words, $R_\Box[x] = R[\uparrow x]$ and $R_\Diamond[x] = \downarrow R[x]$.

Hence, we will often treat **IK**-spaces as $\Box\Diamond$-frames satisfying (I) and (II). We will later see that doing so will prove quite useful.

The fact that axioms 1-4 of Definition 4.1 correspond to $\Box\Diamond$-frames hint at a correspondence between axioms **A** and **B** and conditions (I) and (II) of Lemma 4.9. The proof of this will appear in the following subsection.

By Lemma 4.9, given a **IK**-space (X, R), we may define $R_\Box[x] := R[\uparrow x]$ and $R_\Diamond[x] := \downarrow R[x]$. Then (X, R) is equivalent to $(X, R_\Box, R_\Diamond)$. Thus, in this chapter and the next, we will often choose whichever formulation best suits our purposes. With this in mind, and using (T4) of Definition 4.7, we make sense of condition (6) for **IKS** p-morphisms as the standard back condition for $R_\Box$, namely

(6') If $f(x)S_\Box z$ then $f(x') = z$ for some $x' \in R_\Box[x]$

.

Proposition 4.10. *Let (X, R) and (Y, S) be **IK**-spaces, and define $R_\Box[x] := R[\uparrow x]$ and $S_\Box[x] := S[\uparrow x]$. If $f : X \to Y$ is an order p-morphism, then the following are equivalent:*

(6') If $f(x)S_\Box z$ then $f(x') = z$ for some $x' \in R_\Box[x]$;

(6) If $f(x)(\leq_Y \circ S)z$ then $f(x') \leq_Y z$ for some $x' \in R[\uparrow x]$.

Proof. To see this, let $f(x)S_\Box z$, then there is $x' \in R_\Box[x]$ such that $f(x') \leq z$. By the back condition for $\leq$ (i.e. (2)), there is $w \geq x'$ such that $f(w) = z$.

$$
\begin{tikzcd}
 & & z \\
x' \arrow[r, dotted, "f"'] & f(x') \arrow[ur, "\leq"] & \\
x \arrow[u, dotted, "R_\Box"] \arrow[rr, "f"'] & & f(x) \arrow[uu, "\leq \circ S = S_\Box"']
\end{tikzcd}
\qquad
\begin{tikzcd}
w \arrow[r, dotted, "f"] & z = f(w) \\
x' \arrow[u, dotted, "\leq"] \arrow[r, "f"'] & f(x') \arrow[u, "\leq"]
\end{tikzcd}
$$

And as $xR_\Box x' \leq w$ and $R_\Box[x]$ is an upset, we have that $w \in R_\Box[x]$. Furthermore, the back condition for $R_\Box$ clearly implies (6), as $f(x') = z$ satisfies $f(x') \leq z$. $\Box$

4.2.2 Coalgebraic IK-spaces

Our goal is to find an endofunctor on Esakia spaces such that its coalgebras correspond in a natural way to **IK**-spaces. Note that it is already evident from Definitions 4.7 and 4.8 that finding a coalgebraic representation for **IK**-spaces and morphisms is not straight-forward. Recall that to turn a frame into a coalgebra, we view the relations as maps taking a point to its set of successors, while homomorphisms between coalgebras will correspond to the morphisms between frames. Then note from the definitions that for a **IK**-space $(X, R_\Box, R_\Diamond)$, $R_\Box[-]$ and $R_\Diamond[-]$) are only required to be monotone (Priestley morphisms), whereas **IKS** p-morphisms must additionally be p-morphisms with respect to the order (Esakia morphisms). Thus, if we take coalgebras for a Priestley endofunctor, we will have more coalgebra homomorphisms than **IKS** p-morphisms. If we instead take coalgebras for an Esakia endofunctor, we will have more **IK** frames than coalgebras, as not all relations will correspond to coalgebra maps. This is precisely where the construction from Chapter 3 comes in: we first ensure a 1-1 correspondence between our frames and *positive* coalgebras, and then turn these into intuitionistic coalgebras, thereby excluding any homomorphisms that are not Esakia.

As outlined in the introduction to this chapter, we start by representing $\Box\Diamond$-frames, which correspond to the smallest intuitionistic extension of PML satisfying the normality axioms (Definition 4.1). Thus, our first step is to turn a $\Box\Diamond$-frame into a positive coalgebra.

Recall the upper Vietoris functor $\mathcal{V}^\uparrow$ (Definition 3.8) and lower Vietoris functor $\mathcal{V}^\downarrow$ (Definition 3.9) from Chapter 3. As the relations of a $\Box\Diamond$-frame $(X, R_\Box, R_\Diamond)$ do not interact in any way, we treat them separately. Thus, following the results from [BHM23] presented in Chapter 3, $(X, R_\Box, R_\Diamond)$ can be represented by a coalgebra on $\mathcal{V}^\uparrow(-) \times \mathcal{V}^\downarrow(-)$. We will often use the shorthand $\mathcal{V}^{\uparrow\downarrow}$ when referring to this functor.

Theorem 4.11. *There is a 1-1 correspondence between $\Box\Diamond$-frames and coalgebras for the Priestley endofunctor $(\mathcal{V}^\uparrow(-) \times \mathcal{V}^\downarrow(-))$.*

Proof. Let $(X, R_\Box, R_\Diamond)$ be a $\Box\Diamond$-frame. We can define a coalgebra as $(X, (R_\Box[-], R_\Diamond[-]) : X \to \mathcal{V}^{\uparrow\downarrow}(X))$ where $(R_\Box[-], R_\Diamond[-])(x) = (R_\Box[x], R_\Diamond[x])$. By Lemma 4.3, $R_\Box[-]$ is monotone with respect to $\supseteq$ (the ordering of $\mathcal{V}^\uparrow(X)$) and $R_\Diamond[-]$ with respect to $\subseteq$ (the ordering of $\mathcal{V}^\downarrow(X)$). Thus, if $x \leq y \in X$ then $R_\Box[x] \leq_{\mathcal{V}^\uparrow(X)} R_\Box[y]$ and $R_\Diamond[x] \leq_{\mathcal{V}^\downarrow(X)} R_\Diamond[y]$, so $(R_\Box[x] \times R_\Diamond[x]) \leq_{\mathcal{V}^{\uparrow\downarrow}(X)} (R_\Box[y] \times R_\Diamond[y])$.

To see that it is continuous, let $U' \times V' \subseteq \mathcal{V}^{\uparrow\downarrow}(X)$ be a subbasic open. Then U', V' are of the form $[U]$ or $\langle X - V \rangle$ for U, V clopen upsets of X in the case of U' and clopen downsets in the case of V'. Then we have the following cases:

- $U' = [U] = \{C \in \mathcal{V}^\uparrow(X) | C \subseteq U\}$. Then $R_\Box^{-1}[U] = \{x \in X | R_\Box[x] \subseteq U\} = \Box_{R_\Box} U$. Since U is a clopen upset of X, $\Box_{R_\Box} U$ is also a clopen upset of X given the definition.

- $U' = \langle X - V \rangle = \{C \in \mathcal{V}^{\uparrow}(X) | C \cap U \neq \emptyset\}$. Then $R_{\Box}^{-1}\langle X - V \rangle = \{x \in X | R_{\Box}[x] \cap (X - V) \neq \emptyset\} = X \setminus \Box_{R_{\Box}} V$. As $V \in ClopUp(X)$, $\Box_{R_{\Box}} V \in ClopUp(X)$, so $X \setminus \Box_{R_{\Box}} V$ is clopen.
- $V' = [U] = \{C \in \mathcal{V}^{\downarrow}(X) | C \subseteq U\}$. Then $R_{\Diamond}^{-1}[U] = \{x \in X | R_{\Diamond}[x] \subseteq U\} = \{x \in X | R_{\Diamond}[x] \cap (X \setminus U) = \emptyset\} = X \setminus \Diamond_{R_{\Diamond}}(X \setminus U)$. As U is a clopen downset, $X \setminus U$ is a clopen upset, so $\Diamond_{R_{\Diamond}}(X \setminus U)$ is a clopen upset, so $X \setminus \Diamond_{R_{\Diamond}}(X \setminus U)$ is clopen.
- $V' = \langle X - V \rangle = \{C \in \mathcal{V}^{\downarrow}(X) | C \cap (X - V) \neq \emptyset\}$. Then $R_{\Diamond}^{-1}\langle X - V \rangle = \{x \in X | R_{\Diamond}[x] \cap (X - V) \neq \emptyset\} = \Diamond_{R_{\Diamond}}(X \setminus V)$, which must be a clopen upset of X by definition.

For the converse direction, let $(X, (\alpha, \beta) : X \to \mathcal{V}^{\uparrow\downarrow}(X))$ be a coalgebra. We can define a $\Box\Diamond$-frame $(X, R_{\Box}, R_{\Diamond})$ where the relations are defined respectively by $R_{\Box}[x] = \alpha(x)$ and $R_{\Diamond}[x] = \beta(x)$. We automatically have that $R_{\Box}[x]$ is a closed upset and $R_{\Diamond}[x]$ is a closed downset.

Now let $U \in ClopUp(X)$. We have that $\Box_{R_{\Box}} U = \{x \in X | R_{\Box}[x] \subseteq U\} = R_{\Box}^{-1}[U]$. Since $R_{\Box}[-]$ is continuous and monotone, it follows that $R_{\Box}^{-1}[U] \in ClopUp(X)$. Finally, $\Diamond_{R_{\Diamond}} U = \{x \in X | R_{\Diamond}[x] \cap U \neq \emptyset\} = R_{\Diamond}^{-1}\langle U \rangle$, which is clopen by continuity and monotonicity of $R_{\Diamond}$.

$\Box$

Despite this correspondence on objects, we do not have an equivalence of the categories $\mathbf{Fr}_{\Box\Diamond}$ and $\mathbf{Coalg}(\mathcal{V}^{\uparrow} \times \mathcal{V}^{\downarrow})$. This is because morphisms between $\Box\Diamond$-frames must be p-morphisms (Esakia morphisms), but $\mathcal{V}^{\uparrow} \times \mathcal{V}^{\downarrow}$ is treated as an endofunctor on **Pries**, thus coalgebra homomorphisms need only be Priestley morphisms (i.e. continuous and monotone). Thus, there are more morphisms in $\mathbf{Coalg}(\mathcal{V}^{\uparrow} \times \mathcal{V}^{\downarrow})$ than in $\mathbf{Fr}_{\Box\Diamond}$.

While $\mathcal{V}^{\uparrow} \times \mathcal{V}^{\downarrow}$ also acts as an endofunctor on **Esa** [4], we cannot simply restrict to this category. The problem lies in the coalgebra morphisms corresponding to the relations $R_{\Box}$ and $R_{\Diamond}$, which need not themselves be p-morphisms. Thus, if we were to restrict to **Esa**, we would have more coalgebras than $\Box\Diamond$-frames. However, by applying the functor $\mathcal{V}_G$, we arrive at the following:

Theorem 4.12. *The categories of $\Box\Diamond$-frames and* $\mathbf{CoAlg}(\mathcal{V}_G(\mathcal{V}^{\uparrow\downarrow}))$ *are equivalent.*

The proof below is a special case of the proof of [AB24, Theorem 4.2].

Proof. Let $(X, R_{\Box}, R_{\Diamond})$ be a $\Box\Diamond$-frame, with the corresponding $\mathcal{V}^{\uparrow} \times \mathcal{V}^{\downarrow}$-coalgebra $(X, (R_{\Box}[-], R_{\Diamond}[-]) : X \to \mathcal{V}^{\uparrow\downarrow}(X))$. Let us write $f^R_{\Box\Diamond}$ to denote the coalgebra map $(R_{\Box}[-], R_{\Diamond}[-])$. As X is an Esakia space, $\mathcal{V}^{\uparrow\downarrow}(X)$ a Priestley space, and $f^R_{\Box\Diamond} : X \to \mathcal{V}^{\uparrow\downarrow}(X))$ a Priestley morphism, we have by Proposition 3.22 that there is a unique Esakia morphism $\overline{f^R_{\Box\Diamond}} : X \to \mathcal{V}_G(\mathcal{V}^{\uparrow\downarrow}(X))$ extending $f^R_{\Box\Diamond}$. Recall that $\overline{f^R_{\Box\Diamond}}$ is given by $\overline{f^R_{\Box\Diamond}}(x) = ((f^R_{\Box\Diamond})_0(x), (f^R_{\Box\Diamond})_1(x), ...)$, where $(f^R_{\Box\Diamond})_0 = f^R_{\Box\Diamond}$ and $(f^R_{\Box\Diamond})_{n+1}(x) = (f^R_{\Box\Diamond})_n[\uparrow x]$.

Let $(Y, S_{\Box}, S_{\Diamond})$ be another $\Box\Diamond$-frame, and $h : X \to Y$ be a $\Box\Diamond$ p-morphism. Denote the map $(S_{\Box}[-], S_{\Diamond}[-]) : Y \to \mathcal{V}^{\uparrow\downarrow}(Y)$ by $f^S_{\Box\Diamond}$. Then we have the corresponding $\mathcal{V}^{\uparrow\downarrow}$-coalgebras $(X, R : X \to \mathcal{V}^{\uparrow\downarrow}(X))$ and $(Y, S : Y \to \mathcal{V}^{\uparrow\downarrow}(Y))$. By Proposition 3.22, we can lift these to $(X, \overline{f^R_{\Box\Diamond}} : X \to \mathcal{V}_G(\mathcal{V}^{\uparrow\downarrow}(X)))$ and $(Y, \overline{f^S_{\Box\Diamond}} : Y \to \mathcal{V}_G(\mathcal{V}^{\uparrow\downarrow}(Y)))$.

By the functoriality of $\mathcal{V}_G(\mathcal{V}^{\uparrow\downarrow}(-))$, we can lift h to a map $\overline{h}$, which in turn essentially depends on the map $h^* : \mathcal{V}^{\uparrow\downarrow}(X) \to \mathcal{V}^{\uparrow\downarrow}(Y)$, where

$$h^*(x) = (\uparrow h[R_{\Box}[x]], \downarrow h[R_{\Diamond}[x]])$$

[4] See for example [AB24] for the case of $\mathcal{V}^{\uparrow}$, the case for $\mathcal{V}^{\downarrow}$ is analogous

It is straight-forward to prove that $h^* \circ f^R_{\Box\Diamond} = f^S_{\Box\Diamond} \circ h$ iff h is a $\Box\Diamond$ p-morphism. That the forth conditions for $R_\Box$ and $R_\Diamond$ hold iff $h^* \circ f^R_{\Box\Diamond}(x) \subseteq f^S_{\Box\Diamond} \circ h(x)$ (for all x) is similar to the classical case (see Chapter 3). The proof that the back conditions hold iff $h^* \circ f^R_{\Box\Diamond}(x) \supseteq f^S_{\Box\Diamond} \circ h(x)$ follows from observing that the $R_\Box$ back-condition $h(x)S_\Box y \implies \exists z \in X.xR_\Box z \;\&\; h(z) \leq y$ can be rewritten as $y \in S_\Box[h(x)] \implies y \in {\uparrow}h[R_\Box[x]]$, which is precisely the requirement that $S_\Box[h(x)] \subseteq {\uparrow}h[R_\Box[x]]$, and the $R_\Diamond$ back-condition $h(x)S_\Diamond y \implies \exists z \in X.xR_\Diamond z \;\&\; y \leq h(z)$ can be rewritten as $y \in S_\Diamond[h(x)] \implies y \in {\downarrow}h[R_\Diamond[x]]$, which is equivalent to requiring that $S_\Diamond[h(x)] \subseteq {\downarrow}h[R_\Diamond[x]]$. Furthermore, this means precisely that the required coalgebra diagram commutes, as the liftings will commute with these maps [5]. $\Box$

Thus, we may now capture spaces which dually satisfy axioms 1-4 of Definition 4.1 via coalgebras for the functor $\mathcal{V}^{\uparrow\downarrow}$. To achieve a coalgebraic representation for **IK**-spaces, we will need to look at subspaces which, in addition, dually satisfy axioms **A** and **B**.

Note that in the space $\mathcal{V}^{\uparrow}(X)$, elements $\Box a$ correspond to the clopen upsets $[U]$, and in the space $\mathcal{V}^{\downarrow}(X)$, elements $\Diamond a$ correspond to the clopen upsets $\langle U\rangle$ for $U \in ClopUp(X)$. We start by identifying the following subspace of $\mathcal{V}^{\uparrow\downarrow}(X)$, corresponding to condition (I) of Lemma 4.9:

Definition 4.13. *Let* $FS_1(X) = \{(D, C) \in \mathcal{V}^{\uparrow\downarrow}(X) : C = {\downarrow}(D \cap C)\}$.

Proposition 4.14. $FS_1(X)$ *is the Priestley subspace of* $\mathcal{V}^{\uparrow\downarrow}(X)$ *for which axiom* ***A*** *dually holds, i.e.* $FS_1(X) = \{(D, C) \in \mathcal{V}^{\uparrow\downarrow}(X) \mid \forall U, V \in ClopUp(X).\ (D, C) \in (\mathcal{V}^{\uparrow}(X) \times \langle U \to V\rangle) \cap ([U] \times \mathcal{V}^{\downarrow}(X)) \implies (D, C) \in \mathcal{V}^{\uparrow}(X) \times \langle V\rangle\}$.

Proof. For the left-to-right direction, we proceed with a direct proof. Let $(D, C) \in FS_1(X)$ be arbitrary, and so ${\downarrow}(D \cap C) = C$. Suppose that $(D, C) \in (\mathcal{V}^{\uparrow}(X) \times \langle U \to V\rangle) \cap ([U] \times \mathcal{V}^{\downarrow}(X))$. Unfolding this one step, we have that $C \in \langle U \to V\rangle$ (so $C \cap (U \to V) \neq \emptyset$), and $D \in [U]$ (so $D \subseteq U$). We want to show that $C \in \langle V\rangle$, i.e. $C \cap V \neq \emptyset$. Since $C \cap (U \to V) \neq \emptyset$, fix some $x \in C \cap (U \to V)$. Unfolding the definition of $x \in (U \to V)$, this means that for any z, if $x \leq z$ and $z \in U$ then $z \in V$. Since $x \in C$, by our assumption it follows that $x \in {\downarrow}(D \cap C)$. Then there is some $y \geq x$ in $D \cap C$. Since $D \subseteq U$, it follows that $y \in U$, so by $x \in U \to V$ we have that $y \in V$. So $y \in C$ and $y \in V$, and thus $y \in C \cap V \neq \emptyset$. Thus, $C \in \langle V\rangle$, as desired.

For the converse direction, we proceed by contraposition. Suppose $C \neq {\downarrow}(D \cap C)$. As C is a downset, the inclusion ${\downarrow}(D \cap C) \subseteq C$ always holds, so it must be that $C \not\subseteq {\downarrow}(D \cap C)$. Then $\exists x \in C.\forall y.(x \leq y \implies y \notin D$ or $y \notin C)$. Let us fix this x in what follows. Suppose towards contradiction that axiom **A** holds, i.e. $\forall (D, C) \in \mathcal{V}^{\uparrow\downarrow}(X).\ (D, C) \in (\mathcal{V}^{\uparrow}(X) \times \langle U \to V\rangle) \cap ([U] \times \mathcal{V}^{\downarrow}(X)) \implies (D, C) \in \mathcal{V}^{\uparrow}(X) \times \langle V\rangle$.

Now consider the set

$$A = \{{\uparrow}x\} \cup \{U | D \subseteq U\} \cup \{-V | C \subseteq -V\}$$

for $U, V \in ClopUp(X)$. We claim that if axiom **A** holds, then A has the finite intersection property.

Suppose towards contradiction that A does not have the FIP. Then there are finite subsets $\{U_1, ..., U_n\} \subseteq \{U | D \subseteq U\}$ and $\{-V_1, ..., -V_m\} \subseteq \{-V | C \subseteq -V\}$ such that

$${\uparrow}x \cap U_1 \cap \cdots \cap U_n \cap -V_1 \cap \cdots \cap -V_m = \emptyset$$

[5] See the proof of Theorem 4.18 for a similar, more detailed proof that traces how all the liftings commute.

Let $U = \bigcap\{U_1, ..., U_n\}$ and $-V = \bigcap\{-V_1, ..., -V_m\}$. Then U and V are clopen upsets (as they are finite intersections of clopen upsets). We can thus rewrite the above as $\uparrow x \cap U \cap -V = \emptyset$.

Note that $D \subseteq U$ and $C \subseteq -V$, so $D \in [U]$ and $C \in -\langle V \rangle$. We now claim that $x \in U \to V$, since suppose $x \leq y$ and $y \in U$. Then it must be that $y \notin -V$ (so $y \in V$), otherwise we would have $y \in \uparrow x \cap U \cap -V = \emptyset$, a contradiction. Thus, we have that $\forall y \geq x.(y \in U \implies y \in V)$, which means precisely that $x \in U \to V$.

