The Goldblatt-Thomason theorem for derivative spaces

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Abstract

The Goldblatt-Thomason theorem is a classic result of modal definability of Kripke frames. Its topological analogue for the closure semantics has been proved by ten Cate et al. (2009). In this paper we prove a version of the Goldblatt-Thomason theorem for topological semantics via the Cantor derivative. We work with derivative spaces which provide a natural generalisation of topological spaces on the one hand and of weakly transitive frames on the other.

Keywords: Modal definability, topological semantics, Cantor derivative, Alexandroff extensions.

1 Introduction

The Goldblatt-Thomason theorem [9] is one of the classic results in modal logic. It states that an elementary class of Kripke frames is modally definable iff it is closed under generated subframes, bounded morphic images, disjoint unions and reflects ultrafilter extensions. In [15] van Benthem gave a model theoretic proof of this theorem. Since then a number of Goldblatt-Thomason theorems have been proved in different contexts. Van Benthem provided a

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version of this result for finite Kripke frames [14]. Ten Cate [13] investigated Goldblatt-Thomason theorems for hybrid languages. Gabelaia [8] gave a topological version of this theorem for c-semantics (and [12] proved it for extended languages). The Goldblatt-Thomason theorem for coalgebraic modal logic was established in [10].

In this paper we prove the Goldblatt-Thomason theorem for the topological derived set semantcis (d-semantics). If in the topological c-semantics the modal diamond $\Diamond$ is interpreted as the closure c, in the derived set semantics the diamond $\Diamond$ is interpreted as the Cantor derivative d. Recall that in a topological space a derived set $d(A)$ of a set A consists of the point x such that for every open neighbourhood U_x of x the intersection $A \cap (U_x \setminus \{x\}) \neq \varnothing$. It is well known that the logic of all topological spaces for c-semantics is $\mathsf{S4}$ [11] and that the logic of all topological spaces for the d-semantics is the logic $\mathsf{wK4}$ of all weakly transitive frames (see, e.g., [16]), where a relation is weakly transitive if

$$\forall x \forall y \forall z (Rxy \wedge Ryz) \to (Rxz \vee x = z)).$$

Recall that the *ultrafilter extension* of a Kripke frame (X, R) is the ultrafilter frame of the modal algebra $(\mathcal{P}(X), \Diamond_R)$. In the case of topological c-semantics the role of ultrafilter extensions is played by the Alexandroff extensions [8,12]. The *Alexandroff extension* of a topological space (X, τ) is the Alexandroff space associated with the ultrafilter frame of the $\mathsf{S4}$-algebra $(\mathcal{P}(X), \mathsf{c})$. The topological version of the Goldblatt-Thomason theorem states that an elementary class (i.e., an L_t-definable class, see below) of topological spaces is modally definable in the c-semantics iff it is closed under open subspaces, interior images (i.e., images under continuous and open maps), topological sums and reflects Alexandroff extensions [8,12].

We define the *d-Alexandroff extension* of a topological space (X, τ) as the ultrafilter frame of the $\mathsf{wK4}$-algebra $(\mathcal{P}(X), d)$. However, while the Alexandroff extension of a topological spaces is an Alexandroff topological space, the d-Alexandroff extension of a topological space, may not be a topological space, which complicates the matter. In order to overcome this difficulty, instead of topological spaces we work with *derivative spaces* introduced in [6,1]. Derivative spaces generalize topological spaces on the one hand and weakly transitive Kripke frames on the other. We show that the d-Alexandroff extension of a derivative space is a derivative space (in fact, it is always a weakly transitive frame).

The original Goldblatt-Thomason theorem, as well as its topological variant, gives a characterization of modally definable classes that are elementary. We introduce an appropriate first-order language for studying derivative spaces. Similarly to the language $\mathcal{L}^2$ and its fragment $\mathcal{L}_t$ used to study model theory of topological spaces [7,12], the language $\mathcal{L}_2$ is a two sorted first-order language with two sorts of variables, where one ranges over points of the space and the other ranges over basic subsets.

We also work with d-analogues of generated subframes, p-morphic images and disjoint unions—d-subspaces, bounded morphic images and d-sums. Our main result (Theorem 5.2) states that an $\mathcal{L}_2$-definable class of derivative spaces is modally definable iff it is closed under d-subspaces, d-morphic images and d-sums and reflects the d-Alexandroff extensions.

For proving the result we use the model-theoretic approach of [14,12]. In order to do this we develop some model theory of derivative spaces. We introduce an equivalent presentation of derivative spaces, which we call *based spaces* as they resamble the presentation of topological spaces in terms of their bases. Based space presentation enables us to apply model-theoretic technique to derivative spaces more easily. We define ultraproducts and saturation of based spaces. Using based spaces as an equivalent presentation of derivative spaces and also seeing them as structures of $\mathcal{L}_2$, enable us to apply model-theoretic results, such as compactness, to our models. Utilizing this we show that any derivative space has a saturated ultrapower with a d-map from this ultrapower to the d-Alexandroff extension of the space. This is one of our main technical lemmas for proving the Goldblatt-Thomason theorem.

As a special case we obtain the Goldblatt-Thomason theorem for topological derivative spaces (Theorem 5.6) i.e., for derivative spaces that are topological spaces. Since the d-Alexandroff extension of a topological derivative space is not necessarily a topological derivative space, we consider a reflection from a wider class which contains topological spaces and weakly transitive Kripke frames. Then, we show that an $\mathcal{L}_2$-elementary class of topological spaces is modally definable iff it is closed under disjoint unions, open subspaces, d-morphic images and reflects weak transitive extensions.

Topological spaces with c-semantics satisfy all conditions of the Goldblatt-Thomason theorem, these spaces are definable over the class of all derivative spaces. Indeed, they are definable by the formula $p \to \Diamond p$. On the other hand, the class of all derivative spaces associated with weakly transitive Kripke frames is not modally definable over the class of all derivative spaces, since it does not reflect the d-Alexandroff extension. It also follows from our Goldblatt-Thomason theorem that the class of T_1-spaces is not modally definable over the class of all topological derivative spaces, while the class of T_d-spaces is definable.

2 Preliminaries: Derivative spaces and based spaces

In this section we review the main semantical structures used in the text: derivative spaces, and, specifically, their presentation as based spaces. Derivative spaces were introduced by Fernández-Duque and Iliev [6] as 'convergence spaces', in order to unify topological and Kripke semantics for the logic of the Cantor derivative. They were renamed *derivative spaces* by Baltag et al. [1]. These are a special case of the more general *derivative algebras* of Esakia [5]. Derivative spaces may moreover be presented as *based spaces,* similarly to how topological spaces may be presented in terms of a basis. The latter presentation will be the most convenient for us, since many classic results for first order logic can be readily applied to based spaces.

Derivative spaces are sets of points equipped with an operator satisfying the basic properties of the Cantor derivative, although this operator need not coincide with the 'true' derivative of a topological space.

Definition 2.1 A *derivative space* is a pair (X, d) where $X \neq \varnothing$ is a set of points and $d : \mathcal{P}(X) \to \mathcal{P}(X)$ is a derivative operator that satisfies the following conditions for all $A, B \subseteq X$:

(i) $d(\varnothing) = \varnothing$;

(ii) $d(A \cup B) = d(A) \cup d(B)$;

(iii) $d(d(A)) \subseteq A \cup d(A)$.

The dual of d, called the *co-derivative*, is defined as $\tilde{d}(A) = X - d(X - A)$, for each $A \subseteq X$.

The following two examples show that we can define two different derivative spaces over a given topological space. In other words, a derivative space is a generalized notion of a topological space.

Example 2.2 Let $\mathcal{X} = (X, \tau)$ be a topological space, then (X, c) is a derivative space, where c is the topological closure operator of τ. Any derivative space of this form is called a *topological closure space*. Note that in addition to the conditions of the derivative operator in Definition 2.1, the topological closure operator also satisfies $A \subseteq \mathsf{c}(A)$ and $\mathsf{c}(\mathsf{c}(A)) \subseteq \mathsf{c}(A)$ for all $A \subseteq X$.

In 1944, by interpreting the modal operator $\Diamond$ as the topological closure operator, McKinsey and Tarski introduced a topological semantics of modal logic [11], nowadays named c-semantics.

Example 2.3 Let $\mathcal{X} = (X, \tau)$ be a topological space. The Cantor derivative operator of $\mathcal{X}$, denoted by d_τ, is the operator that assigns to each subset of X the set of its limit points, i.e., for any $A \subseteq X$

$$d_\tau(A) = \{a \in X \mid \forall O \in \tau \ (a \in O \Rightarrow A \cap O - \{a\} \neq \varnothing)\}.$$

Then (X, d_τ) is a derivative space.