Since $x \in C$ by assumption, we have that $C \cap (U \to V) \neq \emptyset$, so $C \in \langle U \to V \rangle$. We also have that $D \in [U]$ and $C \notin \langle V \rangle$, as noted above. Thus, $(D, C) \in \mathcal{V}^{\uparrow} X \times \langle U \to V \rangle \cap [U] \times \mathcal{V}^{\downarrow}(X)$, but $(D, C) \notin \mathcal{V}^{\uparrow}(X) \times \langle V \rangle$, contradicting our assumption that axiom **A** holds.

We may then conclude that under the assumption that axiom **A** holds, the set A has the FIP. Then by compactness of Priestley spaces, $\bigcap A \neq \emptyset$. By Lemma 2.49 (in the preliminaries), $D = \bigcap\{U | D \subseteq U\}$ and $C = \bigcap\{-V | C \subseteq -V\}$. But then there is some y such that $y \in \uparrow x$, $y \in D$, and $y \in C$, a contradiction.

$\square$

That is, for any $(D, C) = (R_\Box[x], R_\Diamond[x])$, then $R_\Diamond[x] = \downarrow(R_\Box[x] \cap R_\Diamond[x])$ for any x (condition (i)) if and only if axiom **A** is dually satisfied. Notice that if X is an Esakia space, then we have implications $U \to V \in ClopUp(X)$ for all $U, V \in ClopUp(X)$ between clopen upsets (recall from Chapter 2 that $U \to V = -\downarrow(U - V)$), so for axiom **A** we only need to add the modal operators via the functor $\mathcal{V}^{\uparrow\downarrow}(X)$, and dually quotient over **A** by taking the subspace $FS_1(X)$. However, axiom **B** involves implications between modal formulas, so in order to quotient over **B** we need a space that has implications between clopen upsets of $\mathcal{V}^{\uparrow\downarrow}(X)$ (or, dually, add elements of the form $\{a \to b | a, b \in D_{FS_1(X)}\}$). To this end, we now take $\mathcal{V}_r(FS_1(X))$. Recall from Definition 3.11 that this yields the Priestley space of the closed, rooted subsets of $FS_1(X)$, ordered by reverse inclusion, whose basis is given by subsets $[U], \langle V \rangle$ for $U, V \in Clop(FS_1(X))$.

Now we may look at a subspace satisfying (II) of 4.9, and show that these dually correspond to distributive lattices satisfying axiom **A**.

Definition 4.15. *Let* $FS_2(X) = \{C \in \mathcal{V}_r(FS_1(X)) | \forall (D, E) \in C,\ y \in D$ *and* $y \leq z$, *there exists* $(D', E') \geq (D, E)$ *in* C *such that* $z \in D' \cap E'\}$

Proposition 4.16. $FS_2(X)$ *is the Priestley subspace of* $V_r(FS_1(X))$ *for which axiom* ***B*** *dually holds, i.e. the set of elements* $FS_2(X) = \{C \in \mathcal{V}_r(FS_1(X)) \mid \forall U, V \in ClopUp(X)\ .\ C \in [-(\mathcal{V}^{\uparrow}(X) \times \langle U \rangle) \cup ([V] \times \mathcal{V}^{\downarrow}(X))] \implies C \in [[U \to V] \times \mathcal{V}^{\downarrow}(X)]$.

Proof. For the left-to-right direction, we proceed with a direct proof. Let $C \in \mathcal{V}_r(\mathcal{V}^{\uparrow\downarrow}(X))$ such that $\forall (D, E) \in C$, $y \in D$ and $y \leq z$, there exists $(D', E') \geq (D, E)$ in C such that $z \in D' \cap E'$. We assume that $C \in [-(\mathcal{V}^{\uparrow}(X) \times \langle U \rangle) \cup ([V] \times \mathcal{V}^{\downarrow}(X))]$, and show that $C \in [[U \to V] \times \mathcal{V}^{\downarrow}(X)]$. Unfolding this, our assumption is that $C \subseteq -(\mathcal{V}^{\uparrow}(X) \times \langle U \rangle) \cup [V] \times \mathcal{V}^{\downarrow}(X)$, and we want to show that $C \subseteq [U \to V] \times \mathcal{V}^{\downarrow}(X)$, i.e. for any $(D, E) \in C$, $D \subseteq U \to V$. Fix $(D, E) \in C$, and let $y \in D$, $y \leq z$, and $z \in U$. We show that $z \in V$, and then we are done (as this shows that $y \in U \to V$). By our assumption that $(D, E) \in FS_2$, $y \in D$ and $y \leq z$ implies the existence of some $(D', E') \geq (D, E)$ in C such that $z \in E' \cap D'$. Then $z \in E' \cap U \neq \emptyset$, so $E' \in \langle U \rangle$ and thus $(D', E') \notin -(\mathcal{V}^{\uparrow}(X) \times \langle U \rangle)$. Since $(D', E') \in C$, we have by assumption that $(D', E') \in -(\mathcal{V}^{\uparrow}(X) \times \langle U \rangle) \cup ([V] \times \mathcal{V}^{\downarrow}(X))$, so it must be that $(D', E') \in [V] \times \mathcal{V}^{\downarrow}(X)$ and thus $D' \subseteq V$. Then $z \in D' \subseteq V$, so $y \in U \to V$.

For the converse direction, we proceed by contraposition. Suppose that for $C \in \mathcal{V}_r(\mathcal{V}^{\uparrow\downarrow}(X))$, there is some $(D, E) \in C$, $y \in D$ and $y \leq z$, such that there is no $(D', E') \geq (D, E)$ in C with $z \in D' \cap E'$. Let us fix $z \geq y \in D$. We will find $U, V \in ClopUp(X)$ such that $C \in [-(\mathcal{V}^{\uparrow}(X) \times \langle U \rangle) \cup ([V] \times \mathcal{V}^{\downarrow}(X))]$ but $C \notin [[U \to V] \times \mathcal{V}^{\downarrow}(X)]$. Unfolding this, this means:

$$C \subseteq -(\mathcal{V}^{\uparrow}(X) \times \langle U \rangle) \cup [V] \times \mathcal{V}^{\downarrow}(X) \text{ and } C \not\subseteq [U \to V] \times \mathcal{V}^{\downarrow}(X) \iff$$
$$\forall (D', E') \in C.(E' \in \langle U \rangle \implies D' \in [V]) \text{ and } \exists (D'', E'') \in C.(D'' \notin [U \to V]) \iff$$
$$\forall (D', E') \in C.(E' \cap U \neq \emptyset \implies D' \subseteq V) \text{ and } \exists (D'', E'') \in C.(D'' \not\subseteq U \to V)$$

Thus, it suffices to show that there is some $U, V \in ClopUp(X)$ with $z \in U - V$ such that for any $(D', E') \in C$, if $E' \cap U \neq \emptyset$ then $D' \subseteq V$ (i.e. if $E' \in \langle U \rangle$ then $D' \in [V]$, the antecedent of axiom **B**). Then the consequent of axiom **B** won't be satisfied, as $y \leq z$ and $z \in U$, but $z \notin V$, so $y \notin U \to V$. Then $D \not\subseteq U \to V$ as $y \in D$, and thus $(D, E) \in C$ is such that $(D, E) \notin [U \to V] \times \mathcal{V}^{\downarrow}(X))$.

Now suppose towards contradiction that there is no such U, V. Then $\forall U, V \in ClopUp(X)$ such that $z \in U - V$, there exists some $(D', E') \in C$ such that $E' \cap U \neq \emptyset$ and $D' \cap -V \neq \emptyset$.

Consider the set

$$B = \{C\} \cup \{(\langle -V \rangle \times \mathcal{V}^{\downarrow}(X)) | z \in -V\} \cup \{(\mathcal{V}^{\uparrow}(X) \times \langle U \rangle) | z \in U\}.$$

We claim that B does not have the FIP (with the intersection taken in the space $FS_1(X)$), which will allow us to derive our contradiction. Suppose towards contradiction that B does have the FIP. Then by compactness there is $(D', E') \in \bigcap B$.

Now consider the sets

$$F_{D'} = \{D'\} \cup \{-V | z \in -V\} \quad \text{and} \quad F_{E'} = \{E'\} \cup \{U | z \in U\}$$

for $U, V \in ClopUp(X)$. We claim that if B has the FIP, then both $F_{D'}$ and $F_{E'}$ have the FIP (in the space X).

Suppose towards contradiction that $F_{D'}$ does not have the FIP. Then $D' \cap -V_0 \cap \cdots \cap -V_m = \emptyset$ for some $\{-V_0, \ldots, -V_m\} \subseteq \{-V | z \in -V\}$. Then $-V = \bigcap\{-V_0, \ldots, -V_m\}$ is a clopen downset containing z (as it is the finite intersection of clopen downsets containing z), and thus $(D', E') \in \langle -V \rangle \times \mathcal{V}^{\downarrow}(X)$ by our assumption that $(D', E') \in \bigcap B$. But then $D' \cap -V \neq \emptyset$, a contradiction. The case for $F_{E'}$ is similar. Thus, assuming that B has the FIP, we may conclude that both $F_{D'}$ and $F_{E'}$ have the FIP.

Then by compactness we have that $\bigcap F_{D'} \neq \emptyset$ and $\bigcap F_{E'} \neq \emptyset$, so there is some $w \in D'$ such that $w \in \bigcap\{-V | z \in -V\} = {\downarrow} z$. Since $z \geq w \in D'$, and D' is an upset, it follows that $z \in D'$. Similarly, there is some $w' \in E'$ such that $w' \in {\uparrow} z$, so since E' is a downset and $z \leq w' \in E'$, we have that $z \in E'$. But then we have that $(D', E') \in C$ and $z \in D' \cap E'$, contradicting our original assumption that there is no such (D', E'). Thus, if there is indeed no such (D', E') containing z, it must be that B does not have the FIP.

Since B does not have the FIP, we have that $C \cap \langle -V_0 \rangle \times \mathcal{V}^{\downarrow}(X) \cap \mathcal{V}^{\uparrow}(X) \times \langle U_0 \rangle \cap \cdots \cap \langle -V_m \rangle \times \mathcal{V}^{\downarrow}(X) \cap \mathcal{V}^{\uparrow}(X) \times \langle U_m \rangle = \emptyset$. Let $U = U_0 \cap \cdots \cap U_m$ and $V = V_0 \cap \cdots \cap V_m$. We have by construction that $z \in U - V$. Then by our earlier assumption there is some $(D', E') \in C$ such that $E' \cap U \neq \emptyset$ and $D' \cap -V \neq \emptyset$. But since $(D', E') \in C$, it must be that $(D', E') \notin \langle -V_i \rangle \times \mathcal{V}^{\downarrow}(X) \cap \mathcal{V}^{\uparrow}(X) \times \langle U_i \rangle$ for some V_i, U_i, as otherwise the intersection would be non-empty. We then have two cases:

- $E' \cap U_i = \emptyset$, which implies $E' \cap U = \emptyset$, or

- $D' \cap -V_i = \emptyset$, which implies $D' \cap -V = \emptyset$,

both of which lead to a contradiction. Thus, we have found U, V such that axiom **B** is refuted, concluding this proof. $\square$

We now have a correspondence between **IK**-spaces (X, R) and $\mathcal{V}_r(\mathcal{V}^{\uparrow} \times \mathcal{V}^{\downarrow})$ coalgebras $(X, \alpha : X \to FS_2(X))$. Note that the coalgebra morphisms must be open relative to the root map $r : FS_2(X) \to FS_1(X)$ (recall Definition 3.13), in order to preserve the layer of relative pseudocomplements that were previously added. To turn these into coalgebras for an appropriate endofuntor on Esakia spaces, we look at the composition $(\mathcal{V}_G^r \circ FS_2)(X)$. Here, recall that the r superscript specifies that we take the r-Vietoris complex (Definition 3.21). In other words, we have already completed the first step of the construction, while incorporating the necessary additional quotienting.

This leads us to our first result:

Theorem 4.17. *Let X be an Esakia space. The following are in 1-1 correspondence:*

*(i) **IK**-spaces (X, R) over X,*

(ii) r-open Priestley maps $f : X \to FS_2(X)$, and

(iii) Esakia morphisms $f' : X \to \mathcal{V}_G^r(FS_2(X))$

Proof.

- For (i) to (ii), let (X, R) be a **IK**-space. Let $R_\Box = \leq \circ R$ and $R_\Diamond = R \circ \geq$, and define the map

$$\begin{aligned} R_X &: X \to FS_1(X) \\ x &\mapsto (R_\Box[x], R_\Diamond[x]). \end{aligned}$$

Then by Lemma 3.17, $R'_X : X \to \mathcal{V}_r(FS_1(X))$ defined by $R'_X(a) = R_X[\uparrow a]$ is the unique r-open, continuous, monotone map making the triangle in Figure 4.1 commute.

Furthermore, we have that for any $x \in X$, $R_X[\uparrow x] \in FS_2(X)$. To see this, note that if $(D, E) \in R_X[\uparrow x]$ then $(D, E) = (R_\Box[x'], R_\Diamond[x'])$ for some $x' \geq x$. Furthermore, as (X, R) is a **IK**-space, $R_\Box[x'] = R[\uparrow x']$. Now let $y \in R_\Box[x'] = R[\uparrow x']$ and $y \leq z$. Then as $R_\Box[x']$ is an upset, $z \in R[\uparrow x']$. Then there is $v \geq x'$ such that $x' \in R_\Box[v] \cap R_\Diamond[v]$. By Lemma 4.3, $(R_\Box[x'], R_\Diamond[x']) \leq (R_\Box[v], R_\Diamond[v])$, and therefore $R_X[\uparrow x] \in FS_2(X)$. Therefore, restricting the codomain of R'_X to $FS_2(X)$ is justified.

$$\begin{tikzcd} X \arrow[rr, "R'_X"] \arrow[dr, "R_X"'] & & \mathcal{V}_r(FS_1(X)) \arrow[dl, "r"] \\ & FS_1(X) \arrow[d, "t"] & \\ & (\{*\}, \leq) & \end{tikzcd}$$

Figure 4.1: Commuting Triangle for R'_X

Where t is the terminal map to the one-element Priestley space. Clearly, R_X is t-open as every map is trivially open relative to the terminal map.

- For (ii) to (i), let $f : X \to FS_2(X)$ be an r-open, monotone, continuous map. Let us define the map $f_R : X \to FS_1(X)$ by

$$f_R := r \circ f.$$

Note that this assignment means precisely that f is the unique lifting of f_R (see [AB24]).

We may now define the **IK**-space (X_f, R_f) where $X_f = X$ and R_f is given by

$$xR_f y \iff y \in \pi_0 f_R(x) \cap \pi_1 f_R(x),$$

where, recalling that $f_R(x) \in FS_1(X)$ is an ordered pair, π_0 and π_1 are the projections to its first and second coordinates, respectively. We will subsequently see that the above assignment of $R_f[x]$ is equivalent to defining $R_f^{\Box}[x] = \pi_0 f_R(x)$ and $R_f^{\Diamond}[x] = \pi_1 f_R(x)$.

Note that $f(x) = f_R[\uparrow x] = \{f(z) | z \geq x\}$, as given that f is monotone, it follows that $f(x)$ is the root of $f[\uparrow x]$.

We now check that this satisfies the conditions of an **IK**-space (see Definition 4.7). Throughout, let $f_R[x] = (C, D)$:

To see that R_f is point-closed, note that because $f_R(x) = (C, D)$ is in FS_1, C is a closed upset and D a closed downset of X. Thus, $\pi_0(f_R(x)) \cap \pi_1(f_R(x)) = C \cap D$ is closed in $X = X_f$.

To show that $R_f[\uparrow x]$ is a closed upset, it suffices to show that $R_f[\uparrow x] = C$ (i.e. $\pi_0(f_R(x)) = R_f[\uparrow x] = \bigcup\{R_f[z] | z \geq x\}$). For the right-to-left inclusion, let $y \in C$. Recall that $(C, D) = r(f(x))$, where $f(x) \in FS_2(X)$. Then $(C, D) \in f(x)$, $y \in C$ and trivially $y \leq y$, so as $f(x) \in FS_2(X)$ we have that there is $(C', D') \geq (C, D) \in f(x)$ such that $y \in C' \cap D'$. Now take $\uparrow(C', D')$ in $FS_1(X)$, which is clearly closed, rooted, and belongs to $FS_2(X)$ as $\uparrow(C', D') \subseteq f(x) \in FS_2(X)$ (since $f(x)$ is an upset and $(C', D') \geq (C, D) \in f(x)$), so the FS_2 condition (Definition 4.15) clearly holds. Then $f(x) \leq_{FS_2(X)} \uparrow(C', D')$ (since $FS_2(X)$ is ordered by $\supseteq$). As f is r-open, there is $z \geq x$ such that $r(f(z)) = r(\uparrow(C', D'))$. Thus, $f_R(z) = (C', D')$, so we have found $z \geq x$ such that $y \in R_f[z]$. For the left-to-right inclusion, let $z \geq x$, with $f_R(z) = (C', D')$, and let $y \in (C' \cap D')$. By monotonicity of f_R, $(C, D) = f_R(x) \leq f_R(z) = (C', D')$, so by the ordering of $\mathcal{V}^{\uparrow\downarrow}(X)$, $C \supseteq C'$, thus $y \in C$.

We further have that $\downarrow R_f[x] = D$, as $(C, D) \in FS_1(X)$. Thus, denote

$$R_f^{\Box}[x] = \pi_0 \circ f_R[x] \text{ and}$$
$$R_f^{\Diamond}[x] = \pi_1 \circ f_R[x].$$

Now let $U \in ClopUp(X)$. We have that $\Box_{R_f^{\Box}} U = \{x \in X | R_f^{\Box}[x] \subseteq U\} = \{x \in X | (\pi_0 \circ r \circ f)[x] \subseteq U\} = (\pi_0 \circ f_R)^{-1}[[U]]$. Then this is clopen, as $(\pi_0 \circ f_R) : X \to \mathcal{V}^{\uparrow}(X)$ is continuous, and $[U]$ is clopen in $\mathcal{V}^{\uparrow}(X)$.

Similarly, $\Diamond_{R_f^{\Diamond}} U = \{x \in X | R_f[x] \cap U \neq \emptyset\} = \{x \in X | (\pi_1 \circ r \circ f)[x] \cap U \neq \emptyset\} = (\pi_1 \circ f_R)^{-1}[\langle U \rangle]$, which is clopen given continuity of $\pi_1 \circ f_R : X \mapsto \mathcal{V}^{\downarrow}(X)$ is continuous, and $\langle U \rangle$ is clopen in $\mathcal{V}^{\downarrow}(X)$.

To see that $R_f[x] = R_f[\uparrow x] \cap \downarrow R_f[x]$, note that, as we have shown, $C = R_f[\uparrow x]$, and as $R_f[x] \in FS_1(X)$ we have that $D = \downarrow(C \cap D) = \downarrow R_f[x]$. Thus, this amounts to $R_f[x] = C \cap D$, which holds by definition.

It is straight-forward to see that $r[R_X[\uparrow -]] = R_X[-]$ by monotonicity of R_X, and $rf[\uparrow -] = f$ by monotonicity of f. Thus, the assigments between **IK**-spaces and $FS_2(-)$-coalgebras are each others' inverses.

- For (iii) to (ii), let $f' : X \to \mathcal{V}_G^r(FS_2(X))$ be an Esakia morphism. Then the map $f : X \to FS_2(X)$ defined by

$$f := \lambda_0 \circ f'$$

which projects everything to the first coordinate is clearly a Priestley morphism. Now we claim that f is furthermore r-open:

Let $a \in X$, $b \in FS_2(X)$, and let $(\lambda_0 \circ f')(a) \leq b$. We show that there is $a' \in X$ such that $a \leq a'$ and $r((\lambda_0 \circ f')(a')) = r(b)$. We have that $b \subseteq (\lambda_0 \circ f')(a)$ (as $\mathcal{V}_G^r$ is ordered by reverse inclusion). Furthermore, by the definition of $\mathcal{V}_G^r(FS_2(X))$, we have $(b_0, b_1, b_2, ...) \in \mathcal{V}_G^r(FS_2(X))$ where $b_0 = b$, $b_i = \uparrow b_{i+1}$ and $b_i \in V_i$. To see this, note that for any x, $\uparrow x$ is clearly rooted, is always closed in Priestley spaces (Proposition 2.51), and is furthermore g-open for any g: let $g : A \to B$, $\uparrow x \subseteq A$, $s \in \uparrow x$, $z \in A$, and $s \leq z$. Then we have that $z \in \uparrow x$ since $s \leq z$ and $\uparrow x$ is an upset. Thus for $s' = z$, we have $s' \geq s$ in $\uparrow x$ such that $g(s') = g(z)$. It is also clear that $f'(a) \leq (b_0, b_1, b_2, ...)$. As f' is a p-morphism and $f'(a) \leq (b_0, b_1, b_2, ...)$, there is $a' \in X$ such that $f'(a') = (b_0, b_1, b_2, ...)$. Clearly, $\lambda_0(b_0, b_1, b_2, ...) = b$, so $r((\lambda_0 \circ f')(a')) = r(b)$, and we have thus shown that $f = (\lambda_0 \circ f')$ is r-open.