In [11], McKinsey and Tarski also noted that the modal operator $\Diamond$ interpreted as the Cantor derivative operator gives another topological semantics for modal logic. It is known as the d-semantics for modal logic. Since $\mathsf{c}(A) = A \cup d_\tau(A)$, the d-semantics is more expressive than c-semantics. For example, the class of T_0 spaces is not definable in c-semantics ([12, Cor. 37]), but it is definable in d-semantics ([2, Cor. 1]).

Examples 2.2 and 2.3 show that the class of derivative spaces contains the class of topological spaces, equipped with either the closure or Cantor derivative operators. On the other hand, any derivative space (X, d) can be seen as a topological space by defining the closure operator c_d as $\mathsf{c}_d(A) = A \cup d(A)$. We denote this induced topology by τ_d. Note that given an arbitrary derivative space (X, d), its derivative operator d does not necessarily coincide with the Cantor derivative of its induced topology τ_d. Precisely, for any $A \subseteq X$ we have

$$d(A) = d_{\tau_d}(A) \cup \mathrm{ref}(A), \tag{1}$$

where d_{τ_d} is the Cantor operator of τ_d and $\mathrm{ref}(A) = \{a \in A \mid a \in d(\{a\})\}$ (see Appendix for the proof). The elements of ref(A) are called *reflexive points.*

We call a subset $A \subseteq X$, *d-closed* if it is closed in τ_d, i.e., $d(A) \subseteq A$. So, $A \subseteq X$ is *d-open* if $A \subseteq \tilde{d}(A) = X - d(X - A)$.

Another special case of derivative spaces comes from weakly transitive Kripke frames. A Kripke frame (W, R) is weakly transitive if $(wRv \wedge vRz \Rightarrow w = z \vee wRz)$, for all $w, v, z \in W$.

Example 2.4 Let (W, R) be a weakly transitive Kripke frame. We obtain a derivative space (W, d_R) by defining, for any $A \subseteq W$,

$$d_R(A) = \{w \in W \mid wRs \text{ for some } s \in A\}.$$

Then, τ_{d_R} is the upset topology over (W, R), i.e., $O \subseteq W$ is open if for any $a \in O$ we have $R(a) = \{s \mid aRs\} \subseteq O$. Also, $\mathrm{ref}(A) = \{a \in A \mid aRa\}$. Therefore, if R is a weakly transitive and irreflexive relation, then d_R is the Cantor derivative operator of τ_R. Also, if R is a transitive and reflexive relation, then d_R is the topological closure operator of τ_R.

As mentioned above, the class of derivative spaces contains some interesting classes of structures, e.g., topological spaces and weakly transitive Kripke frames. On the other hand, the class of derivative spaces is a subclass of the class of monotonic neighbourhood structures.

Definition 2.5 A *neighbourhood derivative space* is a pair $(X, \mathcal{N})$, where $\mathcal{N} : X \to 2^{2^X}$ is a neighbourhood assignment that satisfies the following conditions for any $a \in X$:

(i) $X \in \mathcal{N}(a)$,

(ii) if $A \in \mathcal{N}(a)$ and $A \subseteq B$, then $B \in \mathcal{N}(a)$,

(iii) if $A, B \in \mathcal{N}(a)$, then $A \cap B \in \mathcal{N}(a)$,

(iv) if $a \in A \in \mathcal{N}(a)$, then $\{b \in X \mid A \in \mathcal{N}(b)\} \in \mathcal{N}(a)$.

In [1], it is shown that there is an equivalent presentation of derivative spaces as neighbourhood derivative spaces. For a given derivative space (X, d) and $a \in X$, let

$$\mathcal{N}_d(a) = \{A \subseteq X \mid a \notin d(X - A)\} = \{A \subseteq X \mid a \in \tilde{d}(A)\}.$$

We call the members of $\mathcal{N}_d(a)$ *d-neighbourhoods of* a. Note that every open neighbourhood of a with respect to the topology τ_d is a d-neighbourhood of a, but the converse is not true in general. For example, when d is the Cantor derivative of a topological space, then $A \in \mathcal{N}_d(a)$ if there is an open neighbourhood O of a such that $O - \{a\} \subseteq A$. In other words, A is d-open if for any $a \in A$ we have $A \in \mathcal{N}_d(a)$.

Conversely, for a given derivative neighbourhood space $(X, \mathcal{N})$, we can define an operator d over $\mathcal{P}(X)$ as follows:

$$d(A) = \{a \in X \mid \forall O \in \mathcal{N}(a)\ O \cap A \neq \varnothing\}.$$

Then (X, d) is a derivative space.

In addition to this equivalent presentation of derivative spaces, we can also consider another one which plays a similar role as bases for topological spaces relative to derivative spaces.

Definition 2.6 Let $\mathcal{B} : X \to 2^{2^X}$ be a function over $X \neq \varnothing$. Then $\mathcal{B}$ is called a *basic neighbourhood assignment* if for each $a \in X$ the following conditions hold:

(i) $\mathcal{B}(a) \neq \varnothing$,

(ii) if $A, B \in \mathcal{B}(a)$, then there is $C \in \mathcal{B}(a)$ such that $C \subseteq A \cap B$,

(iii) if $A \in \mathcal{B}(a)$ and $b \in A$, then there is $B \in \mathcal{B}(b)$ such that $B \subseteq A \cup \{a\}$.

Any set $A \in \mathcal{B}(a)$ is called a *basic d-neighbourhood* of a. Also, $\bigcup_{a \in X} \mathcal{B}(a)$ is called a *derivative base* (or set of basic d-neighbourhoods) of X, and $(X, \mathcal{B})$ is called a *based space.*

Note that basic neighbourhood assignments need not be monotone (i.e., closed under supersets), but this does not affect their modal logic, as one may obtain an equivalent monotone structure from them.

Lemma 2.7 *Let $\mathcal{B}$ be a basic neighbourhood assignment over X. For each $a \in X$, let $\mathcal{N}_B(a)$ be the closure of $\mathcal{B}(a)$ under supersets. Then $\mathcal{N}_B$ is a derivative neighbourhood assignment.*

Moreover, the function $d_B : \mathcal{P}(X) \to \mathcal{P}(X)$ defined as

$$d_B(A) = \{a \in X \mid \forall O \in \mathcal{B}(a)\ O \cap A \neq \varnothing\}$$

is a derivative operator, and $\mathcal{N}_{d_B} = \mathcal{N}_B$.

Likewise, we may readily obtain a based space from a given derivative space by assigning to each point its set of 'punctured neighbourhoods'.

Lemma 2.8 *Let (X, d) be a derivative space and $\mathcal{B}_d : X \to 2^{2^X}$ be an operator which assigns to each $a \in X$ the set of its punctured d-neighbourhoods, i.e.,*

$$\mathcal{B}_d(a) = \{O \subseteq X \mid a \in \tilde{d}(O)\ \&\ O \subseteq \tilde{d}(O \cup \{a\})\}.$$

Then $\mathcal{B}_d$ is a basic neighbourhood assignment with $d_B = d$.

So, we can consider based spaces as an equivalent presentation of derivative spaces. In other words, we can identify a class K of derivative spaces as a class K' of based spaces $(X, \mathcal{B})$ such that $(X, d_B) \in K$. There are various technical advantages to working with based spaces.

Example 2.9 Assume that (X, τ) is a topological space and σ is a basis for τ. Then the function $\mathcal{B}_1$ defined as $\mathcal{B}_1(a) = \{O \in \sigma \mid a \in O\}$ for each $a \in X$, is a basic neighbourhood assignment of X and $d_{B_1} = \mathsf{c}$. Also the function $\mathcal{B}_2$ defined as $\mathcal{B}_2(a) = \{O - \{a\} \mid a \in O \in \sigma\}$ is a basic neighbourhood assignment with $d_{B_2} = d_\tau$.

3 Model theory of derivative spaces

Our main focus in this paper is on the basic modal language. Let $\mathbb{P}$ be a countable set of propositional variables. Modal formulas are constructed recursively from $\mathbb{P}$ using Boolean connectives and modal operators $\Diamond, \Box$.

Definition 3.1 A based model is a triple $\mathfrak{M} = (X, \mathcal{B}, [\![\cdot]\!])$ where $\mathcal{X} = (X, \mathcal{B})$ is a based space and $[\![\cdot]\!] : \mathbb{P} \to \mathcal{P}(X)$ is a valuation function.