- For (ii) to (iii), let $f : X \to FS_2(X)$ be an r-open Priestley morphism. As f is r-open by assumption, we have by Lemma 3.17 the existence of the unique Priestley map $f[\uparrow -] : X \to \mathcal{V}_r(FS_2(X))$ (where $\mathcal{V}_r(FS_2(X))$ is the space of closed, rooted, r-open subsets of $FS_2(X)$) that is open relative to the root map $r_1 : \mathcal{V}_r(FS_2(X)) \to FS_2(X)$ making the diagram in Figure 4.2 commute. Define the map $\overline{f}$ by $\overline{f}(x) = (f_0(x), f_1(x), ...)$ where $f_0(x) = f(x)$ and $f_{n+1}(x) = f_n[\uparrow x]$. Then by Proposition 3.22, $\overline{f} : X \to \mathcal{V}_G^r(FS_2(X))$ is the unique Esakia morphism extending f.

$$\begin{tikzcd}
X \arrow[rr, "f[\uparrow -]"] \arrow[dr, "f"'] & & \mathcal{V}_r(FS_2(X)) \arrow[dl, "r_1"] \\
& FS_2(X) \arrow[d, "r"] & \\
& FS_1(X) &
\end{tikzcd}$$

Figure 4.2: Commuting Triangle for $\mathcal{V}_r(FS_2(X))$

Furthermore, it is clear that if $f : X \to FS_2(X)$ is r-open, then $\lambda_0(\overline{f}) = f$ and if $f' : X \to \mathcal{V}_G^r(FS_2(X))$ then $\overline{\lambda_0(f')} = f'$. Thus, these assignments are each others' inverses as well.

$\square$

From this, it follows that the assignments $(X, R_\Box, R_\Diamond) \quad \mapsto \quad \overline{(R_\Box[\uparrow -], R_\Diamond[\uparrow -])} : X \to \mathcal{V}^r_G(FS_2(X))$ and

$f' : X \to \mathcal{V}^r_G(FS_2(X)) \quad \mapsto \quad (X, R_\Box, R_\Diamond)$ where $R_\Box[-] = \pi_0 \circ r \circ \lambda_0 \circ f'$ and $R_\Diamond[-] = \pi_1 \circ r \circ \lambda_0 \circ f'$ are also each others' inverses. This brings us to our first main result:

Theorem 4.18. *The category **IK** is equivalent to the category **CoAlg**($\mathcal{V}^r_G(FS_2(-))$).*

It remains to show that there exist inverse assignments on the morphisms in these categories.

Proof. Let (X, R) and (Y, S) be **IK**-spaces, and $f : X \to Y$ be a **IKS** p-morphism. Then we have the corresponding $(\mathcal{V}^{\uparrow\downarrow}(-)$-coalgebras $(X, R : X \to FS_1(X))$ and $(Y, S : Y \to FS_1(Y))$. [6]

Recall that $\mathcal{V}^{\uparrow\downarrow}(f) = (\uparrow f[-], \downarrow f[-])$. As both $R : X \to FS_1(X)$ and the root map $r_X : FS_2(X) \to FS_1(X)$ are open relative to the terminal map (Lemma 3.18), and likewise for $S : Y \to FS_1(Y)$ and $r_Y : FS_2(Y) \to FS_1(Y)$, the $\mathcal{V}^{\uparrow\downarrow}$-coalgebras lift to $\mathcal{V}_r(\mathcal{V}^{\uparrow\downarrow})$-coalgebras $(X, R[\uparrow -] : X \to FS_2(X))$ and $(Y, S[\uparrow -] : Y \to FS_2(Y))$, where $r_X \circ R[\uparrow -] = R$ and $r_Y \circ S[\uparrow -] = S$ by Lemma 3.17. Then by Lemma 3.20, we then also have the map $\mathcal{V}_r(\uparrow f[-], \downarrow f[-]) = (\uparrow f[-], \downarrow f[-])[-]$, where $(\uparrow f[-], \downarrow f[-]) \circ r_X = r_Y \circ (\uparrow f[-], \downarrow f[-])[-]$. Let $\lambda^X_0 : \mathcal{V}^r_G(FS_2(X)) \to FS_2(X)$ and $\lambda^Y_0 : \mathcal{V}^r_G(FS_2(Y)) \to FS_2(Y)$ denote the projections from $\mathcal{V}^r_G(FS_2(-))$ to the first coordinate.

Again, as both λ^X_0 and $R[\uparrow -]$ are open relative to r_X (and analogously for Y), these coalgebras in turn lift to $\mathcal{V}^r_G(\mathcal{V}_r(\mathcal{V}^{\uparrow\downarrow}))$-coalgebras $(X, \overline{R[\uparrow -]} : X \to \mathcal{V}^r_G(FS_2(X)))$ and $(Y, \overline{S[\uparrow -]} : Y \to \mathcal{V}^r_G(FS_2(Y)))$, where $\lambda^X_0 \circ \overline{R[\uparrow x]} = R[\uparrow x]$ by Proposition 3.22. Finally, we have $\mathcal{V}^r_G(\mathcal{V}_r(\mathcal{V}^{\uparrow\downarrow}(f))) = \overline{\uparrow f[-], \downarrow f[-])[-]} : \mathcal{V}^r_G(FS_2(X)) \to \mathcal{V}^r_G(FS_2(Y))$ as defined in Lemma 3.23, where $(\uparrow f[-], \downarrow f[-])[-] \circ \lambda^X_0 = \lambda^Y_0 \circ \overline{(\uparrow f[-], \downarrow f[-])[-]}$. Furthermore, the $\mathcal{V}^r_G(FS_2(-))$-coalgebra diagram commutes if and only if the inner square of Figure 4.3 commutes, i.e. if $(\uparrow f[-], \downarrow f[-]) \circ R = S \circ f$. To see this, note that all of the lifted maps are defined uniquely such that they yield a commuting diagram, so everything but the FS_1 sub-diagram is always forced to commute. That is, $\lambda^X_0 \circ \overline{R[\uparrow x]} = R[\uparrow x]$ and $r_X \circ R[\uparrow x] = R$ so $r_X \circ \lambda^X_0 \circ \overline{R[\uparrow x]} = R$. Likewise, $r_Y \circ \lambda^Y_0 \circ \overline{S[\uparrow x]} = S$, and similarly $r_Y \circ \lambda^Y_0 \circ \overline{(\uparrow f[-], \downarrow f[-])[-]} = (\uparrow f[-], \downarrow f[-]) \circ r_X \circ \lambda^X_0$. Thus, showing that $\overline{(\uparrow f[-], \downarrow f[-])[-]} \circ \overline{R[\uparrow x]} = \overline{S[\uparrow x]} \circ f$ amounts to showing that $(\uparrow f[-], \downarrow f[-]) \circ R = S \circ f$. We now show that this in turn means precisely that f is an **IKS** p-morphism.

Let $f : X \to Y$ be a **IKS** p-morphism. We must show that (i) $S_\Box[f(x)] = \uparrow f[R_\Box[x]]$ and (ii) $S_\Diamond[f(x)] = \downarrow f[R_\Diamond[x]]$. For (i), let $y \in S_\Box[f(x)]$. As f is a **IKS** p-morphism, it follows immediately that $y \in \uparrow f[R_\Box[x]]$ (see Definition 4.8, (6)). Now let $f(y) \in f[R_\Box[x]]$ for $y \in R_\Box[x]$. Then as X is a **IK** frame and therefore $R_\Box[x] = R[\uparrow x]$, there is w such that $xRw \leq y$. Then, as f is a **IKS** p-morphism, by conditions (4) and (2) $f(x)Sf(w) \leq f(y)$, so $f(x)S_\Box f(y)$ and thus $f(y) \in S_\Box[f(x)]$, as desired.

For (ii), let $y \in S_\Diamond[f(x)]$. Then $y \in \downarrow(S_\Box[f(x)] \cap S_\Diamond[f(x)])$, so there is $z \geq y$ such that $z \in S[f(x)]$. Then (by (5) of Definition 4.8) $z \leq f(x')$ for some $x' \in R[x] \subseteq R_\Diamond[x]$, so $f(x') \in f[R_\Diamond[x]]$. As $y \leq z \leq f(x')$, $y \in \downarrow f[R_\Diamond[x]]$.

Now let $f(y) \in f[R_\Diamond[x]]$ for $y \in R_\Diamond[x]$. As $y \in R_\Diamond[x]$, there is $z \geq y$ such that $z \in R[x]$. Then $f(y) \leq f(z)$, and $f(z) \in S[f(x)]$, so $f(y) \in \downarrow S[f(x)] = S_\Diamond[f(x)]$. Thus we have shown that if f is a **IKS** p-morphism, the FS_1 diagram commutes.

For the converse direction, let f yield a commuting diagram. As f is an Esakia morphism, it is clear that f is continuous, as well as a p-morphism with respect to $\leq$.

[6] here and in what follows, we are restricting the codomains of the relevant maps to the desired subspaces (FS_1 or FS_2).

Let $y \in R_\Box[x] \cap R_\Diamond[x]$. Then $f(y) \in \uparrow f[R_\Box[x]]$ and $f(y) \in \downarrow f[R_\Diamond[x]]$. As the diagram commutes, $f(y) \in S_\Box[f(x)]$ and $f(y) \in S_\Diamond[f(x)]$, and thus $f(y) \in S[f(x)]$, as desired. This shows that condition (4) of Definition 4.8 holds.

Now let $z \in S_\Box[f(x)]$. We must find $x' \in R_\Box[x]$ such that $f(x') = z$. As the diagram commutes, $z \in \mathcal{V}^\uparrow(f)(\lambda_0(R(x))) = \uparrow f[R_\Box[x]]$. As f is a p-morphism and thus preserves upsets (and $R_\Box[x] \in \mathcal{V}^\uparrow(X)$ is an upset), this is equivalent to $f[R_\Box[x]]$. Then $z = f(x')$ for some $x' \in R_\Box[x]$, so condition (6) of Definition 4.8 holds.

Finally, let $z \in S_\Box[f(x)] \cap S_\Diamond[f(x)]$. We must find $x' \in R_\Box[x] \cap R_\Diamond[x]$ such that $f(x') \geq z$. As the diagram commutes, $z \in \uparrow f[R_\Box[x]] \cap \downarrow f[R_\Diamond[x]]$. Then there is $w \in R_\Diamond[x]$ such that $f(w) \geq z$. Recall that as $(R_\Box[x], R_\Diamond[x]) \in FS_1(X)$, we have that $R_\Diamond[x] = \downarrow(R_\Box[x] \cap R_\Diamond[x])$, so there is $x' \geq w$ such that $x' \in R_\Box[x] \cap R_\Diamond[x]$. Then by monotonicity of f, $f(x') \geq f(w) \geq z$. Thus, we have found $x' \in R[x]$ such that $f(x') \geq z$, concluding the proof that f is a **IKS** p-morphism.

Thus, $f : X \to Y$ a **IKS** p-morphism from $(X, R_\Box, R_\Diamond)$ to $(Y, S_\Box, S_\Diamond)$ if and only if f is is a coalgebra homomorphism from $(X, \overline{(R_\Box[\uparrow -], R_\Diamond[\uparrow -])} : X \to \mathcal{V}_G^r(FS_2(X)))$ to $(Y, \overline{(S_\Box[\uparrow -], S_\Diamond[\uparrow -])} : Y \to \mathcal{V}_G^r(FS_2(Y)))$.

□

$$
\begin{tikzcd}
 & & X \arrow[r, "f"] \arrow[d, "R"] \arrow[ddl, "R[\uparrow -]"'] \arrow[dddll, "\overline{R[\uparrow -]}"'] & Y \arrow[d, "S"'] \arrow[ddr, "S[\uparrow -]"] \arrow[dddrr, "\overline{S[\uparrow -]}"] & & \\
 & & FS_1(X) \arrow[r, "(\uparrow f[-], \downarrow f[-])"'] & FS_1(Y) & & \\
 & FS_2(X) \arrow[ur, "r_X"'] \arrow[rrr, "(\uparrow f[-], \downarrow f[-])[-]"'] & & & FS_2(Y) \arrow[ul, "r_Y"] & \\
\mathcal{V}_G^r(FS_2(X)) \arrow[ur, "\lambda_0^X"'] \arrow[rrrrr, "\overline{(\uparrow f[-], \downarrow f[-])[-]}"'] & & & & & \mathcal{V}_G^r(FS_2(Y)) \arrow[ul, "\lambda_0^Y"]
\end{tikzcd}
$$

Figure 4.3: Commuting diagram for $\mathcal{V}_G^r(FS_2(-))$-coalgebras

4.3 The discrete setting: image-finite Kripke frames

The method outlined in the previous section extends to image-finite Kripke frames, using the duality between $\mathbf{ImFinPos}_p$ of image-finite posets with p-morphisms and profinite Heyting algebras [BB08].

Definition 4.19. *An algebra $(H, \wedge, \vee, \to, \Box, \Diamond, \top, \bot)$ is called an $\mathbf{K_{IK}}$-algebra if $(H, \wedge, \vee, \to, \top, \bot)$ is a profinite Heyting algebra (see Definition 2.31), and it satisfies the axiomatization given in Definition 4.1 of **IK**-algebras.*

4.3.1 Image-finite Kripke frames for Fischer Servi logic

Definition 4.20 ($\Box\Diamond$ Kripke-frame)**.** *A* $\Box\Diamond$ Kripke frame *or* $K_{\Box\Diamond}$-frame *is a triple $(X, R_\Box, R_\Diamond)$ such that X is an image-finite poset and the following conditions hold:*

1. *$R_\Box[x]$ is an upset*

2. $R_\Diamond[x]$ *is a downset*

(i) $R_\Box = \leq \circ R_\Box \circ \leq$

(ii) $R_\Diamond = \geq \circ R_\Diamond \circ \geq$

Definition 4.21 ($K_{\Box\Diamond}$ p-morphisms)**.** *Let* $(X, R^X_\Box, R^X_\Diamond)$ *and* $(Y, R^Y_\Box, R^Y_\Diamond)$ *be two* $K_{\Box\Diamond}$*-frames. A map* $f : X \to Y$ *is a* $K_{\Box\Diamond}$ p-morphism *provided*

1. *f is an p-morphism with respect to* $\leq$*;*
2. $xR^X_\Diamond z$ *implies* $f(x)R^Y_\Diamond f(z)$*;*
3. $xR^X_\Box z$ *implies* $f(x)R^Y_\Box f(z)$*;*
4. $f(x)R^Y_\Diamond y$ *implies there exists* $z \in X$ *such that* $xR^X_\Diamond z$ *and* $y \leq f(z)$*.*
5. $f(x)R^Y_\Box y$ *implies there exists* $z \in X$ *such that* $xR^X_\Box z$ *and*$f(z) = y$*.*

Note that these conditions are the same as in the topological case, but dropping the requirement that f be continuous.

Definition 4.22 (K_{FS}-frame)**.** *Let* $(X, \leq, R)$ *be a triple where* $(X, \leq)$ *is an image-finite poset and* $R \subseteq X \times X$*. We say this is a* K_{FS}-frame *if the following conditions are satisfied (see for example [Pal04b, Definition 2.0.6]):*

1. $(R\circ \leq) \subseteq (\leq \circ R)$
2. $(\geq \circ R) \subseteq (R\circ \geq)$

We will also make use of the following lemma and corollary due to [Pal04b], pp.2:

Lemma 4.23. *Let* $(X, \leq, R)$ *be a* K_{FS}*-frame. Then we have the following equivalences:*

a. $(\leq \circ R) \subseteq (R\circ \leq) \iff Up(X)$ *is closed under* $\Box_R$*.*

b. $(\geq \circ R) \subseteq (R\circ \geq) \iff Up(X)$ *is closed under* $\Diamond_R$*.*

c. $(R\circ \leq) \subseteq (\leq \circ R) \iff$ *for every* $x \in X$*,* $R[\uparrow x] \in Up(X)$*.*

Definition 4.24 (K_{FS} p-morphism)**.** *Let* (X, R) *and* (Y, S) *be* K_{FS}*-frames. A* K_{FS} p-morphism *is a map* $f : X \to Y$ *such that the following conditions are satisfied:*

1. *f is monotone;*
2. *f is a p-morphism w.r.t* $\leq$*.*
3. xRz *implies* $f(x)Sf(z)$*;*
4. *If* $f(x)Sz$ *then* $z \leq_Y f(x')$ *for some* $x' \in R[x]$*.*
5. *If* $f(x)(\leq_Y \circ S)z$ *then* $f(x') \leq_Y z$ *for some* $x' \in R[\uparrow x]$*.*

We denote by $\mathbf{K_{FS}}$ the category of K_{FS}-frames and K_{FS} p-morphisms.

The following lemma is precisely analogous to Lemma 4.9 used for the topological setting.

Lemma 4.25. *Let $(X, R_\Box, R_\Diamond)$ be a $K_{\Box\Diamond}$-frame. Then (X, R) is a K_{FS}-frame where $R := R_\Box \cap R_\Diamond$ provided the following conditions hold:*

(I) $R_\Diamond = \downarrow(R_\Box \cap R_\Diamond)$ *and*

(II) $R_\Box = \leq \circ (R_\Box \cap R_\Diamond)$

4.3.2 Coalgebraic K_{FS}-frames

We will begin by defining discrete analogues for the functors $\mathcal{V}^\uparrow$ and $\mathcal{V}^\downarrow$:

Definition 4.26. *Let $(X, \leq)$ and $(Y, \leq)$ be posets and $f : X \to Y$ a monotone map. The endofunctors $Up(X)$ and $Down(X)$ on **Pos** (called the* upset *functor and* downset *functor respectively) are defined as follows:*

On objects:

1. $Up(X) = \{C \subseteq X | C \text{ is an upset}\}$*, ordered by reverse inclusion.*
2. $Down(X) = \{C \subseteq X | C \text{ is a downset}\}$*, ordered by inclusion.*

On morphisms:

1. $Up(f) : Up(X) \to Up(Y)$ *is given by* $Up(f)(A) := \uparrow f[A] = \{y' \in Y | y' \geq y\} \text{ for some } y \in f[A]\}$*.*
2. $Down(f) : Down(X) \to Down(Y)$ *is given by* $Down(f)(A) := \downarrow f[A] = \{y' \in Y | y' \leq y\} \text{ for some } y \in f[A]\}$*.*

Lemma 4.27. *Let X be a poset. Then the posets $(Up(X), \supseteq)$ and $(Down(X), \subseteq)$ are order-isomorphic.*

Proof. Let X be a poset, and define the map $c : Up(X) \to (Down(X))$ by $c(U) = X - U$. Then $X - U$ is clearly a downset, and thus an element of $Down(X)$. Now let $U \leq_{Up} V$, i.e. $U \supseteq V$. It is easily verified that $U \supseteq V$ if and only if $(X - U) \subseteq (X - V)$, and thus $c(U) \leq_{Down} c(V)$. Clearly, this defines a bijection. □

Lemma 4.28. *$Up(-)$ and $Down(-)$ restrict to endofunctors on $\textbf{\textit{ImFinPos}}_p$*

Proof. It is well-known that $Up(-)$ is an endofunctor on $\mathbf{ImFinPos}_p$ (see for example [AB24]). It then follows from Lemma 4.27 that if X is image-finite then $(Down(X), \subseteq)$ is also image-finite. Let $f : X \to Y$ be a p-morphism between image-finite posets. Monotonicity of $Down(f)$ is immediate. For the back condition, let $\downarrow f[A] \subseteq B$. Then we claim that $f^{-1}[B]$ is a downset of X. To see this, let $x \in f^{-1}[B]$ and $x' \leq x$, then by monotonicity $f(x') \leq f(x) \in B$ so $x' \in f^{-1}[B]$. Furthermore, we have that $f[A] \subseteq B$, so $A \subseteq f^{-1}[B]$. As B is a downset, clearly $\downarrow f[f^{-1}[B]] = B$. Thus, the back condition is satisfied. □

However, we again have the issue that K_{FS} relations only need to be monotone, and not p-morphisms. Thus, just as we treated $\mathcal{V}^\uparrow$ and $\mathcal{V}^\downarrow$ as endofunctors on **Pris** rather than **Esa**, we will treat $Up(-)$ and $Down(-)$ as endofunctors on **Pos**.

As in the topological setting, elements $\Box a$ in the dual distributive lattice correspond to upsets $[A] = \{B \in Up(X) | B \subseteq A\}$ and elements $\Diamond a$ correspond to upsets $\langle A \rangle = \{B \in Down(X) | B \cap A \neq \emptyset\}$.

Theorem 4.29. *There is a 1-1 correspondence between $K_{\Box\Diamond}$-frames and coalgebras for the **Pos** endofunctor ($Up \times Down$). By the results in [AB24], the categories of $K_{\Box\Diamond}$-frames and $\mathbf{CoAlg}(P_G(Up \times Down))$ are isomorphic.*

Proof. Let $(X, R_\Box, R_\Diamond)$ be a $K_{\Box\Diamond}$-frame. Define the coalgebra $(X, (R_\Box[-], R_\Diamond) : X \to Up(X) \times Down(X))$. As $(X, R_\Box, R_\Diamond)$ is a $K_{\Box\Diamond}$-frame, $R_\Box$ and $R_\Diamond$ satisfy the mix laws (i) and (ii) of Definition 4.20. Then by Lemma 4.4, the function $R_\Box[-]$ is monotone with respect to $\supseteq$ (the ordering of $Up(X)$) and $R_\Diamond$ is monotone with respect to $\subseteq$ (the ordering of $Down(X)$). Thus, for $x \leq y$ in X, $(R_\Box[x], R_\Diamond[x]) \leq (R_\Box[y], R_\Diamond[y])$ in $Up(X) \times Down(X)$.

For the converse direction, let $(X, (\alpha, \beta) : X \to Up(X) \times Down(X))$ be a coalgebra, and define the frame $(X, R_\Box, R_\Diamond)$ where $R_\Box[x] = \alpha(x)$ and $R_\Diamond[x] = \beta(x)$. Then clearly $R_\Box[x]$ is an upset and $R_\Diamond[x]$ a downset of X, and as (α, β) is monotone, the mix laws hold by Lemma 4.4.

$\Box$

To achieve a coalgebraic representation for K_{FS}-frames, we will again need to look at subsets which satisfy conditions (I) and (II) (dually axioms **A** and **B**). These will be entirely analogous to the topological case (Definitions 4.13 and 4.15). As we will be evident from the proofs of Propositions 4.31 and 4.33 below, this setting is somewhat simpler, in that image-finiteness will not be required for the verifications that (I) and (II) correspond dually to **A** and **B**. It will only be necessary to ensure later on that the p-morphisms lift appropriately.

With this said, we start by identifying the subset of $Up(X) \times Down(X)$ that satisfies (I), analogous to $FS(X)$ defined in 4.13:

Definition 4.30. *Let $KFS_1(X) = \{(D, C) \in Up(X) \times Down(X) : C = {\downarrow}(D \cap C)\}$.*

Proposition 4.31. *$KFS_1(X)$ is the subset of $Up(X) \times Down(X)$ for which axiom **A** dually holds, i.e. $KFS_1(X) = \{(D, C) \in Up(X) \times Down(X) \mid \forall A, B \in Up(X) \,.\, (D, C) \in (Up(X) \times \langle A \to B\rangle) \cap ([A] \times Down(X)) \implies (D, C) \in Up(X) \times \langle B\rangle\}$.*

Proof. For the left-to-right direction, we proceed with a direct proof. Let ${\downarrow}(D \cap C) = C$, and suppose that $(D, C) \in (Up(X) \times \langle A \to B\rangle) \cap ([A] \times Down(X))$. Unfolding this one step, we have that $C \in \langle A \to B\rangle$ (so $C \cap (A \to B) \neq \emptyset$), and $D \in [A]$ (so $D \subseteq U$). We want to show that $C \in \langle B\rangle$, i.e. $C \cap B \neq \emptyset$. Since $C \cap (A \to B) \neq \emptyset$, fix some $x \in C \cap (A \to B)$. Unfolding the definition of $x \in (A \to B)$, this means that for any z, if $x \leq z$ and $z \in A$ then $z \in B$. Since $x \in C$, by our assumption it follows that $x \in {\downarrow}(D \cap C)$. Then there is some $y \geq x$ in $D \cap C$. Since $D \subseteq A$, it follows that $y \in A$, so by $x \in A \to B$ we have that $y \in B$. So $y \in C$ and $y \in B$, and thus $y \in C \cap B \neq \emptyset$. Thus, $C \in \langle B\rangle$, as desired.

For the right-to-left-direction, we proceed by contraposition. Suppose that $C \not\subseteq {\downarrow}(D \cap C)$ for some $C \in Up(X) \times Down(X)$. Then there exists $x \in C$ such that $x \leq y$ implies $y \notin D$ or $y \notin C$. We fix this x. In other words, $\{{\uparrow}x\} \cap D \cap C = \emptyset$. We have that $D \in [D]$ (as $D \subseteq D$) and $C \in -\langle -C\rangle$ (as $C \cap -C = \emptyset$), and furthermore $x \in D \to -C$: let $x \leq y$ and $y \in D$. Since we have that $\{{\uparrow}x\} \cap D \cap C = \emptyset$, it must be that $y \in -C$. Thus, $x \in C \cap (D \to -C) \neq \emptyset$. Then $(D, C) \in Up(X) \times \langle D \to -C\rangle \cap [D] \times Down(X)$ but $(D, C) \notin Up(X) \times \langle -C\rangle$, so axiom **A** does not dually hold. $\Box$

Notice that if X is an image-finite poset, then its dual lattice is a profinite Heyting algebra and thus has all relative pseudocomplements. In fact, we have all implications, as everything is discrete.

Nevertheless, in order to avoid making non-free identifications when quotienting over axiom **B**, we must still add a layer of relative pseudocomplements. Thus, we first take the space $P_r(KFS_1(X))$ (Definition 3.25), forming the set of rooted, finite subsets of X ordered by reverse inclusion. Now we may look at a subspace satisfying (ii):

Definition 4.32. *Let* $KFS_2(X) = \{C \in P_r(FS_1(X)) | \forall (D, E) \in C,\ y \in D$ *and* $y \le z$, *there exists* $(D', E') \ge (D, E)$ *in* C *such that* $z \in D' \cap E'\}$

Proposition 4.33. $KFS_2(X)$ *is the subset of* $P_r(KFS_1(X))$ *for which axiom* ***B*** *dually holds, i.e.* $KFS_2(X) = \{C \in P_r(KFS_1(X)) \mid \forall A, B \in Up(X)\ .\ C \in [-(Up(X) \times \langle A\rangle) \cup [B] \times Down(X)] \implies C \in [[A \to B] \times Down(X)]\}$.

Proof. For the left-to-right direction, we proceed with a direct proof. Let $C \in P_r(Up(X) \times Down(X))$ such that $\forall (D, E) \in C$, $y \in D$ and $y \le z$, there exists $(D', E') \ge (D, E)$ in C such that $z \in D' \cap E'$. We assume that $C \in [-(Up(X) \times \langle A\rangle) \cup [B] \times Down(X)]$, and show that $C \in [[A \to B] \times Down(X)]$.

Unfolding this, our assumption is that $C \subseteq -(Down \times \langle A\rangle) \cup [B] \times Down(X)$, and we want to show that $C \subseteq [A \to B] \times Down(X)$, i.e. for any $(D, E) \in C$, $D \subseteq A \to B$. Fix $(D, E) \in C$, and let $y \in D$, $y \le z$, and $z \in A$. We must show that $z \in B$, and then we are done. By our assumption that $(D, E) \in KFS_2$, $y \in D$ and $y \le z$ implies the existence of some $(D', E') \ge (D, E)$ in C such that $z \in D' \cap E'$. Then $z \in E' \cap A \neq \emptyset$, so $E' \in \langle A\rangle$ and thus $(D', E') \notin -(Up(X) \times \langle A\rangle)$. Since $(D', E') \in C$, we have by assumption that $(D', E') \in -(Up(X) \times \langle A\rangle) \cup [B] \times Down(X)$, so it must be that $(D', E') \in [B] \times Down(X)$ and thus $D' \subseteq B$. Then $z \in D' \subseteq B$, as desired. Thus, $y \in A \to B$.

For the converse direction, we proceed by contraposition. Suppose that for $C \in P_r(Up(X) \times Down(X))$, there is some $(D, E) \in C$, $y \in D$ and $y \le z$, such that for any $\forall (D', E') \ge (D, E) \in C.(z \notin D' \cap E')$. Let us fix $z \ge y \in D$. We will find $A, B \in Up(X)$ such that $C \in [-(Up(X) \times \langle A\rangle) \cup [B] \times Down(X)]$ but $C \notin [[A \to B] \times Down(X)]$. That is, $\forall (D', E') \in C.(E' \cap A \neq \emptyset \implies D' \subseteq B)$ and $\exists (D'', E'') \in C.(D'' \not\subseteq A \to B)$. Analogously to the topological case, it suffices to show that there is some $A, B \in Up(X)$ with $z \in A - B$ such that for any $(D', E') \in C$, if $E' \cap A \neq \emptyset$ then $D' \subseteq B$. We claim that this is satisfied by the upsets $A = \uparrow z$ and $B = X \setminus \downarrow z$.

Suppose towards contradiction that there is $(D', E') \in C$ such that $E' \cap \uparrow z \neq \emptyset$ and $D' \cap \downarrow z \neq \emptyset$ (i.e. $D' \not\subseteq (X \setminus \downarrow z)$). Then there is some $w \in D'$ such that $w \in \downarrow z$. Since $z \ge w \in D'$, and D' is an upset, it follows that $z \in D'$. Similarly, there is some $w' \in E'$ such that $w' \in \uparrow z$, so since E' is a downset and $z \le w' \in E'$, we have that $z \in E'$. Now let $D'' = D \cap D'$, so we have $z \in D''$ (as $z \in D$ by assumption). Let $E'' = E \cup E'$, which also contains z. But then we have $(D'', E'') \ge (D, E) \in C$ (as $D'' \subseteq D$ and $E \subseteq E''$) with $z \in D'' \cap E''$, contradicting our original assumption and concluding the proof. □

We now have a 1-1 correspondence between K_{FS}-frames (X, R) and maps $\alpha : X \to KFS_2(X)$ which commute with the root map $r : KFS_2(X) \to KFS_1(X)$. To turn these into coalgebras for an appropriate endofuntor on $\mathbf{ImFinPos}_p$, we look at the composition $P^r_G \circ KFS_2(X)$. As in the previous section, the r superscript specifies that we take the inverse limit of the r-discrete complex (Definition 3.29). This ensures that the layer of implications added before taking $KFS_2(X)$ are preserved. This leads us to the result analogous to that in Theorem 4.17:

Theorem 4.34. *Let X be an image-finite poset. The following are in 1-1 correspondence:*

1. *K_{FS}-frames (X, R),*

2. *r-open monotone maps $f : X \to KFS_2(X)$, and*

3. *p-morphisms $f' : X \to P^r_G(KFS_2(X))$*

Proof.

- For (i) to (ii), let (X, R) be a K_{FS}-frame. Let $R_\Box =\leq \circ R$ and $R_\Diamond = R\circ \geq$, and define the map

$$R_X : X \to FS_1(X)$$
$$x \mapsto (R_\Box[x], R_\Diamond[x]).$$

Then by Lemma 3.27, $R'_X : X \to P_r(KFS_1(X))$ defined by $R'_X(a) = R_X[\uparrow a]$ is the unique r-open monotone map making the triangle in figure 4.4 commute. Note that $R_X[\uparrow a]$ is finite, as X is image-finite and thus $\uparrow a$ is finite. Furthermore, as in the proof of Theorem 4.17, it is straightforward to show that for any $x \in X$, $R_X[\uparrow x] \in KFS_2(X)$. Thus, restricting the codomain of R'_X to $KFS_2(X)$ clearly preserves r-openness.

$$\begin{tikzcd}
X \arrow[rr, "R'_X"] \arrow[dr, "R_X"'] & & KFS_2(X) \arrow[dl, "r"] \\
& KFS_1(X) \arrow[d, "t"] & \\
& (\{*\}, \leq) &
\end{tikzcd}$$

Figure 4.4: Commuting Triangle for R'_X

Where t is the terminal map to the one-element poset. Clearly, R_X is t-open as every map is trivially open relative to the terminal map (Lemma 3.18).

- For (ii) to (i), let $f : X \to KFS_2(X)$ be an r-open monotone map. Let us define the map $f_R : X \to KFS_1(X)$ by

$$f_R := r \circ f$$

Note that by the results in [Alm24], f is the unique lifting of f_R. Now define the K_{FS}-frame (X_f, R_f) where $X_f = X$ and R_f is given by

$$xR_fy \iff y \in \pi_0 f_R(x)) \cap \pi_1 f_R(x))$$

Note that this necessarily means that $f(x) = f_R[\uparrow x] = \{f(z) | z \geq x\}$, as given that f is monotone, $f(x)$ must bet the root of $f[\uparrow x]$, so $f(x) = rf[\uparrow x] = f_R[\uparrow x]$.

We now show that this assignment will satisfy the conditions for K_{FS}-frames given in Definition 4.22. To see that $(R\circ \leq) \subseteq (\leq \circ R)$, let $y \in \uparrow R_f[x]$. Then there is some $z \leq y \in X$ such that $z \in R_f[x]$. We must show that for some $w \geq x$, wR_fy. Denote $(C, D) := rf(x)$. Then (C, D) is the root of $f(x) = \{(C, D), (C_1, D_1), ...\}$. As $f(x) \in KFS_2(X)$, $z \in C \cap D$ and $z \leq y$, there exists some $(C_i, D_i) \geq (C, D) \in f(x)$ such that $z \in C_i \cap D_i$. We furthermore have that $\uparrow(C_i, D_i)$

is a rooted upset of $KFS_1(X)$ that is open relative to any map with domain $KFS_1(X)$, and a subset of $f(x)$. Thus, as KFS_2 is ordered by $\supseteq$, we have that $f(x) \leq \uparrow(C_i, D_i)$ in KFS_2. Furthermore, $z \in r(\uparrow(C_i, D_i))$. As f is r-open, it follows that there is $w \geq x \in X$ such that $r(f(w)) = r(C_i, D_i)$. Thus, we have found $w \geq x$ such that $wR_f y$

To see that $(\geq \circ R) \subseteq (R\circ \geq)$, let $y \in R_f[\downarrow x]$. Then there is some $z \leq x$ such that $zR_f y$. We must find some $w \geq y$ such that $xR_f w$. Denote $(D, C) := f_R(z)$ and $(D', C') := f_R(x)$. By monotonicity of f_R, we have that $(D, C) \leq (D', C')$, and thus $D \supseteq D'$ and $C \subseteq C'$. Then $y \in C \subseteq C' =\downarrow (D' \cap C')$ as this is in $KFS_1(X)$. Thus, there must be some $w \geq y \in D' \cap C'$, and we are done.

Furthermore, these assignments are clearly reversing.

- For (iii) to (ii), let $f' : X \to P^r_G(KFS_2(X))$ be a p-morphism. Then the map $f : X \to FS_2(X)$ defined by

$$f := \pi_0 \circ f'$$

which projects everything to the first coordinate is clearly monotone. Furthermore, we claim that f is r-open. Let $a \in X$, $b \in KFS_2(X)$, and let $(\pi_0 \circ f')(a) \leq b$. We must find $a' \in X$ such that $a \leq a'$ and $r((\pi_0 \circ f')(a')) = r(b)$. We have that $b \subseteq (\pi_0 \circ f')(a)$ (as P^r_G is ordered by reverse inclusion). Furthermore, by the definition of $P^r_G(KFS_2(X))$, we have $(b_0, b_1, b_2, ...) \in P^r_G(FS_2(X))$ where $b_0 = b$, $b_i = \uparrow b_{i+1}$ and $b_i \in P_i$. To see this, recall that for any x and any g, $\uparrow x$ is g-ope (and is clearly a rooted upset). Furthermore, we have that $f'(a) \leq (b_0, b_1, b_2, ...)$. As f' is a p-morphism and $f'(a) \leq (b_0, b_1, b_2, ...)$, there is $a' \in X$ such that $f'(a') = (b_0, b_1, b_2, ...)$. Clearly, $\pi_0(b_0, b_1, b_2, ...) = b$, so $r((\pi_0 \circ f')(a')) = r(b)$, and we have thus shown that $f = (\pi_0 \circ f')$ is r-open.

- For (ii) to (iii), let $f : X \to KFS_2(X)$ be an r-open monotone map. Analogousy to the topological case [AB24], there is a unique p-morphism $f' : X \to P^r_G(KFS_2(X))$ extending it:

$$\begin{tikzcd}
X \arrow[rr, "{f[\uparrow -]}"] \arrow[dr, "f"'] & & P_r(KFS_2(X)) \arrow[dl, "r_1"] \\
& KFS_2(X) \arrow[d, "r"] & \\
& KFS_1(X) &
\end{tikzcd}$$

As f is r-open by assumption, we have by Lemma 57 in [Alm24] the existence of the r-open monotone map $f[\uparrow -]$. Note that for any $a \in X$, $f[\uparrow a]$ is a finite subset as X is image-finite. We then proceed with the same construction as in Proposition 3.30 to the map $\overline{f} : X \to P^r_G(KFS_2(X))$.

As in the topological case, it is clear by definition that the assignments between (ii) and (iii) are each others' inverses.

$\square$

Theorem 4.35. *The category* $\mathbf{K_{FS}}$ *is equivalent to the category* $\boldsymbol{CoAlg}(\mathcal{V}_G^r(FS_2(-)))$.

It again remains to show that there are inverse assignments on morphisms.

Proof. This proof is precisely analogous to that of Theorem 4.18, so we will only provide a sketch, tracing analogies. Let (X, R) and (Y, S) be K_{FS}-frames, and $f : X \to Y$ be a K_{FS} p-morphism. Then we have the corresponding $(Up(-) \times Down(-))$-coalgebras $(X, R : X \to Up(X) \times Down(X))$ and $(Y, S : Y \to Up(Y) \times Down(Y))$, and the map $Up(f) \times Down(f) = ({\uparrow}f[-], {\downarrow}f[-])$. By Lemma 3.27,we lift the coalgebras to $P_r(FS_1(-))$ coalgebras, and by 3.28, $\mathcal{P}_r({\uparrow}f[-], {\downarrow}f[-]) = ({\uparrow}f[-], {\downarrow}f[-])[-]$, where $({\uparrow}f[-], {\downarrow}f[-]) \circ r_X = r_Y \circ ({\uparrow}f[-], {\downarrow}f[-])[-]$. Again, using that the projection $\lambda_0^X : P_G^r(KFS_2(X)) \to KFS_2(X)$ and the map $R[\uparrow -]$ are r_X-open, and $\lambda_0^Y : P_G^r(KFS_2(Y)) \to KFS_2(Y)$, and $S[\uparrow -]$ are r_Y-open, by Proposition 3.30, the coalgebras are lifted to $P_G^r(FS_2(-))$ coalgebras $(X, \overline{R[\uparrow -]} : X \to P_G^r(KFS_2(X)$ and $(Y, \overline{S[\uparrow -]} : Y \to P_G^r(KFS_2(Y)$. Then f is lifted to the map $P_G^r(P_r((Up(-) \times Down(-))(f))) = \overline{({\uparrow}f[-], {\downarrow}f[-])}$ by Lemma 3.31.

The liftings are all defined such that they commute, so showing that the diagram for $P_G^r(KFS_2(-))$ commutes reduces to showing that the diagram for $KFS_1(-)$ commutes, as in the topological setting. Furthermore, the $KFS_1(-)$ diagram commutes if and only if f is a K_{FS} p-morphism. The proof of this is the same as for the topological case, as K_{FS} p-morphisms must satisfy the same conditions as **IKS** p-morphisms (except for continuity). $\square$

$$\begin{tikzcd}
 & & X \arrow[rr, "f"] \arrow[d, "h_R"] \arrow[ddl, "h_R[\uparrow -]"'] \arrow[dddll, "\overline{h_R[\uparrow -]}"'] & & Y \arrow[d, "h_S"'] \arrow[ddr, "h_S[\uparrow -]"] \arrow[dddrr, "\overline{h_S[\uparrow -]}"] & & \\
 & & KFS_1(X) \arrow[rr, "({\uparrow}f[-],{\downarrow}f[-])"'] & & KFS_1(Y) & & \\
 & KFS_2(X) \arrow[ur, "r_X"'] \arrow[rrrr, "({\uparrow}f[-],{\downarrow}f[-])[-]"'] & & & & KFS_2(Y) \arrow[ul, "r_Y"] & \\
P_G^r(KFS_2(X)) \arrow[ur, "\pi_0^X"'] \arrow[rrrrrr, "\overline{{\uparrow}f[-],{\downarrow}f[-])[-]}"'] & & & & & & P_G^r(KFS_2(Y)) \arrow[ul, "\pi_0^Y"]
\end{tikzcd}$$

Figure 4.5: Commuting diagram for $P_G^r(KFS_2(-))$-coalgebras

We now have a coalgebraic representation for both **IK**-spaces and image-finite **IK** Kripke frames. We remark that, if the frame we start with is finite, these constructions are virtually the same, as the topology for a finite **IK**-space (X, R) will be discrete. Thus, in the finite case, the constructions differ only when taking the projective limit, as P_G takes only the image-finite part. We will now provide an example, to illustrate the first few steps of our construction when applied to a finite frame.