The satisfaction of formulas is defined by structural induction. For propositional variables and Boolean connectives, we have the standard definitions, and for modal operator we have

$$\mathfrak{M}, a \models \Diamond\varphi \text{ iff } A \cap [\![\varphi]\!] \neq \varnothing \text{ for all } A \in \mathcal{B}(a).$$

Then we have $\mathfrak{M}, a \models \Box\varphi$ iff there exists $A \in \mathcal{B}(a)$ such that $A \subseteq [\![\varphi]\!]$.

In other words, for any derivative model $\mathfrak{M} = (X, d, [\![\cdot]\!])$ we have

$$\mathfrak{M}, a \models \Diamond\varphi \text{ iff } a \in d([\![\varphi]\!]),$$

and thus, $\mathfrak{M}, a \models \Box\varphi$ iff $a \in \tilde{d}([\![\varphi]\!])$.

3.1 Corresponding language

The original Goldblatt-Thomason theorem [9] provides a characterization of modally definable conditions for classes of Kripke frame that are elementary, i.e., definable in the corresponding first-order language. Ten Cate et al. [12] also prove a version of the Goldblatt-Thomason theorem for those topological closure spaces that are definable in a suitable corresponding language $\mathcal{L}_t$. In this section, we introduce a first-order language which is appropriate for studying derivative spaces. This language is interpreted over based neighbourhood spaces as an equivalent presentation of derivative spaces.

Let $\mathcal{L}_2$ be a two-sorted first-order language. The first sort ranges over the set of points of a space, and we denote its variables by $x, y, \ldots$. The second sort ranges over basic subsets of the space, and we denote its variables by $U, V, \ldots$. The language $\mathcal{L}_2$ contains two binary relations ε and ν, where ε relates point variables with basic subset variables and ν relates basic subset variables with point variables. Also, $\mathcal{L}_2$ contains a unary predicate P_p for each proposition p. The formulas of $\mathcal{L}_2$ are defined as follows: [3]

$$\varphi ::= x = y \mid x \,\varepsilon\, U \mid U \,\nu\, x \mid P_p(x) \mid \neg\varphi \mid \varphi \wedge \varphi \mid \exists x\varphi \mid \exists U\varphi.$$

Any based model $\mathfrak{M} = (X, \mathcal{B}, [\![\cdot]\!])$ can be seen as an $\mathcal{L}_2$-structure

$$(X, \bigcup_{a \in X} \mathcal{B}(a), \varepsilon^{\mathfrak{M}}, \nu^{\mathfrak{M}}, \{P_p \mid p \in \mathbb{P}\}),$$

where $a \,\varepsilon^{\mathfrak{M}}\, O$ means that $a \in O$ and $O \,\nu^{\mathfrak{M}}\, a$ means that $O \in \mathcal{B}(a)$.

Clearly, not every $\mathcal{L}_2$-structure is a based model. However, the class of basic neighbourhood models is characterized by an $\mathcal{L}_2$-theory.

[3] The language $\mathcal{L}_2$ is similar to the language $\mathcal{L}^2$, introduced for studying topological spaces [7], except that it has an extra predicate ν.

Assume that $\mathcal{M} = (D_1, D_2, \varepsilon^{\mathcal{M}}, \nu^{\mathcal{M}}, \{P_p \mid p \in \mathbb{P}\})$ is an $\mathcal{L}_2$-model. For any $A \in D_2$, let $\|A\| = \{a \in D_1 \mid a \; \varepsilon^{\mathcal{M}} \; A\}$.

Lemma 3.2 *There exists a finite set of $\mathcal{L}_2$-formulas Γ_{basic} such that if $\mathfrak{M} = (D_1, \mathcal{B}, [\![\cdot]\!])$ is any $\mathcal{L}_2$-model such that $\mathcal{M} \models \Gamma_{basic}$, then $\mathfrak{M}$ is a based model with a basic neighbourhood assignment $\mathcal{B}(a) = \{\|A\| \mid A \in D_2 \;\&\; A \; \nu^{\mathcal{M}} \; a\}$.*

Then the function d defined over $\mathcal{P}(D_1)$ as

$$d(A) = \{a \in D_1 \mid \forall O \in D_2 \; (O \; \nu^{\mathcal{M}} \; a \Rightarrow \|O\| \cap A \neq \varnothing)\},$$

is a derivative operator with $d_{\mathcal{B}} = d$.

Just as the standard translation interprets modal formulas with their Kripke semantics as first order formulas, we may interpret the modal language in $\mathcal{L}_2$ with respect to its derivational semantics.

Definition 3.3 [Translation] Given a designated first order variable x, we recursively define a translation Tr_x from modal formulas to $\mathcal{L}_2$-formulas as follows:

$$\begin{aligned} Tr_x(p) &= P_p(x) \\ Tr_x(\neg\varphi) &= \neg Tr_x(\varphi) \\ Tr_x(\varphi \wedge \psi) &= Tr_x(\varphi) \wedge Tr_x(\psi) \\ Tr_x(\Diamond\varphi) &= \forall U \; (U \; \nu \; x \rightarrow \exists y(y \; \varepsilon \; U \wedge Tr_y(\varphi))) \\ Tr_x(\Box\varphi) &= \exists U(U \; \nu \; x \wedge \forall y(y \; \varepsilon \; U \rightarrow Tr_y(\varphi))) \end{aligned}$$

Proposition 3.4 *For any based model $\mathfrak{M}$ and any modal formula φ we have*

$$\mathfrak{M}, a \models \varphi \textit{ iff } \mathfrak{M} \models Tr_x(\varphi)[a].$$

Remark 3.5 In the translation of modal formulas, the quantifiers over basic subset variables appear in restricted forms. Let $\mathcal{L}_d$ be a language obtained by restricting the use of quantifiers over set variables of $\mathcal{L}_2$ as follows:

- $\exists U(U \; \nu \; x \wedge \alpha)$, where α is negative in U,
- $\forall U(U \; \nu \; x \rightarrow \alpha)$, where α is positive in U.

Then one can easily see that $\mathcal{L}_d$ is invariant under basic d-neighbourhoods, i.e., the satisfaction of $\mathcal{L}_d$-formulas is independent of interpreting $U \; \nu \; x$ as 'U is a d-neighbourhood of x' or as 'U is a *basic* d-neighbourhood of x'. [4]

3.2 Ultraproducts

Ultraproducts are an essential tool in estabilshing the original Goldblatt-Thomason theorem, as well as its topological variant; they will also be used throughout our own proof. In this section, we review their definitions and basic properties.

Let I be a non-empty set. A set $\mathcal{D} \subseteq \mathcal{P}(I)$ is a *filter* over I, if $I \in \mathcal{D}$ and $\mathcal{D}$ is closed under finite intersections and supersets. A filter $\mathcal{D}$ is called

[4] Similarly, over the class of topological spaces, the language $\mathcal{L}_t$ is introduced as a fragment of $\mathcal{L}^2$ which is invariant under topological bases, see [7,12].

an *ultrafilter* if for all $A \subseteq I$, either $A \in \mathcal{D}$ or $I - A \in \mathcal{D}$. The ultrafilter theorem states that any subset of $\mathcal{P}(I)$ with the finite intersection property can be extended to an ultrafilter over I; in other words, if $\mathcal{D}$ is a filter such that whenever $X_1, \ldots, X_n \in \mathcal{D}$, it follows that $X_1 \cap \ldots \cap X_n \neq \varnothing$, then there is an ultrafilter $\mathcal{U} \supseteq \mathcal{D}$.

Let $(X_i : i \in I)$ be a family of non-empty sets and $\prod_{i\in I} X_i$ be the Cartesian product of this family, i.e., $\prod_{i\in I} X_i = \{(a_i)_{i\in I} | a_i \in X_i\}$. Two elements $(a_i)_{i\in I}, (b_i)_{i\in I} \in \prod_{i\in I} X_i$ are $\mathcal{D}$-equivalent, denoted by $(a_i)_{i\in I} \sim_{\mathcal{D}} (b_i)_{i\in I}$, if $\{i \in I | a_i = b_i\} \in \mathcal{D}$. It is clear that $\sim_{\mathcal{D}}$ is an equivalence relation. We denote the equivalence class of $(a_i)_{i\in I}$ by $[(a_i)]$. Let $\prod_{\mathcal{D}} X_i$ be the set of all equivalence classes.