Example 4.36. *Let* (X, R) *be a* ***IK****-space where* X *is the finite Esakia space depicted in Figure 4.6.*

$$\begin{tikzcd}
y & & z \\
 & x \arrow[ul, "\leq"] \arrow[ur, "\leq"'] &
\end{tikzcd}$$

Figure 4.6: Example finite **IK**-space

The topology on X is given by the subbasis

$$\mathcal{B} = \{\emptyset, \{y\}, \{z\}, \{y, z\}, \{x, y, z\}\},$$

which is precisely the upsets. Thus, considering (X, R) as a K_{FS}-frame will yield the same results in the following steps. We now take the hyperspaces $(\mathcal{V}^{\uparrow}(X) \supseteq)$ and $(\mathcal{V}^{\downarrow}(X), \subseteq)$, as in figure 4.7.

$$\begin{tikzcd}
 & \emptyset & \\
\{y\} \arrow[ur, "\supseteq"] & & \{z\} \arrow[ul, "\supseteq"'] \\
 & \{y, z\} \arrow[ul, "\supseteq"] \arrow[ur, "\supseteq"'] & \\
 & \{x, y, z\} \arrow[u, "\supseteq"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \{x, y, z\} & \\
\{x, y\} \arrow[ur, "\subseteq"] & & \{x, z\} \arrow[ul, "\subseteq"'] \\
 & \{x\} \arrow[ul, "\subseteq"] \arrow[ur, "\subseteq"'] & \\
 & \emptyset \arrow[u, "\subseteq"] &
\end{tikzcd}$$

(1) Closed upsets, $\supseteq$ (2) Closed downsets, $\subseteq$

Figure 4.7: Taking the hyperspaces

Which is again precisely the same as the construction using the functors Up and Down. Already at this stage, taking the product results in 25 points – so the construction will quickly get out of hand. Quotienting by taking $FS_1(X)$ makes the results slightly more manageable, so we include this here. We calculate the pairs (D, C) such that $C = {\downarrow}(D \cap C)$ in Table 4.2, yielding the poset depicted in Figure 4.8.

Downsets					
$X =$	${\downarrow}(X \cap X)$	${\downarrow}(\{y, z\} \cap X)$			
$\{x, y\} =$	${\downarrow}(X \cap \{x, y\})$	${\downarrow}(\{y, z\} \cap \{x, y\})$	${\downarrow}(\{y\} \cap \{x, y\})$		
$\{x, z\} =$	${\downarrow}(X \cap \{x, z\})$	${\downarrow}(\{y, z\} \cap \{x, z\})$	${\downarrow}(\{z\} \cap \{x, z\})$		
$\{x\} =$	${\downarrow}(X \cap \{x\})$				
$\emptyset =$	${\downarrow}(X \cap \emptyset)$	${\downarrow}(\{y, z\} \cap \emptyset)$	${\downarrow}(\{z\} \cap \emptyset)$	${\downarrow}(\{y\} \cap \emptyset)$	$(\emptyset, \emptyset)$

Table 4.2: Pairs in $FS_1(X)$

At this stage, taking the rooted subsets will result in far too many points to calculate manually. However, the first few steps here serve to aid visualization, and to illustrate that the constructions coincide in the finite case.

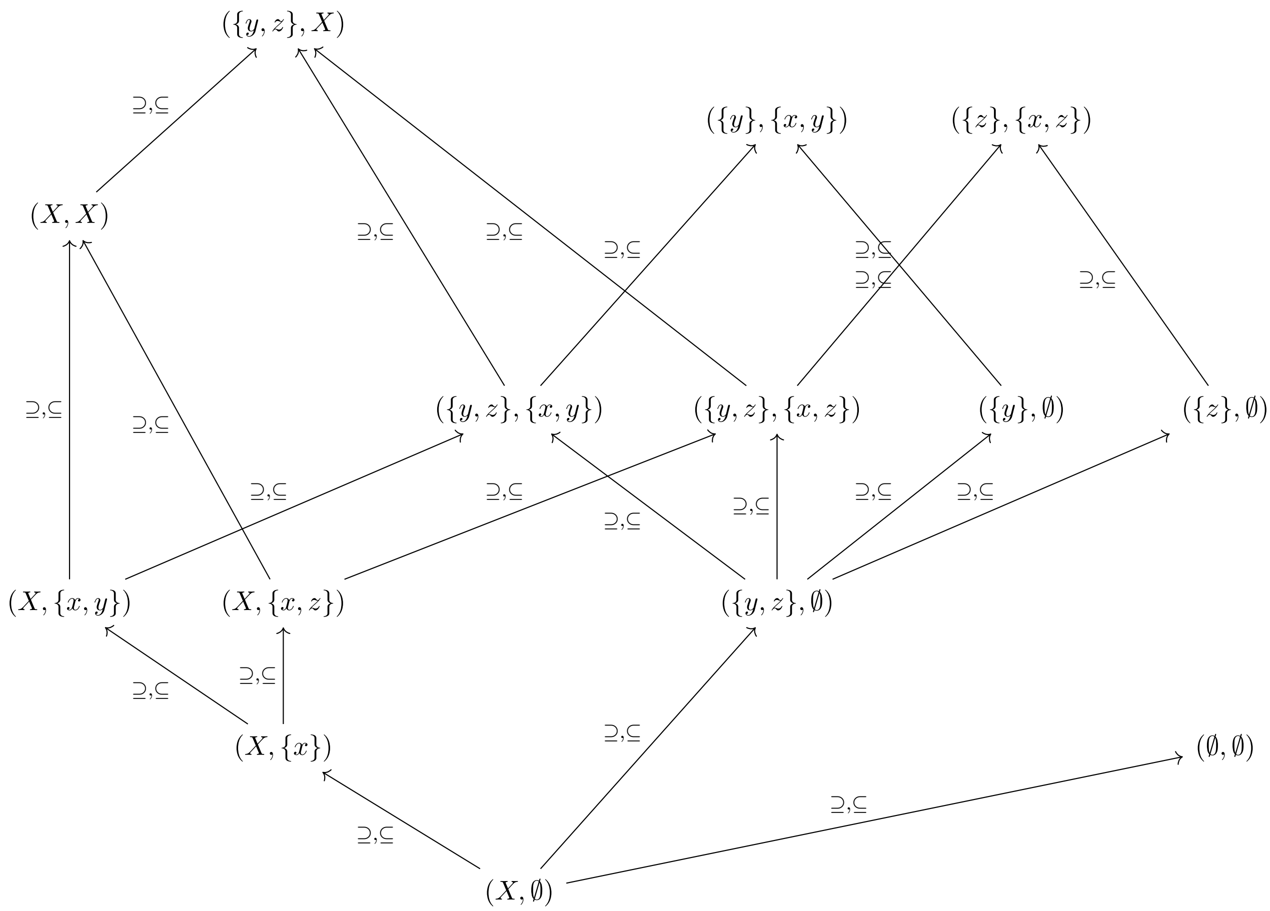

Figure 4.8: Poset for $FS_1(X)$

Chapter 5

Applications: Bisimulations, Free IK-Algebras, and Rank-1 Axioms

In this chapter, we show some of the consequences that can be derived from coalgebraic completeness for Fischer Servi logic. In particular, we derive a notion of bisimulation for **IK**-spaces, give a construction for the free **FS**-algebra on X generators, and provide an example of how our method can be used for extensions of Fischer Servi logic with rank-1 axioms. This serves to highlight the contribution of our work: our coalgebraic formulation leads naturally to a notion of bisimulation for Fischer Servi frames, and to characterizing the dual spaces of free **IK**-algebras.

5.1 Bisimulation

To our knowledge, there has not yet been a characterisation of bisimulation between Fischer Servi frames. We present a notion here, which springs naturally from analyzing our coalgebra morphisms, and show that this formulation indeed provides a truth-invariant relation between **FS**- and $\mathbf{K_{FS}}$ models. Thus, the contribution of this section is two-fold: we provide a characterisation of bisimulation, which in turn illustrates the correctness of our coalgebraic representation.

Recall from Definition 3.6 that a bisimulation between F-coalgebras (X, α) and (Y, γ) is a relation $B \subseteq X \times Y$ such that the diagram in Figure 5.1 commutes:

$$\begin{tikzcd}
X \arrow[d, "\alpha"] & B \arrow[l, "\pi"'] \arrow[r, "\pi'"] \arrow[d, dotted, "\beta"] & Y \arrow[d, "\gamma"] \\
FX & FB \arrow[l, "F\pi"] \arrow[r, "F\pi'"'] & FY
\end{tikzcd}$$

Figure 5.1: Bisimulation between coalgebras

We refer the reader to [Ven07, Chapter 11] for a detailed overview of coalgebra bisimulations.

5.1.1 Bisimulation for IK-spaces

Definition 5.1. *(Fischer Servi bisimulation) Let (X, R) and (Y, S) be two **IK**-spaces. We say that a relation $\sim \subseteq X \times Y$ is a* **IKS**-bisimulation *if $\sim$ is closed in the subspace topology of $X \times Y$* [1] *, and the*

[1] See [BFV07] for why bisimulations for modal spaces must be closed.

below conditions are met. Throughout, let $x, x' \in X$ and $y, y' \in Y$.

Forth:

(i) *Whenever $x \leq x'$ and $x \sim y$, there is some $y' \geq y$ such that $x' \sim y'$;*

(ii) *Whenever xRx' and $x \sim y$, there is some $y' \in S[y]$ such that $x' \sim y'$.*

Back:

(i) *Whenever $y \leq y'$ and $x \sim y$, there is some $x' \geq x$ such that $x' \sim y'$;*

(ii) *Whenever ySy' and $x \sim y$, there is some $x' \in R[x]$ such that $x' \sim y'$.*

Throughout, we will follow Lemma 4.9 and define $R_\Box = \leq \circ R$ and $R_\Diamond = R \circ \geq$ for any **IK**-space (X, R), and use the according semantics for $\Box$ and $\Diamond$.

This definition matches the usual definition of bisimulations as in Definition 3.7, but to our knowledge there is no existing characterization of **IKS**-bisimulations in the literature. Furthermore, the semantics for $\Box$ and $\Diamond$ differ from the classical case. Thus, we will preface this section by proving that our definition corresponds to truth-invariance for **IK**-models.

Proposition 5.2. *Let $M = (X, R, V)$ and $M' = (Y, S, V')$ be two **IK**-models, and B a bisimulation (between their underlying frames) such that if $x \sim y$ for $x \in X$ and $y \in Y$, $M, x \models \phi$ iff $M, x \models \phi$ for any formula ϕ.*

Proof. Let $M = (X, R, V)$ and $M' = (Y, S, V')$ be two **IK**-models, and B a bisimulation between them. Recall that valuations for intuitionistic logic must be persistent. We prove by induction that this means invariance of truth for any formula ϕ. We focus on the modal connectives $\Box$ and $\Diamond$, as the other cases are standard.

Let $\phi = \Diamond\psi$, and suppose $M, x \models \Diamond\psi$ and $x \sim y$. Then there is x' such that $xR_\Diamond x'$ and $x' \models \psi$. As $R_\Diamond[x] = \downarrow R[x]$, there is $v \geq x'$ such that $v \in R[x]$. By the persistence of valuations, $M, v \models \psi$. By the forth condition for R, there is $y' \in R[y]$ such that $v \sim y'$. As $R[y] = R_\Diamond[y] \cap R_\Box[y]$, $y' \in R_\Diamond[y]$. By the induction hypothesis, $M', y' \models \psi$ and thus $M', y \models \Diamond\psi$. The other direction is similar.

Now let $\phi = \Box\psi$, and suppose $M, x \models \Box\psi$ and $x \sim y$. As $R_\Box[x] = R[\uparrow x]$, we have that $\forall v, x'. x \leq vR_\Box x' \implies M, x' \models \psi$. Let $y \leq wRy'$, and show that $M', y' \models \psi$. By the back condition for $\leq$, as $y \leq w$, there is $u \geq x$ such that $u \sim w$. Then by the back condition for R, as wRy', there is $x' \in R[v]$ such that $x' \sim y'$, so by assumption $M, x' \models \psi$. Then by the induction hypothesis, $M', y' \models \psi$. As w and y' were arbitrary, $M', y \models \Box\psi$. The other direction is similar. $\Box$

Theorem 5.3. *Let $(X, R_\Box, R_\Diamond)$ and $(Y, S_\Box, S_\Diamond)$ be two **IK**-spaces. Then the following are in one-to-one correspondence:*

(i) ***IKS**-bisimulations between X and Y*

(ii) *Bisimulations for the endofunctor $V_G(FS_2(-))$.*

Proof. Let $B \subseteq X \times Y$ be a **IKS**-bisimulation. We proceed step-by-step through the construction to show that this corresponds to a commuting diagram for the functor $V_G^r(FS_2(-))$ (i.e. a coalgebra bisimulation).

Define a map $(\beta, \gamma) : B \to FS_1(B)$ sending a pair (x, y) to the pair $((R_\Box[x] \times S_\Box[y]) \cap B, (R_\Diamond[x] \times S_\Diamond[y]) \cap B)$. Monotonicity and continuity of (β, γ) follow from the fact that the maps $(R_\Box[-], S_\Box[-]) : (X \times Y) \to \mathcal{V}^\uparrow(X) \times \mathcal{V}^\uparrow(Y)$ and $(R_\Diamond[-], S_\Diamond[-]) : (X \times Y) \to \mathcal{V}^\downarrow(X) \times \mathcal{V}^\downarrow(Y)$ are continuous and monotone, and the restriction to the subspace topology. We first show that indeed $(\beta, \gamma)(x, y) \in FS_1(B)$. The fact that $R_\Box[x] \times S_\Box[y] \cap B$ and $R_\Diamond[x] \times S_\Diamond[y] \cap B$ are closed in B is immediate from the subspace topology inherited from $X \times Y$. That $R_\Box[x] \times S_\Box[y] \cap B$ is an upset in B is immediate from the fact that $R_\Box[x] \times S_\Box[y]$ is an upset in $X \times Y \supseteq B$, and likewise that $R_\Diamond[x] \times S_\Diamond[y] \cap B$ is a downset. Thus, $(\beta, \gamma)(x, y) \in \mathcal{V}^{\uparrow\downarrow}(B)$.

To see that the FS_1 condition is satisfied, it suffices to show that $(R_\Diamond[x] \times S_\Diamond[y]) \cap B \subseteq {\downarrow}(((R_\Box[x] \times S_\Box[y]) \cap B) \cap ((R_\Diamond[x] \times S_\Diamond[y]) \cap B))$, since the converse inclusion always holds as we showed that $(R_\Diamond[x] \times S_\Diamond[y]) \cap B$ is a downset. As (X, R) and (Y, S) are **IK**-spaces, we have that $R_\Diamond[x] \subseteq {\downarrow}(R_\Box[x] \cap R_\Diamond[x])$, and likewise $S_\Diamond[y] \subseteq {\downarrow}(S_\Box[y] \cap S_\Diamond[y])$, hence also $R_\Diamond[x] \times S_\Diamond[y] \subseteq {\downarrow_X}(R_\Box[x] \cap R_\Diamond[x]) \times {\downarrow_Y}(S_\Box[y] \cap S_\Diamond[y])$. This in turn can be rewritten as ${\downarrow_{X \times Y}}((R_\Box[x] \cap R_\Diamond[x]) \times (S_\Box[y] \cap S_\Diamond[y])) = {\downarrow_{X \times Y}}((R_\Box[x] \times S_\Box[y]) \cap (R_\Diamond[y] \times S_\Diamond[y]))$. Then certainly $(R_\Diamond[x] \times S_\Diamond[y]) \cap B \subseteq {\downarrow_B}((R_\Box[x] \times S_\Box[y]) \cap B \cap (R_\Diamond[y] \times S_\Diamond[y]))$. Thus, $(\beta, \gamma)(x, y) \in FS_1(B)$.

We now show that $(\mathcal{V}^{\uparrow\downarrow})(\pi_Y) \circ (\beta, \gamma) = (S_\Box, S_\Diamond) \circ \pi_Y$, i.e. the right square of the diagram in Figure 5.2 commutes. The proof that the left square commutes is similar, and we omit it.

We first claim that $S_\Box[y] \cap S_\Diamond[y] = {\uparrow}\pi_Y(\beta(x, y)) \cap {\downarrow}\pi_Y(\gamma(x, y))$. Let $(x, y) \in B$, and $y' \in {\uparrow}\pi_Y(\beta(x, y)) \cap {\downarrow}\pi_Y(\gamma(x, y))$, i.e. $y \in {\uparrow}\pi_Y[(R_\Box[x] \times S_\Box[y]) \cap B]$ and $y' \in {\downarrow}\pi_Y[(S_\Diamond[x] \times S_\Diamond[y]) \cap B]$. Then $y' \in {\uparrow}S_\Box[y]$ and $y' \in {\downarrow}S_\Diamond[y]$, and as (Y, S) is a **IK**-space, $y' \in S[y]$.

For the converse direction, let $y' \in S[y]$. Then $y' \in S_\Box[y] \cap S_\Diamond[y]$ as (Y, S) is a **IK**-space. As B is a **IKS**-bisimulation and $(x, y) \in B$, there is $(x', y') \in B$ such that $x' \in R[x]$. Then also $x' \in R_\Box[x] \cap R_\Diamond[x]$, so $(x', y') \in (R_\Box[x] \times S_\Box[y]) \cap B$ and $(x', y') \in (R_\Diamond[x] \times S_\Diamond[y]) \cap B$. Then $y' \in \pi_Y(R_\Box[x] \times S_\Box[y]) = S_\Box[y] = {\uparrow}S_\Box[y]$ and $y' \in \pi_Y(R_\Diamond[x] \times S_\Diamond[y]) = S_\Diamond[y] = {\downarrow}S_\Diamond[y]$, so $y' \in {\uparrow}\pi_Y(\beta(x, y)) \cap {\downarrow}\pi_Y(\gamma(x, y))$, as desired. Thus, we have shown that $S[y] = {\uparrow}\pi_Y(R_\Box[x] \times S_\Box[y] \cap B) \cap {\downarrow}\pi_Y(R_\Diamond[x] \times S_\Diamond[y] \cap B)$.

It now remains to show that $S_\Diamond[y] = {\downarrow}\pi_Y[\gamma(x, y)]$ and $S_\Box[y] = {\uparrow}\pi_Y[\beta(x, y)]$. To see that $S_\Diamond[y] = {\downarrow}\pi_Y[\gamma(x, y)]$, recall that $S_\Diamond[y] = {\downarrow}S[y] = {\downarrow}({\uparrow}\pi_Y(\beta(x, y)) \cap {\downarrow}\pi_Y(\gamma(x, y)))$. Then as ${\downarrow}\pi_Y[\gamma(x, y)]$ is a downset in Y, clearly ${\downarrow}({\uparrow}\pi_Y(\beta(x, y)) \cap {\downarrow}\pi_Y(\gamma(x, y))) \subseteq {\downarrow}\pi_Y[\gamma(x, y)]$. For the other inclusion, we have that $\pi_Y[\gamma(x, y)] \subseteq S_\Diamond[y]$ so clearly ${\downarrow}\pi_Y[\gamma(x, y)] \subseteq {\downarrow}S_\Diamond[y] = S_\Diamond[y]$.