Assume $(\mathcal{X}_i : i \in I)$ is a family of based spaces, and $\mathcal{D}$ is an ultrafilter over I. Define the function $\mathcal{B}_{\mathcal{D}}$ over $\prod_{\mathcal{D}} X_i$ as

$$\mathcal{B}_{\mathcal{D}}([(a_i)]) = \{\prod_{\mathcal{D}} U_i \mid \{i \in I \mid U_i \in \mathcal{B}_i(a_i)\} \in \mathcal{D}\},$$

where $\mathcal{B}_i$ is a basic neighbourhood assignment of X_i. Then one can easily see that $(\prod_{\mathcal{D}} X_i, \mathcal{B}_{\mathcal{D}})$ is a based space.

Definition 3.6 The *d-ultraproduct* of a family of based models $(\mathfrak{M}_i : i \in I)$ is a model $\prod_{\mathcal{D}} \mathfrak{M}_i = (\prod_{\mathcal{D}} X_i, \mathcal{B}_{\mathcal{D}}, [\![\cdot]\!]_{\mathcal{D}})$, where $[\![p]\!]_{\mathcal{D}} = \prod_{\mathcal{D}} [\![p]\!]_i$ for any proposition $p \in \mathbb{P}$.

If $\mathfrak{M}_i = \mathfrak{M}$ for each $i \in I$, then $\prod_{\mathcal{D}} \mathfrak{M}_i$ is called an *ultrapower* of $\mathfrak{M}$. We denote by $\hat{a}$ the class $[(a_i)]$ where $a_i = a$ for each $i \in I$.

Example 3.7 If $(\mathcal{X}_i : i \in I)$ is a family of topological spaces (X_i, σ_i) where σ_i is a topological base, then $\mathcal{B}_{\mathcal{D}}$ is a topological base over $\prod_{\mathcal{D}} X_i$ (see Definition 15 in [12]).

Proposition 3.8 *For any family of based models* $(\mathfrak{M}_i : i \in I)$ *and for any modal formula* φ*, we have*

$$\prod_{\mathcal{D}} \mathfrak{M}_i, [(a_i)] \models \varphi \quad \textit{iff} \quad \{i \in I \mid \mathfrak{M}_i, a_i \models \varphi\} \in \mathcal{D}.$$

Since the ultraproduct of based spaces is a based space, the Łoś Theorem applied to the two-sorted first order language $\mathcal{L}_2$-formulas implies that $\mathcal{L}_2$ has the compactness property over that class of based models.

3.3 d-Saturation

The next key ingredient in a proof for the Goldblatt-Thomason theorem is saturation; essentially, a structure is saturated if any set $\Gamma(x)$ of formulas with one free variable whose finite subsets are satisfied on some point of the structure is uniformly satisfied on a single point. By modifying the notion of $\mathcal{L}_t$-saturation introduced in [12] for topological closure spaces, we provide an appropriate notion for derivative spaces.

Definition 3.9 [Derivative Saturated] Let $\mathfrak{M} = (X, \mathcal{B}, [\![\cdot]\!])$ be a based model. A subset $A \subseteq X$ is *point-saturated* if for any $\mathcal{L}_2$-type $\Gamma(x)$, i.e., a set of $\mathcal{L}_2$-formulas with one point free variable x and without any basic subset variable, we have $\Gamma(x)$ is satisfiable in A provided that it is finitely satisfiable in A. A based model $(X, \mathcal{B}, [\![\cdot]\!])$ is *derivative saturated* (or *d-saturated* for short) if it satisfies the following conditions:

(i) X is point-saturated.

(ii) For any $a \in X$, any $O \in \mathcal{B}(a)$ is point-saturated.

(iii) For any $a \in X$ there exists $O_a \in \mathcal{B}(a)$ such that for any formula $\varphi(x)$ which is true in all members of some neighbourhood of a, is true over all members of O_a.

Proposition 3.10 *For any based model $\mathfrak{M}$, there is an ultrafilter $\mathcal{D}$ such that $\prod_{\mathcal{D}} \mathfrak{M}$ is d-saturated.*

The proof is given in the Appendix.

4 Definability

Our main goal in this paper is to study modal definability conditions for classes of derivative spaces. As mentioned above, based spaces provide an equivalent presentation of derivative spaces, and we investigate the definability conditions of based spaces. This will allow us to have more flexibility on the corresponding language, as well as granting us access to standard techniques for first order logic. More precisely, for any given class $\mathcal{C}$ of derivative spaces, we identify $\mathcal{C}$ with the class $\mathcal{C}'$ of based spaces $(X, \mathcal{B})$ such that (X, d_B) belongs to $\mathcal{C}$.