Now we show that $S_\Box[y] = {\uparrow}\pi_Y[\beta(x, y)]$. As π_Y is a p-morphism, ${\uparrow}\pi_Y[\beta(x, y)] = \pi_Y[\beta(x, y)] = \pi_Y(R_\Box[x] \times S_\Box[y] \cap B)$. Now let $y' \in S_\Box[y]$. As (Y, S) is a **IK**-space, there is $z \geq y$ such that $y \in S_\Box[z] \cap S_\Diamond[z]$. By the back condition for $\leq$, there is $w \geq x$ such that $(w, z) \in B$. Then, by the back condition for R, there is $x' \in R[w]$ such that $(x', y') \in B$. As $x \leq wRx'$, it follows that $xR_\Box x'$. Thus, we have found $x' \in R_\Box[x]$ such that $(x', y') \in B$. Then $(x', y') \in \beta(x, y)$ so $y' \in \pi_Y[\beta(x, y)]$. The other inclusion always holds, as $\beta(x, y) \subseteq R_\Box[x] \times S_\Box[y]$ so $\pi_Y[\beta(x, y)] \subseteq S_\Box[y]$.

$$\begin{tikzcd}
X \arrow[d, "{(R_\Box[-], R_\Diamond[-])}"'] & B \arrow[l, "\pi_X"'] \arrow[r, "\pi_Y"] \arrow[d, "{(\beta,\gamma)}"] & Y \arrow[d, "{(S_\Box[-], S_\Diamond[-])}"] \\
FS_1(X) & FS_1(B) \arrow[l, "(\mathcal{V}^{\uparrow\downarrow})(\pi_X)"] \arrow[r, "(\mathcal{V}^{\uparrow\downarrow})(\pi_Y)"'] & FS_1(Y)
\end{tikzcd}$$

Figure 5.2: Commuting diagram for $FS_1(-)$

Note that $FS_1(B)$ is a Priestley space as it is a closed subspace of $\mathcal{V}^{\uparrow\downarrow}(B)$. Then given our map

$(\beta,\gamma) : B \to FS_1(B)$, by Lemma 3.17, there is a unique r-open, continuous, order-preserving map $(\beta,\gamma)' : B \to \mathcal{V}_r(FS_1(B))$, defined by $(\beta,\gamma)'(a) = (\beta,\gamma)[\uparrow a]$, such that $r \circ (\beta,\gamma)' = (\beta,\gamma)$, i.e. the diagram in Figure 5.3 commutes.

$$\begin{tikzcd}
B \arrow[rr, "(\beta,\gamma)'"] \arrow[dr, "(\beta,\gamma)"'] & & V_r(FS_1(B)) \arrow[dl, "r"] \\
& FS_1(B) &
\end{tikzcd}$$

Figure 5.3: Lifting for FS_2

We show that this factors through $FS_2(B)$. Let $(x,y) \in B$. We must show that $(\beta,\gamma)[\uparrow(x,y)] \in FS_2(B)$. That it is a rooted upset of $FS_1(B)$ holds by definition, thus we must show that it meets the FS_2 condition from Definition 4.15. Recall that $(\beta,\gamma)[\uparrow(x,y)] = \{(R_\Box[x'] \times S_\Box[y']) \cap B, (R_\Diamond[x'] \times S_\Diamond[y']) \cap B | (x',y') \geq_B (x,y)\}$ $x' \geq x$ & $y' \geq y\}$. Let $(x',y') \geq (x,y)$ in B, $(z,w) \in \beta(x',y') = (R_\Box[x'] \times S_\Box[y']) \cap B$, and suppose $(z,w) \leq (z',w')$ in B. Then as (X,R) and (Y,S) are **IK**-spaces, there must be $(x'',y'') \geq (x',y')$ such that $(z',w') \in (R_\Box[x''] \times S_\Box[y'']) \cap (R_\Diamond[x''] \times S_\Diamond[y'']) \cap B$, so the FS_2 condition holds and thus $(\beta,\gamma)[\uparrow(x,y)] \in FS_2(B)$.

By uniqueness of $(\beta,\gamma)'$, the diagram in Figure 5.4 commutes.

$$\begin{tikzcd}
& B \arrow[r, "\pi_Y"] \arrow[d, "(\beta,\gamma)"] \arrow[dl, "(\beta,\gamma)'"'] & Y \arrow[d, "S"] \arrow[dr, "S'"] & \\
& FS_1(B) \arrow[r, "(\mathcal{V}^{\uparrow\downarrow})(\pi_Y)"'] & FS_1(Y) & \\
FS_2(B) \arrow[ur, "r"'] \arrow[rrr, "(\mathcal{V}^{\uparrow\downarrow})(\pi_Y)[-]"'] & & & FS_2(Y) \arrow[ul, "r_Y"]
\end{tikzcd}$$

Figure 5.4: Step for FS_2

Note that the outer triangles in Figure 5.4 commute by Lemma 3.17, and the lower square commutes by Lemma 3.20. Finally, using Proposition 3.22,we lift $(\beta,\gamma)'$ to a map $\overline{(\beta,\gamma)'} : B \to \mathcal{V}_G^r(FS_2(X \times Y))$, which makes the diagram in Figure 5.5 commute, again by uniqueness.

$$\begin{tikzcd}
& & B \arrow[r, "\pi_Y"'] \arrow[d, "\beta"] \arrow[dl, "(\beta,\gamma)'"] \arrow[ddll, "\overline{(\beta,\gamma)'}"'] & Y \arrow[d, "S"] \arrow[dr, "S'"'] \arrow[ddrr, "\overline{S'}"] & & \\
& & FS_1(B) \arrow[r, "(\mathcal{V}^{\uparrow\downarrow})(\pi_Y)"'] & FS_1(Y) & & \\
& FS_2(B) \arrow[ur, "r_B"'] \arrow[rrr, "(\mathcal{V}^{\uparrow\downarrow})(\pi_Y)[-]"'] & & & FS_2(Y) \arrow[ul, "r_Y"] & \\
V_G^r(FS_2(B)) \arrow[ur, "\pi_0^B"'] \arrow[rrrrr, "\overline{(\mathcal{V}^{\uparrow\downarrow})(\pi_Y)[-]}"'] & & & & & V_G^r(FS_2(Y)) \arrow[ul, "\pi_0^Y"]
\end{tikzcd}$$

Figure 5.5: Step for V_G

Conversely, let $(B, h : B \to FS_1(B))$ yield a commuting diagram for $(X, \alpha), (Y, \gamma)$. Then we claim that B is a **IKS**-bisimulation. Given $(X, \alpha : X \to V_G^r(FS_2(X)))$ and $(Y, \gamma : Y \to V_G^r(FS_2(Y)))$, we recover the relation for the corresponding **IK**-spaces $(X, R_\Box, R_\Diamond)$ and $(Y, S_\Box, S_\Diamond)$ by setting $(R_\Box, R_\Diamond)(x) = (r_X \circ \lambda_0 \circ \alpha)(x)$, and $R = R_\Box \cap R_\Diamond$. Likewise for S, and let $h = r_X \circ \lambda_0 \circ \beta$. That is, by projecting to the first coordinate, then taking the root as in Theorem 4.17. By doing the same for β, we get a commuting diagram for FS_1. We show that the forth conditions hold, as the back conditions are similar.

- Let $x, x' \in X$, $x \leq x'$, and $(x, y) \in B$ for $y \in Y$. We show there is $y' \in Y$ such that $y \leq y'$ and $(x', y') \in B$. We have that π_X and π_Y are required to be p-morphisms with respect to $\leq$. Thus, by $(x, y) \in B$ and $x = \pi_X(x, y)$ and $x \leq x'$, there is $(w, y') \in B$ such that $x' = \pi_X(w, y')$ (so $w = x'$) and $(x, y) \leq_B (x', y')$, so also $y \leq y'$ in Y, as desired.
- Let xRx' and $x \sim y$. We show that there is $y' \in R[y]$ such that $(x', y') \in B$. We have $x' \in R_\Box[x] \cap R_\Diamond[x]$, and as the diagram commutes, $x' \in \pi_X[\beta(x, y)] \cap {\downarrow}\pi_X[\gamma(x, y)]$.

  As $(\gamma, \beta)(x, y) \in FS_1(B)$, we have that $\gamma(x, y) = {\downarrow}(\beta(x, y) \cap \gamma(x, y))$. Thus, $\pi_X[\beta(x, y)] \cap {\downarrow}\pi_X[\gamma(x, y)] = \pi_X[\beta(x, y)] \cap {\downarrow}_X\pi_X({\downarrow}_B(\beta(x, y) \cap \gamma(x, y)))$.

  As $\beta(x, y), \gamma(x, y) \subseteq X \times Y$, we have that $\pi_X[\beta(x, y)] \cap {\downarrow}\pi_X({\downarrow}(\beta(x, y) \cap \gamma(x, y))) = \pi_X[\beta(x, y)] \cap {\downarrow}\pi_X((\beta(x, y) \cap \gamma(x, y)))$, which in turn is equal to $\pi_X[\beta(x, y) \cap \gamma(x, y)]$, and thus $x' \in \pi_X[\beta(x, y) \cap \gamma(x, y)]$. Then there is $(x', y') \in \beta(x, y) \cap \gamma(x, y)$ (so $(x', y') \in B$) and therefore $y' \in S_\Box[y] \cap S_\Diamond[y] = S[y]$ as the diagram commutes.

To see that B is a closed subspace of $X \times Y$, recall that B must be a Priestley space and thus compact. Then as B is a compact subspace of the Priestley space $X \times Y$, it also must be a closed subspace. This concludes the proof of the correspondence between **FS** bisimulations and bisimulations for $\mathcal{V}_G(FS_2(-))$-coalgebras.

$\Box$

Bisimulation for $\mathbf{K_{FS}}$-frames is defined in the same way as for **IK**-spaces (Definition 5.1), dropping the topological requirements. Note that if (X, R) and (Y, S) are $\mathbf{K_{FS}}$-frames, then $B \subseteq X \times Y$ is a poset with the pointwise order, and clearly if X and Y are image finite, then B is image finite. The proof of the correspondence between bisimulations for P_G^r-coalgebras and bisimulations for $\mathbf{K_{FS}}$-frames is essentially the same as in the above section, but dropping all topological requirements. For a similar proof regarding bisimulation with the P_G functor, see [AB24, Theorem 5.2].

5.2 Free IK-algebras

In this section, we show how our construction can be used to generate free **IK** algebras. The construction we use is due to [BGJ14], and is analogous to that in [AB24] to construct the free $\Box$-intuitionistic algebra.

Definition 5.4. *Let X be an Esakia space. Define the following sequence:*

$$(M_0(X), M_1(X), ..., M_n(X), ...)$$

and a sequence of morphisms $\pi_k : M_k(X) \to M_{k-1}(X)$ for $k > 0$ and $\pi_0 : M_0(X) \to M_0(X)$ defined as follows:

(i) $M_0(X) = X$;

(ii) $M_{n+1}(X) := X \times V_G^r(FS_2(M_n(X)))$

(iii) $\pi_0 = id_{M_0}$ and $\pi_1(x, C) = x$;

(iv) $\pi_{n+1}(x, C) = (x, (V_G^r(\mathcal{V}_r(\mathcal{V}^{\uparrow\downarrow}(\pi_n)))(C)))$.

We denote the inverse limit (in **Pris**) of this system by $M_\infty(X)$.

Note that each π_k is a p-morphism, given that π_0 and π_1 are clearly p-morphisms, and the action of $\mathcal{V}_G$ on a Priestley map (from an Esakia space to a Priestley space) yields a p-morphism given Proposition 3.22.

Let us fix some notation. Given $C \in \mathcal{V}_G^r(FS_2(M_k(X)))$ is a sequence $(C_0, C_1, ...)$, define $\lambda_0[C] = C_0$. For the sake of brevity, we will write C_0 in place of $\lambda_0[C]$ whenever this does not lead to ambiguity.

In this section, we will be using a lot of projections from different products. Thus, let us fix here what each one means, to avoid confusion later on. We will denote the projections from an ordered pair (A, B) as p_0 and p_1, mapping it to A and B respectively. This will be used primarily for elements belonging to $FS_1(M_k(X))$ (for some k). The projections from $M_\infty(X)$ to each of its coordinates will be denoted in boldface, by $\mathbf{p}$. We will also use the shorthand $x(n)$ in place of $\mathbf{p}_n(x)$, for $x \in M_\infty(X)$. These projections are not to be confused with the inverse system morphisms π_k, although they do relate to each other, as noted in the preliminaries, by the equality $\mathbf{p}_n = \pi_{n+1} \circ \mathbf{p}_{n+1}$. Finally, as above, λ will be used to denote projections from projective limits of the form $\mathcal{V}_G^r(FS_2(M_k(X)))$. Any further projections will be defined explicitly wherever they are used.

For each k, we now define the relations $R_k^\Box \subseteq (X \times \mathcal{V}_G^r(FS_2(M_k(X)) \times M_k(X)$ and $R_k^\Diamond \subseteq (X \times \mathcal{V}_G^r(FS_2(M_k(X)) \times M_k(X)$, given as:

$$\begin{aligned}(x, C)R_k^\Box y &\iff y \in p_0(r(\lambda_0[C]))\\ (x, C)R_k^\Diamond y &\iff y \in p_1(r(\lambda_0[C]))\end{aligned}$$

We can furthermore define the relation $R_k \subseteq (X \times \mathcal{V}_G^r(FS_2(M_k(X)) \times M_k(X)$, given canonically as the intersection of the $R_k^\Box$ and $R_k^\Diamond$:

$$(x, C)R_k y \iff y \in R_k^\Box[(x, C)] \cap R_k^\Diamond[(x, C)]$$

Where $r : FS_2(M_k(X)) \to FS_1(M_k(X))$ is the root map. This very quickly becomes difficult to keep track of, so for the sake of intuition, the idea is that the desired relations always live in $FS_1(-) \subseteq \mathcal{V}^\uparrow(-) \times \mathcal{V}^\downarrow(-)$. So we first take $\lambda_0[C] \in FS_2(M_k(X))$, where the relation does not yet arise transparently as this is a closed upset of pairs in $FS_1(M_k(X))$. Thus, we take $r(\lambda_0[C]) \in FS_1(M_k(X))$, which gives us the pair $(R_k^\Box[(x, C)], R_k^\Diamond[(x, C)]$, and then (if we want) we may take the intersection of these relations to get $R_k[(x, C)]$. Doing so will be informative when we prove things about **IKS** p-morphisms.

Definition 5.5. *Let $x, y \in M_\infty(X)$ We define the relations $R_\omega^\Box$, $R_\omega^\Diamond$, and R_ω as follows:*

$$\begin{aligned}xR_\omega^\Box y &\iff \forall k \in \omega, x(k+1)R_k^\Box y(k)\\ xR_\omega^\Diamond y &\iff \forall k \in \omega, x(k+1)R_k^\Diamond y(k)\\ xR_\omega y &\iff \forall k \in \omega, x(k+1)R_k y(k)\end{aligned}$$

We clearly have $R_\omega[x] = R^\Box_\omega[x] \cap R^\Diamond_\omega[x]$, as desired.

Lemma 5.6. *If $x \in M_\infty(X)$ and $y \in M_n(X)$ for some $n \geq 1$, then*

(i) If $x(n+1)R^\Box_n y$, then $x(n)R^\Box_{n-1}\pi_n(y)$.

(ii) If $x(n+1)R^\Diamond_n y$, then $x(n)R^\Diamond_{n-1}\pi_n(y)$

Proof. To see that (i) holds, let $x(n+1)R^\Box_n y$ for $y \in M_n(X)$ and denote $x(n+1) = (x', E)$ and $x(n) = (x', C)$. As $x \in M_\infty(X)$, we know that $(x', C) = \pi_{n+1}(x', E)$. Unfolding the definitions, this means that $C_0 = \mathcal{V}^{\uparrow\downarrow}\pi_n[E_0]$. By the monotonicity of π_n, $r(C_0) = (\uparrow\pi_n[-], \downarrow\pi_n[-])[r(E_0)]$, and therefore $p_0(r(C_0)) = \uparrow\pi_n[p_0(r(E_0))]$. Since $(y \in p_0(r(C_0))$ by assumption, it follows immediately that $\pi_n(y) \in \uparrow\pi_n[p_0(r(E_0))]$, and thus $\pi_n(y) \in p_0(r(C_0))$.

To see that (ii) holds, let $x(n+1)R^\Diamond_n y$, and denote $x(n+1) = (x', E)$ and $x(n) = (x', C)$. As $x \in M_\infty(X)$, we know that $(x', C) = \pi_{n+1}(x', E)$. Then we again have that $C_0 = \mathcal{V}^{\uparrow\downarrow}\pi_n[E_0]$, so $r(C_0) = (\uparrow\pi_n[-], \downarrow\pi_n[-])[r(E_0)]$, and therefore that $p_1(r(C_0)) = \downarrow\pi_n[p_1(r(E_0))]$. Since $y \in p_1(r(C_0)$ by assumption, it follows immediately that $\pi_n(y) \in \downarrow\pi_n[p_1(r(E_0))]$, and thus $\pi_n(y) \in p_1(r(C_0))$.

$\Box$

Lemma 5.7. *Let $x \in M_\infty(X)$ and $y \in M_n(X)$. Then we have the following:*

1. *If $x(n+1)R^\Box_n y$ then there exists an extension y^ω of y such that $y^\omega(n) = y$ and $xR^\Box_\omega y^\omega$.*
2. *If $x(n+1)R^\Diamond_n y$ then there exists an extension y^ω of y such that $y^\omega(n) \geq y$ and $xR^\Diamond_\omega y^\omega$.*

Proof. We begin by showing (1). Let $x \in M_\infty(X)$ and denote $x(n+1) = (x', C)$. Suppose that $(x(n+1))R^\Box_n y$ for $y \in M_n(X)$. Then by construction $y \in p_0(r(C_0))$.

Now denote $x(n+2) = (x', D)$ in $M_{n+2}(X)$. As $x \in M_\infty(X)$, we have that $(x', C) = \pi_{n+2}(x', D)$. Recall that π_{n+2} is defined as

$$\pi_{n+2} : M_{n+2} \to M_{n+1}$$
$$(x, C) \mapsto (x, \mathcal{V}^r_G(\mathcal{V}_r(\mathcal{V}^{\uparrow\downarrow}(\pi_{n+1})))(C)),$$

so $C = \mathcal{V}^r_G(\mathcal{V}_r(\mathcal{V}^{\uparrow\downarrow}(\pi_{n+1})))(D)$.

Recall that the action of $\mathcal{V}^r_G$ on a morphism is to apply it "as is" on the first coordinate. Thus, we have that $C_0 = \mathcal{V}_r(\mathcal{V}^{\uparrow\downarrow}(\pi_{n+1}))(D_0)$. Recall further that the action of $\mathcal{V}_r$ on a morphism is to take its direct image. Thus, $\mathcal{V}_r(\mathcal{V}^{\uparrow\downarrow}(\pi_{n+1}))(D_0) = \mathcal{V}^{\uparrow\downarrow}(\pi_{n+1}))[D_0] = (\uparrow\pi_{n+1}[-], \uparrow\pi_{n+1}[-])[D_0]$. As $(\uparrow\pi_{n+1}[-], \downarrow\pi_{n+1}[-])$ is monotone, we have that $(\uparrow\pi_{n+1}[-], \uparrow\pi_{n+1}[-])(r(D_0)) = r(C_0)$. Denote $r(C_0)$ as the pair (A, B) and $r(D_0)$ as the pair (E, F). By assumption $y \in A = \uparrow\pi_{n+1}[E]$, so there must be $z \in E$ such that $\pi_{n+1}(z) \leq y$. As $\pi_{n+1} : M_{n+1} \to M_n$ is a p-morphism, there is $w \in M_{n+1}(X)$ such that $z \leq w$ and $\pi_{n+1}(w) = y$. Finally, recall that $(D_0) \in FS_2(M_{n+1}(X))$, so $E \in \mathcal{V}^\uparrow(M_{n+1}(X))$, and as E is an upset in M_{n+1}, it follows that $w \in E$. Then we have by construction that $(x', D) = x(n+2)R^\Box_{n+1}w$. We thus set $y^\omega(n+1) = w$. Note that this is a well-defined index for an element of the projective limit $M_\infty(X)$, as $\pi_{n+1}(w) = y$, i.e. $\pi_{n+1}(y^\omega(n+1)) = y^\omega(n)$. Then this also holds for any index $m \geq n$, and by Lemma 5.6, for all indexes lower than n. Then by compactness of $M_\infty(X)$, we have shown that there is $y^\omega \in M_\infty(X)$ such that $xR_\omega y^\omega$ and $\mathbf{p}_n(y^\omega) = y$.

We now show (2). The initial steps are quite similar. Let us again take $x \in M_\infty(X)$, and will again denote $x(n+1) = (x', C)$ and $x(n+2) = (x', D)$, and assume now that $(x(n+1))R^{\Diamond}_n y$, i.e. $y \in p_1(r(C_0))$.

As in the case for $\Box$, we have that $(\uparrow\pi_{n+1}[-], \downarrow\pi_{n+1}[-])(r(D_0)) = r(C_0)$, and again denote $r(C_0) = (A, B)$ and $r(D_0) = (E, F)$.

By assumption $y \in B = \downarrow\pi_{n+1}[F]$, so there must be $z \in F$ such that $\pi_{n+1}(z) \geq y$. Furthermore, as $z \in F$, we have by construction that $x(n+2)R^{\Diamond}_{n+2}z$. So $z \in R_n[x(n+2)] \cap \pi^{-1}_{n+1}[\uparrow y] \neq \emptyset$. Then note here that it follows that for any index k, $v \in R^{\Diamond}_{k+1}[x(k+1)]$ implies $v \leq \pi_{k+1}(w)$ for some $w \in R^{\Diamond}_{k+2}[x(k+2)]$.

Thus, there is some $w_2 \in R^{\Diamond}_{n+2}[x(n+2)]$ such that $z \leq \pi_{n+2}(w_2)$. Then by induction there is such a $w_k \in R^{\Diamond}_{n+m}[x(n+m)]$ such that $w_{k-1} \leq \pi_{n+m}(w_k)$, for all indexes $n+m$.

Then, as sketched in Figure 5.6, for any such $w_k \in R^{\Diamond}_{n+m}[x(n+m)]$, by monotonicity of the projective limit morphisms, $\pi_{n+m} \circ \pi_{n+m-1} \circ ... \circ \pi_{n+1}(w_k) \geq y$.

Then it holds that for any $k > n$, $R_k[x(k+1)] \cap \pi^{-1}_k \circ \pi^{-1}_{k-1} \circ ... \circ \pi^{-1}_{n+1}[\uparrow y] \neq \emptyset$, and thus that $R_k[x(k+1)] \cap \mathbf{p}^{-1}_n[\uparrow y] \neq \emptyset$.

Then it follows that the set $\{R_k[x(k+1)] | k \geq n+1\} \cup \mathbf{p}^{-1}_n[\uparrow y]$ has the FIP. Then by compactness, and by Lemma 5.6, there is y^ω such that $xR_\omega y^\omega$ and $\mathbf{p}_n(y^\omega) \geq y$ (i.e. $y^\omega(n) \geq y$).

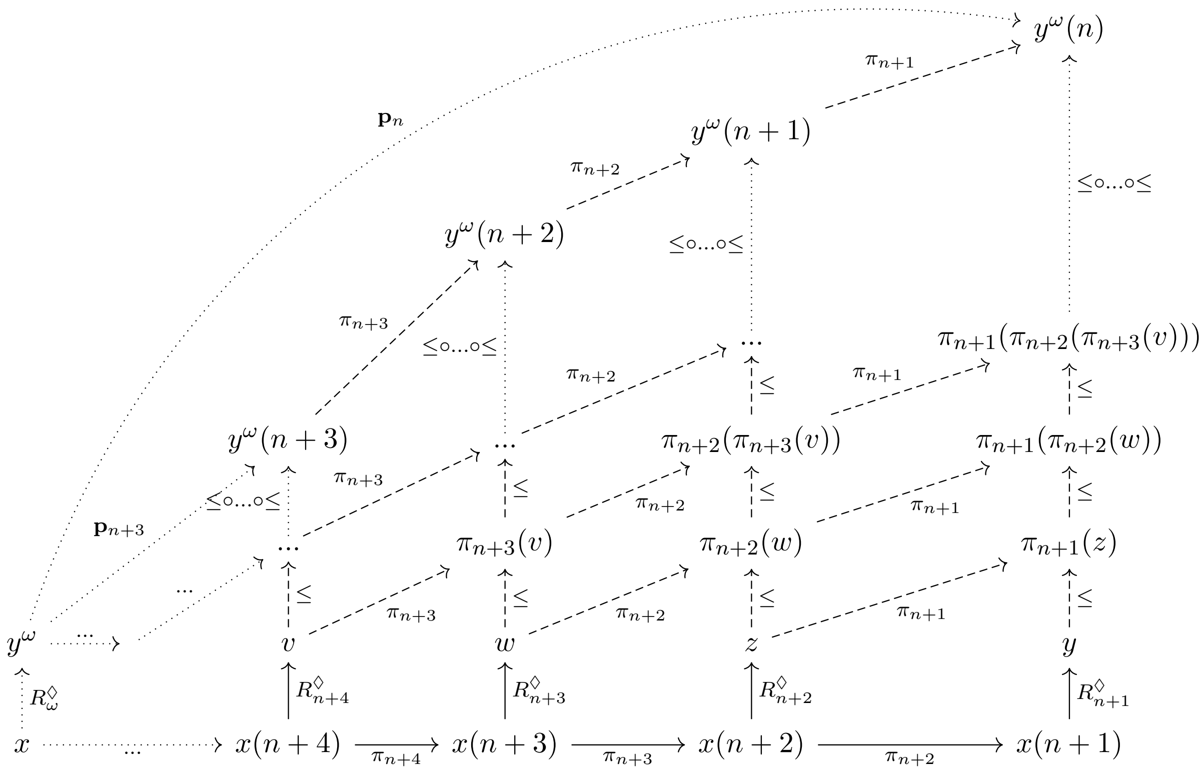

Figure 5.6: Extension of y for $R^{\Diamond}_\omega$

$\Box$

Intuitively, item (1) of Lemma 5.7 allows us to say is that if $x(n+1)R^{\Box}_n y$, then we can infinitely extend y to an element $y^\omega \in M_\infty$ which projects to y and is seen by x (i.e. $xR^{\Box}_\omega y^\omega$). Item (2) says something similar, but is weaker in that it does not require equality. Note that this essentially shows that $\mathbf{p}_n$ meets the "back" conditions for **IKS** p-morphisms (see (5) and (6) of Definition 4.8), as illustrated in Figure 5.7. Note that (1)-(3) are satisfied as $\mathbf{p}_n$ is continuous and an order p-morphism, and (4) is satisfied as $xR_\omega y$ implies $x(n+1)R_{n+1}y(n)$ for all n by definition.

$$\begin{tikzcd} y^\omega \arrow[r, dashed, "\mathbf{p}_n"] & z = \mathbf{p}_n(y^\omega) \\ x \arrow[u, dashed, "R^\Box_\omega"] \arrow[r, "\mathbf{p}_{n+1}"'] & x(n+1) \arrow[u, "R^\Box_{n+1}"'] \end{tikzcd} \qquad \begin{tikzcd} & \mathbf{p}_n(y^\omega) \\ y^\omega \arrow[ur, dashed, "\mathbf{p}_n"] & z \arrow[u, dashed, "\leq"'] \\ x \arrow[u, dashed, "R^\Diamond_\omega"] \arrow[r, "\mathbf{p}_{n+1}"'] & x(n+1) \arrow[u, "R^\Diamond_{n+1}"'] \end{tikzcd}$$

Figure 5.7: Conditions (5) and (6) for the projections

The concept of this projective limit $M_\infty(X)$ which is in turn constructed from other projective limits $\mathcal{V}_G(...)$ may seem on the surface quite unwieldy. However, we will find that we can for the most part prove things locally, by moving along the projections $\mathbf{p}_n : M_\infty(X) \to M_n(X)$. The lemmas we have been proving so far in this section are for this purpose.

Recall from 2.56 that the topology of $M_\infty(X)$ is given by the base

$$\bigcup_{n\in\omega} \{\mathbf{p}_n^{-1}[U] | U \in ClopUp(M_n(X))\},$$

and from Lemma 2.57 that $U \in ClopUp(M_\infty(X))$ if and only if $\mathbf{p}_n[U] \in ClopUp(M_n(X))$. With this in mind, we prove the following lemma:

Lemma 5.8. *If $U = \mathbf{p}_n^{-1}[V]$ for $V \in ClopUp(M_n(X))$, then the following equalities hold:*

$$\textit{\textbf{(i)}}\ \Box_{R^\Box_\omega} U = \mathbf{p}_{n+1}^{-1}[\Box_{R^\Box_n} V].$$
$$\textit{\textbf{(ii)}}\ \Diamond_{R^\Diamond_\omega} U = \mathbf{p}_{n+1}^{-1}[\Diamond_{R^\Diamond_n} V].$$

Proof. **(i)** For the right-to-left inclusion, let $x \in \mathbf{p}_{n+1}^{-1}[\Box_{R^\Box_n} V]$, and suppose $xR^\Box_\omega y$. Recall that $x \in \Box_{R^\Box_\omega} U$ iff $R^\Box_\omega[x] \subseteq U$. Given $xR^\Box_\omega y$, we have by construction that $x(n+1)R^\Box_n y(n)$, which, together with our assumption that $x \in \mathbf{p}_{n+1}^{-1}[\Box_{R^\Box_n} V]$, implies that $y(n) \in V$. Then $y = \mathbf{p}_n^{-1}(y(n)) \in \mathbf{p}_n^{-1}[V] = U$. Thus, we have shown that $R^\Box_\omega[x] \subseteq U$, as desired.

For the left-to-right inclusion, let $x \in \Box_{R^\Box_\omega} U$. We must show that $\mathbf{p}_{n+1}(x) \in \Box_{R^\Box_n} V$, i.e. that $R^\Box_n[x(n+1)] \subseteq V$. So let $y \in M_n(X)$ and suppose that $x(n+1)R^\Box_n y$. We have by Lemma 5.7 that there exists some y^ω such that $\mathbf{p}_n(y^\omega) = y$ and $xR^\Box_\omega y^\omega$. By our assumption that $x \in \Box_{R^\Box_\omega} U$, it then follows that $y^\omega \in U = \mathbf{p}_n^{-1}[V]$, so $\mathbf{p}_n(y^\omega) = y \in V$. We have thus shown that $x \in \mathbf{p}_{n+1}^{-1}(\Box_{R^\Box_n} V)$.

(ii) For the right-to-left inclusion, let $x \in \mathbf{p}_{n+1}^{-1}[\Diamond_{R^\Diamond_n} V]$, and recall that $x \in \Diamond_{R^\Diamond_\omega} U$ iff $R^\Diamond_\omega[x] \cap U \neq \emptyset$. Unfolding our assumption, $\mathbf{p}_{n+1}(x) \in \Diamond_{R^\Diamond_n} V$, i.e. $R^\Diamond_n[x(n+1)] \cap V \neq \emptyset$. Now fix $y \in M_n(X)$ such that $y \in R^\Diamond_n[x(n+1)] \cap V$. We must find $y' \in M_\infty(X)$ such that $y' \in R^\Diamond_\omega[x] \cap U$. By Lemma 5.7 there is $y^\omega \in M_\infty(X)$ such that $xR^\Diamond_\omega y^\omega$ and $\mathbf{p}_n(y^\omega) \geq y$. As $y \in V$ and V is a clopen upset, it follows that $\mathbf{p}_n(y^\omega) \in V$. Then $y^\omega \in \mathbf{p}_n^{-1}[V] = U$. We have thus found $y^\omega \in R^\Diamond_\omega[x] \cap U \neq \emptyset$.

For the left-to-right inclusion, let $x \in \Diamond_{R^\Diamond_\omega} U$, and fix $y \in M_\infty(X)$ such that $y \in R^\Diamond_\omega[x] \cap U$. We must show that $\mathbf{p}_{n+1}(x) \in \Diamond_{R^\Diamond_n} V$, i.e. that $R^\Diamond_n[x(n+1)] \cap V \neq \emptyset$. Given $xR^\Diamond_\omega y$, we have by construction that $x(n+1)R^\Diamond_n y(n)$. Furthermore, as $y \in U = \mathbf{p}_n^{-1}[V]$ by assumption, it follows that $y(n) = \mathbf{p}_n(y) \in V$. Thus, we have found $y(n) \in R^\Diamond_n[x(n+1)] \cap V \neq \emptyset$. $\square$

This lemma will be very useful in that it allows us to prove things locally. That is, rather than reasoning about $M_\infty(X)$, we can instead look at the topology of some $M_n(X)$. With this in place, we now show that our construction indeed defines an **FS**-space.

Proposition 5.9. *Given any Esakia space X, $M_\infty(X)$ is an* ***FS****-space.*

Proof. The fact that the inverse limit is an Esakia space follows straightforwardly from duality, and the fact that each of π_k is a p-morphism.

To see that this is the case, note that each $\mathcal{V}^r_G(\mathcal{V}_r(\mathcal{V}^{\uparrow\downarrow}(M_n(X)))$ is Esakia by Theorem 3.7 in [AB24]). Thus, every π_n is a map from an Esakia space to a Priestley space, so $\mathcal{V}^r_G(\mathcal{V}_r(\mathcal{V}^{\uparrow\downarrow}(\pi_n))$ is a p-morphsim by Proposition 3.22.It is also clear from the definition that π_0 and π_1 are p-morphisms.

To see that $R_\omega[x]$ is closed, let $x \in M_\infty(X)$. Then $R_\omega[x] = \bigcap_{k\in\omega} \{y|x(k+1)R_k y(k)\}$, so to show this is point-closed it suffices to show that R_k is point-closed. We have that $x(k+1) = (z, C)$ where $C \in V^r_G(FS_2(M_k(X)))$, so $y \in R_k[x(k+1)]$ iff $y \in p_0(r(\lambda_0[C])) \cap p_1(r(\lambda_0[C]))$. Thus, $R_k[x(k+1)] = p_0(r(\lambda_0[C])) \cap p_1(r(\lambda_0[C]))$, which is closed in $M_k(X)$. Since the projection $\mathbf{p}_k$ is continuous, it is closed in $M_\infty(X)$.

To show that $R_\omega[\uparrow x]$ is a closed upset, we show that $R_\omega[\uparrow x] = R^\Box_\omega[x]$ and prove closedness for the latter. Recall that $R_k[\uparrow(x, C)] = \bigcup\{p_0(r(E_0))|(y, E) \geq_{M_k} (x, C)\}$, and let $(y, E) \geq_{M_k} (x, C)$. Then $C \leq_{V_G} E$, so $C_0 \leq_{FS_2} E_0$, so $r(C_0) \leq_{FS_1} r(E_0)$. Finally, $p_0(r(C_0)) \leq_{\mathcal{V}^\uparrow} p_0(r(E_0))$. As $\mathcal{V}^\uparrow$ is ordered by reverse inclusion, $p_0(r(E_0)) \subseteq p_0(r(C_0))$. Thus, $\bigcup\{p_0(r(E_0))|(y, E) \geq_{M_k} (x, C)\} = p_0(r(C_0)) = R^\Box_k[(x, C)]$.

To see that $R^\Box_\omega[x]$ is a closed upset, recall that $R^\Box_\omega[x] = \bigcap_{k\in\omega} \{y|x(k+1)R^\Box_k y(k)\}$. Thus, it suffices to show that $R^\Box_k[x(k+1)]$ is closed. Letting $x(k+1) = (x', C)$, $R^\Box_k[x(k+1)] = p_0(r(C_0))$, which is closed in $M_k(X)$ and thus also in $M_\infty(X)$.

Let $U \in ClopUp(M_\infty(X))$, and recall that then $U = \mathbf{p}_n^{-1}[V]$ for some $V \in ClopUp(M_n(X))$, and $\mathbf{p}_n[U] \in ClopUp(M_n(X))$. We show that $\Box_{R^\Box_\omega} U$ is clopen upset. As this proof gets a bit unwieldy, we will preface it with a sketch.

Proof sketch. We are showing clopenness for a subset of the inverse limit, which in turn is itself constructed from a series of products and inverse limits, so we will use Lemma 2.57 together with the continuity of the projections from each product. This, together with Lemma 5.8, will allow us to show that $\Box_{R^\Box_\omega} U \in ClopUp(M_\infty(X))$ by showing that something in $\mathcal{V}^{\uparrow,\downarrow}$ is a clopen upset. Schematically, by Lemma 5.8, $\Box_{R_\omega} U$ will be clopen so long as $\Box_{R_{n+1}} U$, which is clopen if $\Box_{R_{n+1}} U = p_n^{-1}(S)$ for some $S \in ClopUp(\mathcal{V}^r_G(FS_2(M_n(X))))$, which is clopen if it is the preimage of a clopen subset at one of its coordinates, and so on. We now give the full proof.

As $R^\Box_n[(x, C)] = p_0(r(\lambda_0(C)))$, we write

$$\Box_{R^\Box_n} \mathbf{p}_n[U] = \{(x, C) \in M_{n+1}|p_0(r(\lambda_0(C))) \subseteq \mathbf{p}_n[U]\}$$

Now let $p_{\mathcal{V}^r_G(FS_2(M_n(X)))}$ denote the projection from $M_{n+1}(X) = X \times \mathcal{V}^r_G(FS_2(X))$ to $\mathcal{V}^r_G(FS_2(X))$. Then consider the set

$$S = \{C \in \mathcal{V}^r_G(FS_2(M_n(X)))|p_0(r(\lambda_0(C))) \subseteq \mathbf{p}_n[U]\}$$

It is straightforward to see that $\Box_{R^\Box_{n+1}}(\mathbf{p}_n[U]) = p^{-1}_{\mathcal{V}^r_G(FS_2(M_n(X)))}(S)$. Thus, it suffices to show that S is clopen. Once again, as $\mathcal{V}^r_G(FS_2(M_n(X)))$ is a projective limit, the subset S is clopen if and only if $S = \lambda_m^{-1}[V]$ for some index m and some $V \in ClopUp(\mathcal{V}_m(FS_2(M_n(X))))$. Then we claim that

$S = \lambda_0^{-1}[\lambda_0[S]]$. To see this, note that $\lambda_0[S] = \{\lambda_0[C] | p_0(r(\lambda_0(C))) \subseteq \mathbf{p}_n[U]\}$ for $C \in \mathcal{V}_G^r(FS_2(M_n(X)))$, so

$$\lambda_0^{-1}[\lambda_0[S]] = \{C \in \mathcal{V}_G^r(FS_2(M_n(X))) | \lambda_0[C] \in \lambda_0[S]\} = \\ \{C \in \mathcal{V}_G^r(FS_2(M_n(X))) | p_0(r(\lambda_0[C])) \subseteq \mathbf{p}_n[U]\} = S$$

Thus, it suffices to show that $\lambda_0[S]$ is clopen in $FS_2(M_n(X))$. We can go down yet another level to $FS_1(M_n(X))$ by similarly observing that

$$\lambda_0[S] = r^{-1}(\{r(\lambda_0(C)) | C \in S\})$$

where $r : FS_2(M_n(X)) \to FS_1(M_n(X))$ is the root map. Since r is continuous, to show that $\lambda_0[S]$ is clopen in $FS_2(M_n(X))$ it suffices to show that $\{r(\lambda_0(C)) | C \in S\}$ is clopen in $FS_1(M_n(X))$. But then this is precisely the set

$$\{p_0(r(\lambda_0(C))) | C \in S\} \times \mathcal{V}^{\downarrow}(M_n(X)) = \\ \{p_0(r(\lambda_0(C))) | p_0(r(\lambda_0(C))) \subseteq \mathbf{p}_n[U]\} \times \mathcal{V}^{\downarrow}(M_n(X)) = \\ [\mathbf{p}_n[U]] \times \mathcal{V}^{\downarrow}(M_n(X)),$$

which is a clopen upset by construction. Thus, $\Box_{R_n^{\Box}} \mathbf{p}_n[U]$ is a clopen upset.

As $\mathbf{p}_{n+1}$ is continuous, $\mathbf{p}_{n+1}^{-1}[\Box_{R_n^{\Box}} \mathbf{p}_n[U]] \in ClopUp(M_\infty(X))$. Recall that $U = \mathbf{p}_n^{-1}[V]$ and $\mathbf{p}_n[U] = \mathbf{p}_n[\mathbf{p}_n^{-1}[V]]$, so by Lemma 5.8

$$\Box_{R_\omega^{\Box}} U = \mathbf{p}_{n+1}^{-1}[\Box_{R_n^{\Box}} \mathbf{p}_n[\mathbf{p}_n^{-1}[V]]] = \mathbf{p}_{n+1}^{-1}[\Box_{R_n^{\Box}} \mathbf{p}_n[U]] \in ClopUp(M_\infty(X)).$$

We have thus shown that $\Box_{R_\omega^{\Box}} U \in ClopUp(M_\infty(X))$, as desired.

To see that $\Diamond_{R_\omega^{\Diamond}} U$ is a clopen upset, recall that $R_n^{\Diamond}[(x, C)] = p_1(r(\lambda_0(C)))$, so write

$$\Diamond_{R_n^{\Diamond}}(\mathbf{p}_n[U]) = \{(x, C) \in M_{n+1}(X) | p_1(r(\lambda_0(C))) \cap \mathbf{p}_n[U] \neq \emptyset\}.$$

Then as in the case for $\Box_{R_\omega^{\Box}}$, we have that $\Diamond_{R_{n+1}^{\Diamond}}(\mathbf{p}_n[U]) = \mathbb{P}^{-1}_{\mathcal{V}_G^r(FS_2(M_n(X)))}(S)$, for

$$S = \{C \in \mathcal{V}_G^r(FS_2(M_n(X))) | p_1(r(\lambda_0(C))) \cap \mathbf{p}_n[U] \neq \emptyset\}$$

so this proof amounts to showing that $\{r(\lambda_0(C)) | C \in S\}$ is clopen in $FS_1(M_n(X))$. Here, the set $\{r(\lambda_0(C)) | C \in S\}$ can be rewritten as follows:

$$\mathcal{V}^{\uparrow}(M_n(X)) \times \{p_1(r(\lambda_0(C))) | C \in S\} = \\ \mathcal{V}^{\uparrow}(M_n(X)) \times \{p_1(r(\lambda_0(C))) | p_1(r(\lambda_0(C))) \cap \mathbf{p}_n[U] \neq \emptyset\} = \\ \mathcal{V}^{\uparrow}(M_n(X)) \times \langle \mathbf{p}_n[U] \rangle,$$

which is clopen by construction. Thus, $\Diamond_{R_n^{\Diamond}}(\mathbf{p}_n[U])$ is a clopen upset. By Lemma 5.8, we have that

$$\Diamond R_\omega^{\Diamond} U = \mathbf{p}_{n+1}^{-1}[\Diamond_{R_n^{\Diamond}} \mathbf{p}_n[U]],$$

so $\Diamond R^{\Diamond}_{\omega} U \in ClopUp(M_{\infty}(X))$.

To see that $R_\omega[x] = R_\omega[\uparrow x] \cap \downarrow R_\omega[x]$, we show that $\downarrow R_\omega[x] = R^{\Diamond}_{\omega}[x]$. Since we showed above that $R_\omega[\uparrow x] = R^{\Box}_{\omega}[x]$, this will amount to $R_\omega[x] = R^{\Box}_{\omega}[x] \cap R^{\Diamond}_{\omega}[x]$, which is true by definition.

Recall that $R_k[(x, C)] = p_0(r(C_0)) \cap p_1(r(C_0))$. As $(p_0(r(C_0)), p_1(r(C_0))) \in FS_1(M_k(X))$, we have that

$$R^{\Diamond}_{k}[(x, C)] = p_1(r(C_0)) = \downarrow(p_0(r(C_0)) \cap p_1(r(C_0))) = \downarrow R_k[x],$$

as desired.

$\Box$

Now suppose that $(Y, S_\Box, S_\Diamond)$ is a **FS**-space, and assume that $f : Y \to X$ is a p-morphism.

As in the proof of Theorem 4.17, we start with the map

$$\begin{aligned}(S_\Box, S_\Diamond) : Y &\to FS_1(Y)\\ x &\mapsto (S[\uparrow x], \downarrow S[x])\end{aligned}$$

Then define the map $\overline{f} : Y \to FS_1(X)$

$$\overline{f}(y) = (f[S_\Box[y]], \downarrow f[S_\Diamond[y]]);$$

We show that this is well-defined as an element of $FS_1(X)$.

Proof. First note that as $S_\Box[y]$ and $S_\Diamond[y]$ are closed subsets of Y, it follows by Proposition* that $f[S_\Box[y]]$ and $f[S_\Diamond[y]]$ are both closed. Furthermore, the downset of a closed set is closed in Priestley spaces. Thus, it follows immediately that $\downarrow f[S_\Diamond[y]]$ is a downset, and thus $\downarrow f[S_\Diamond[y]] \in \mathcal{V}^{\downarrow}(X)$.

To see that $f[S_\Box[y]] \in \mathcal{V}^{\uparrow}(X)$, let $z \in S_\Box[y]$ and $f(z) \leq w$. Then by the p-morphism condition, there is some $z' \geq z$ such that $f(z') = w$. Then $z' \in S_\Box[y]$ as $S_\Box[y]$ is an upset, and thus $w \in f[S_\Box[y]]$. Hence, $f[S_\Box[y]]$ is a (closed) upset.

It remains to show that $\downarrow f[S_\Diamond[y]] = \downarrow(f[S_\Box[y]] \cap \downarrow f[S_\Diamond[y]])$. The right-to-left inclusion is trivial as $\downarrow f[S_\Diamond[y]]$ is a downset, so it suffices to show that $\downarrow f[S_\Diamond[y]] \subseteq \downarrow(f[S_\Box[y]] \cap \downarrow f[S_\Diamond[y]])$. Let $y' \in \downarrow f[S_\Diamond[y]]$. We must find $w \geq y'$ such that $w \in f[S_\Box[y]] \cap \downarrow f[S_\Diamond[y]]$. By our assumption, there is $z \in S_\Diamond[y]$ such that $y' \leq f(z)$. Since $z \in S_\Diamond[y]$ and $(Y, S_\Box, S_\Diamond)$ is a **FS**-space, it follows that there is some $v \geq z$ such that $v \in S_\Box[y] \cap S_\Diamond[y]$. Furthermore, by monotonicity of f, since $z \leq v$ we have that $f(z) \leq f(v)$, and thus $y' \leq f(v)$. So we have found $f(v) \geq y'$ such that $f(v) \in f[S_\Box[y]] \cap \downarrow f[S_\Diamond[y]]$. This concludes the proof that $(f[S_\Box[y]], \downarrow f[S_\Diamond[y]]) \in FS_1(X)$.

$\Box$

To see that $\overline{f}$ is continuous, let $U \times V$ be clopen in $\mathcal{V}^{\uparrow\downarrow}(X)$. Note that as $FS_1(X)$ is equipped with the subspace topology, it suffices to show that the preimages of clopens in $\mathcal{V}^{\uparrow\downarrow}(X)$ are clopen, as this will also hold for the restriction to $FS_1(X)$. Now letting $U', V' \in ClopUp(X)$ and $U'', V'' \in ClopDown(X)$, we have the following possible forms for U and V:

- $U = [U'] = \{C \in \mathcal{V}^{\uparrow}(X) | C \subseteq U'\}$. Then $f[S_\Box[-]]^{-1}[U] = \{y \in Y | f[S_\Box[y]] \subseteq U'\} = \{y \in Y | S_\Box[y] \subseteq f^{-1}[U']\} = \Box_{S_\Box} f^{-1}[U']$. Then since U' is a clopen upset and f is continuous, $f^{-1}[U']$ is also a clopen upset. Then $\Box_{S_\Box} f^{-1}[U']$ is a clopen upset as Y is a **FS**-space.

- $U = \langle X - V' \rangle = \{C \in \mathcal{V}^{\uparrow}(X) | C \cap (X - V') \neq \emptyset\}$. Then $f[S_\Box[-]]^{-1}\langle X - V' \rangle = \{y \in Y | f[S_\Box[y]] \cap (X - V') \neq \emptyset\} = \{y \in Y | S_\Box[y] \cap f^{-1}[X - V'] \neq \emptyset\} = Y - \Box_{R_\Box} f^{-1}V'$, which is clopen.

- $V = [U''] = \{C \in \mathcal{V}^{\downarrow}(X) | C \subseteq U''\}$. Then $f[S_\Diamond[-]]^{-1}[U] = \{y \in Y | {\downarrow} f[S_\Diamond[y]] \subseteq U'\} = \{y \in Y | S_\Diamond[y] \subseteq {\downarrow} f^{-1}[U']\} = \{y \in Y | S_\Diamond[y] \cap (Y - {\downarrow} f^{-1}[U']) = \emptyset\} = Y - \Diamond S_\Diamond (Y - {\downarrow} f^{-1}[U'])$.

  As Y is an Esakia space and f is continuous, ${\downarrow} f^{-1}[U']$ is a clopen downset, so $Y - {\downarrow} f^{-1}[U']$ is a clopen upset, and thus $Y - \Diamond S_\Diamond (Y - {\downarrow} f^{-1}[U'])$ is clopen.

- $V = \langle X - V'' \rangle = \{C \in \mathcal{V}^{\downarrow}(X) | C \cap (X - V'') \neq \emptyset\}$. Then ${\downarrow} f[S_\Diamond[-]]^{-1}\langle X - V'' \rangle = \{y \in Y | {\downarrow} f[S_\Diamond[y]] \cap (X - V'') \neq \emptyset\} = \{y \in Y | S_\Diamond[y] \cap {\downarrow} f^{-1}[X - V''] \neq \emptyset\} = \Diamond S_\Diamond {\downarrow} f^{-1}[X - V'']$.

  As $(X - V)$ is a clopen upset and f is continuous, $f^{-1}[X - V'']$ is a clopen upset. Furthermore, as Y is an Esakia space, ${\downarrow} f^{-1}[X - V'']$.

To see that $\overline{f}$ is monotone, let $y \leq y'$ in Y. We show that $(f[S_\Box[y]], {\downarrow} f[S_\Diamond[y]]) \leq (f[S_\Box[y']], {\downarrow} f[S_\Diamond[y']])$ in $FS_1(X)$. By Lemma 4.3, since $(Y, S_\Box, S_\Diamond)$ is a $\Box\Diamond$-frame, we have that $S_\Box$ and $S_\Diamond$ are monotone. Thus, $S_\Box[y] \supseteq S_\Box[y']$ and $S_\Diamond[y] \subseteq S_\Diamond[y']$, so $S_\Box[y] \leq_{\mathcal{V}^\uparrow} S_\Box[y']$ and $S_\Diamond[y] \leq_{\mathcal{V}^\downarrow} S_\Diamond[y']$. Then $f[S_\Box[y]] \supseteq f[S_\Box[y']]$, and ${\downarrow} f[S_\Diamond[y]] \subseteq {\downarrow} f[S_\Diamond[y']]$, so $f[S_\Box[y]] \leq_{\mathcal{V}^\uparrow} f[S_\Box[y']]$ and ${\downarrow} f[S_\Diamond[y]] \leq_{\mathcal{V}^\downarrow} {\downarrow} f[S_\Diamond[y']]$. This concludes the proof that $\overline{f}$ is a Priestley map.

Using Lemma 3.17, we obtain the unique continuous, monotone, r-open map $f_1 : Y \to FS_2(X)$ making the diagram in Figure 5.8 commute.

$$
\begin{tikzcd}
Y \arrow[rr, "f_1"] \arrow[dr, "\overline{f}"'] & & V_r(FS_1(X)) \arrow[dl, "r"] \\
& FS_1(X) &
\end{tikzcd}
$$

Figure 5.8: Commuting Triangle

Finally, using Proposition 3.22, we obtain a p-morphism $f_1' : Y \to V_G^r(FS_2(X))$, and hence a map

$$
\begin{aligned}
\overline{\overline{f}} : Y &\to X \times V_G^r(FS_2(X)) \\
y &\mapsto (f(y), f_1'(y)).
\end{aligned}
$$

which is clearly a p-morphism.

This allows us to define a sequence of p-morphisms $f_n : Y \to M_n(X)$, with $f_{n+1} : Y \to M_{n+1}(X)$, given by:

(i) $f_0 = f$

(ii) $f_{n+1} = \overline{\overline{f_n}}$

Let us go through this for the sake of clarity. Given that $f_n : Y \to M_n(X)$ is a p-morphism and $(Y, S_\Box, S_\Diamond)$ is a **FS**-space, we may construct $\overline{\overline{f_n}}$ as above, with $\overline{\overline{f_n}} : Y \to X \times \mathcal{V}_G^r(FS_2(M_n(X)))$. Recalling that $M_{n+1}(X) = X \times \mathcal{V}_G^r(FS_2(M_n(X))$, we then see that f_{n+1} indeed defines a map from Y to $M_{n+1}(X)$.

This sequence of p-morphisms now induces a unique map $f_\infty : Y \to M_\infty(X)$, which is likewise a p-morphism.

Proposition 5.10. *Given X an Esakia space, Y a **FS**-space, and $f : Y \to X$ a p-morphism, the unique lifting $f_\infty : Y \to M_\infty(X)$ is a **IKS** p-morphism from $(Y, S_\Box, S_\Diamond)$ to $(M_\infty(X), R^\Box_\omega, R^\Diamond_\omega)$.*

Proof. It is given by construction that p_∞ is continuous and a p-morphism with respect to $\leq$, so we need only check the conditions involving the relations.

To see that xSy implies that $f_\infty(x)R_\omega f_\infty(y)$, it suffices to show that if xSy, then $f_{k+1}(x)R^\Box_k f_k(y)$ and $f_{k+1}(x)R^\Diamond_k f_k(y)$. Recall that $f_{k+1}(x) = (f_k(x), \mathcal{V}_G\mathcal{V}_r(f_k[S_\Box[x]], {\downarrow} f_k[S_\Box[x]]))$. Thus, we must show that $f_k(y) \in f_k[S_\Box[x]] \cap {\downarrow} f_k[S_\Diamond[x]]$. By assumption $y \in S_\Box[x]$ and $y \in S_\Diamond[x]$, so clearly $f_k(y) \in f_k[S_\Box[y]] \cap {\downarrow} f_k[S_\Diamond[y]]$.

Now we check that $f_\infty(x)R_\omega z$ implies that $z \leq f_\infty(x')$ for some $x' \in S[x]$. We let $f_{k+1}(x)R_k z(k)$ and show that $z(k) \leq f_k(x')$ for some $x' \in S[x]$. Recall that $f_{k+1}(x) = (f_k(x), \mathcal{V}_G\mathcal{V}_r(f_k[S_\Box[x]], {\downarrow} f_k[S_\Diamond[x]]))$. Then by assumption $z(k) \in f_k[S_\Box[x]] \cap {\downarrow} f_k[S_\Diamond[x]])$, so there is some $w \in f_k[S_\Diamond[x]]$ such that $w \geq z(k)$. As $(f_k[S_\Box[x]], {\downarrow} f_k[S_\Diamond[x]]) \in FS_1(M_k(X))$, there is $v \geq w$ such that $v \in f_k[S_\Box[x]] \cap {\downarrow} f_k[S_\Diamond[x]]) \in FS_1(M_k(X)$. This in turn gives us that $v \leq f_k(v')$ for some $v' \in S_\Diamond[x]$. As $S_\Diamond[x] = {\downarrow}(S_\Box[x] \cap S_\Diamond[x])$, there is $x' \geq v' \in S_\Box[x] \cap S_\Diamond[x]$. As f_k is monotone, $f_k(v') \leq f_k(x')$. Then, as $z(k) \leq w \leq v \leq f_k(v') \leq f_k(x')$, we have found $x' \in S[x]$ such that $z(k) \leq f_k(x')$.

Finally we show that $f_\infty(x)R^\Box_\omega z$ implies that $f_\infty(x') = z$ for some $x' \in S_\Box[x]$. Let $f_{k+1}(x)R^\Box_k z(k)$. We must find $x' \in S_\Box[x]$ such that $f_k(x') = z(k)$. By assumption, $z(k) \in f_k[S_\Box[x]]$, so it follows immediately that $z(k) = f_k(v)$ for some $v \in S_\Box[x]$.

$\Box$

Theorem 5.11. *Let X be a set of generators, and let $\mathbb{X}_{F_D(X)}$ denote the Priestley dual of the free distrbutive lattice $F_D(X)$ over X. Then $M_\infty(\mathbb{X}_{F_D(X)})$ is the dual to the free **FS**-algebra on X many generators.*

Thus, using our coalgebraic treatment of Fischer Servi logic, we have provided an explicit characterization of the dual space to the free **IK**-algebra. We remark that our construction does not obviously yield normal forms for Fischer Servi logic, as it requires infinitely many applications of the $\mathcal{V}_G$ construction. However, it may do so for related logics for which the construction terminates after finitely many steps (see [Alm24, Section 5.3]). One could thus hope to use these results to obtain normal forms for semilinear Fischer Servi logic, as well as other related systems.

5.3 Rank-1 axioms

In this section, we discuss how our general technique provides a template for studying intuitionistic modal logics. We exemplify this by considering the additional rank-1 axiom of seriality, providing the dual space to the resulting algebra. We will do this by exploiting our coalgebraic representation, with modifications made only at the quotienting steps.

Recall that an algebra satisfies seriality if satisfies the following modal axiom:

$$\textbf{(D)}\ \Box p \to \Diamond p,$$

where the corresponding frame condition is $\forall w.\exists v.(wRv)$.

Definition 5.12. *Define $\mathbf{IKA_D}$ as the subcategory of **IK**-algebras that satisfy the **(D)** axiom. We denote by $\mathbf{IKS_D}$ the subcategory of Esakia spaces dual to $\mathbf{IKA_D}$.*

Let $(X, R_\Box, R_\Diamond)$ be a $\Box\Diamond$-frame (as in Definition 4.2). Recall $\Box\Diamond$-frames are dual to Heyting algebras satisfying the normality axioms for $\Box$ and $\Diamond$. As we did in Chapter 4 (see Propositions 4.14 and 4.16), we will quotient over **IK** interaction axioms **A** and **B**, and then additionally over axiom **(D)**.

Intuitively, **(D)** dually corresponds to the condition $\forall x(R_\Box \cap R_\Diamond)[x] \neq \emptyset$. That is, every point has a successor which it sees through both relations.

Definition 5.13. *Define* $S(X) = \{C \in FS_2(X) | \forall (D, E) \in C. D \cap E \neq \emptyset\}$

Proposition 5.14. *$S(X)$ is the Priestley subspace of $FS_2(X)$ which dually satisfies **(D)**, i.e.* $C \in [[U] \times V^{\downarrow}(X)] \implies C \in [V^{\uparrow}(X) \times \langle U \rangle]$

Proof. Let $C \in S(X)$ and suppose $C \in [[U] \times V^{\downarrow}(X)]$. We show that $C \in [V^{\uparrow}(X) \times \langle U \rangle]$. By our assumption, $C \subseteq [U] \times V^{\downarrow}(X)$. Now fix some $(D, E) \in C$, then $D \subseteq U$. By our assumption, $D \cap E \neq \emptyset$, so there is some $x \in D \cap E$. Since $D \subseteq U$, it follows that $x \in U$, so $x \in E \cap U \neq \emptyset$ and thus $(D, E) \in V^{\uparrow}(X) \times \langle U \rangle$, as desired.

For the converse direction, assume for contraposition that for some $C \in FS_2(X)$, there is $(D, E) \in C$ such that $D \cap E = \emptyset$. Then by $(D, E) \in FS_1(X)$, we have that $E = \downarrow (D \cap E) = \downarrow \emptyset = \emptyset$ (as $V^{\downarrow}(X)$ is ordered by inclusion). But then we have that $\forall (D', E') \in C$, $D' \subseteq X$, but $E \cap X = \emptyset$. Thus we have found a clopen upset U (namely the entire set X) such that $C \in [[U] \times V^{\uparrow}(X)]$ but $C \notin [V^{\uparrow}(X) \times \langle U \rangle]$. This concludes the proof.

□

Note that while we chose to quotient at $\mathcal{V}_r(FS_1(X))$, we could also do this at step $\mathcal{V}^{\uparrow\downarrow}(X)$, as the **(D)** residuates to $\Box p \leq \Diamond p$. It is also of note that the right-to-left direction of the proof uses the property determined by FS_1. This makes sense, as it establishes compatibility between $\Box$ and $\Diamond$.

We now claim that $\mathbf{IKS_D}$ spaces (X, R) are in one-to-one correspondence with $\mathcal{V}_r(\mathcal{V}^{\uparrow\downarrow})$-coalgebras $(X, f : X \to S(X))$, where again f is required to be open relative to the root map $r : S(X) \to FS_1(X)$. Finally, to turn these into coalgebras for an appropriate endofunctor on Esakia spaces, we apply our favourite functor $\mathcal{V}_G$. We claim that the category $\mathbf{IKS_D}$ is equivalent to the category $\mathbf{Coalg}(\mathcal{V}_G^r(S(-)))$.

Theorem 5.15. *The following are in 1-1 correspondence:*

(i) $\mathbf{IKS_D}$*-frames* (X, R) *over* X

(ii) r*-open Priestley maps* $f : X \to S(X)$

(iii) Esakia morphisms $f' : X \to \mathcal{V}_G^r(S(X))$

Theorem 5.16. *The category* $\mathbf{IKS_D}$ *of* $\mathbf{IKS_D}$*-spaces and p-morphisms is equivalent to the category* ***Coalg***$(\mathcal{V}_G^r(S(-)))$.

The proofs of these theorems are similar to those of 4.17 and 4.18 in Chapter 4. The case for K_{FS}-frames is analogous.

Proof. The proof is similar to that of Theorem 4.18. □

The same can be done for serial $\mathbf{K_{FS}}$-frames over image-finite posets, as in Chapter 4.

Thus, our method of quotienting in stages while applying the $\mathcal{V}_G$ construction subsumes rank-1 extensions of Fischer Servi logic. This furthermore means that our dual characterization of the free **IK**-algebra can also be applied to these logics, providing a uniform way of treating them.

Chapter 6

Conclusions and Further Work

In this thesis, we presented a new coalgebraic semantics for Fischer Servi logic, thereby situating intuitionistic modal logic more firmly within the uniform coalgebraic framework for modal logic. Building on the constructions from [Alm24], we developed a method of treating intuitionistic modal logics which are not the least intuitionistic extension of a positive reduct, by performing additional quotienting within the step-by-step approach. Following the work in [AB24], we derived coalgebraic representations both for modal spaces and image-finite Kripke frames for Fischer Servi logic. As remarked in [Alm24], this method cannot be extended to treat arbitrary intuitionistic Kripke frames, as there does not exist a right adjoint to the inclusion of $\mathbf{Pos}_p$ into $\mathbf{Pos}$ (see [BK24]). Thus, the question of how to represent arbitrary intuitionistic Kripke frames coalgebraically remains open.

We highlighted the contribution of our construction by deriving results that follow from coalgebraic completeness. Our basic theory lead naturally to a notion of bisimulation for Fischer Servi logic, which to our knowledge is novel. Furthermore, our construction of the free **IK**-algebra lays a foundation for several interesting lines of research. For instance, having shown that our approach subsumes rank-1 extensions of **IK**, one could investigate whether the free algebras for these logics are intuitionistic tense algebras, as is often the case with free modal algebras (see e.g. [Alm24]). It would be interesting to investigate in the future whether our approach can be modified to accommodate axioms of higher rank. Furthermore, one might expect that normal forms can be derived if one restricts to a locally tabular logic, where the $\mathcal{V}_G$ construction will terminate after finitely-many steps. We also expect that our approach can be extended to frame conditions of special interest, such as monadic intuitionistic propositional calculus (MIPC) (see e.g. [Bez98], [BZ97]) and intuitionistic S4 (see [Fis84], [PS86], [Sim94]). Overall, it is worth investigating whether our method provides a general recipe for dealing with such logics.

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