Definition 4.1 Let $\mathcal{C}$ be a class of based spaces. A class $K \subseteq \mathcal{C}$ is *modally definable over* $\mathcal{C}$ if there is a set of modal formulas Σ such that for any based space $\mathcal{X} \in \mathcal{C}$ we have $\mathcal{X} \in K$ iff $\mathcal{X} \models \Sigma$.

A class K is *modally definable* if it is modally definable over the class of all based spaces.

Example 4.2 For a based space $\mathcal{X} = (X, \mathcal{B})$, let $\sigma = \bigcup_{a \in X} \mathcal{B}(a) \cup \{\varnothing\}$. Then $\mathcal{X} \models p \to \Diamond p$ iff σ is a topological base and $d_{\mathcal{B}}$ is the closure operator over the topology generated by σ. Note that σ is a topological base iff for any $a \in X$ and $A \in \mathcal{B}(a)$ we have $a \in A$. In other words, the class of all topological closure spaces, i.e., derivative spaces (X, d) for which $d = \mathsf{c}$, is defined by $p \to \Diamond p$.

As mentioned in Example 2.3, the d-semantics is more expressive than the c-semantics. The following is another example witnessing the more expressive power of d-semantics.

Example 4.3 A topological space (X, τ) is a T_d-space if every $x \in X$ is an intersection of an open and a closed set; equivalently if $d_\tau(A)$ is closed for all $A \subseteq X$. The class of T_d-spaces is not definable in c-semantics (see [12, Cor. 37]). But, it is definable over the class of all topological derivative spaces by $\Box p \to \Box\Box p$ [2]. More generally, this formula defines the class of all derivative spaces with the property that $d(d(A)) \subseteq d(A)$ for each $A \subseteq X$.

Example 4.4 It is easy to see that the Löb formula $\Box(\Box p \to p) \to \Box p$ is valid on derivative space $\mathcal{X} = (X, d)$ iff $d(A) = d(A - d(A))$, for any $A \subseteq X$. Also, for any topological space $\mathcal{X}$ we have $\mathcal{X}$ is scattered iff $d_\tau(A) = d_\tau(A - d_\tau(A))$ for any $A \subseteq X$. Thus, for any derivative space $\mathcal{X} = (X, d)$, if (X, τ_d) is scattered and $d_{\tau_d} = d$, then the Löb formula is valid on $\mathcal{X}$ (see, e.g., [16]).

We can also show the converse of that, i.e., if the Löb formula is valid on the derivative space $\mathcal{X}$, then (X, τ_d) is scattered and $d_{\tau_d} = d$. So the class of all scattered spaces is definable by the Löb formula over the class of topological derivative spaces.

4.1 Invariant Results

Any modally definable class must be invariant over any operation that preserves modal formulas. We identify three constructions over based spaces (resp. derivative spaces) that have this property; our Goldblatt-Thomason theorem will then state that these conditions precisely characterize modal definability.

Definition 4.5 Let $(\mathcal{X}_i : i \in I)$ be a family of disjoint based spaces. The *d-sum* of this family is a space $\biguplus_{i \in I} \mathcal{X}_i = (X, \mathcal{B})$, where $X = \biguplus_{i \in I} X_i$ and $\mathcal{B} : X \to 2^{2^X}$ is defined by $\mathcal{B}(a) = \mathcal{B}_i(a)$, for $a \in X_i$.

Proposition 4.6 *Let $(\mathcal{X}_i : i \in I)$ be a family of disjoint based spaces. Then for any formula φ, we have $\biguplus_{i \in I} \mathcal{X}_i \models \varphi$ iff $\mathcal{X}_i \models \varphi$ for all $i \in I$.*

Example 4.7 The class of finite based spaces is not modally definable, since an infinite sum of finite spaces will validate any formula valid on finite spaces. Similarly, the class of all derivative spaces with a finite derivative operator, i.e., with a derivative operator which assigns to each subset a finite subset, is not modally definable.

Definition 4.8 Let $\mathcal{X}$ be a based space. A *d-open subspace* of $\mathcal{X}$ is a space $\mathcal{O} = (O, \mathcal{B}_O)$, where O is a d-open subset of X (i.e. for any $a \in O$ there is $O' \in \mathcal{B}(a)$ such that $O' \subseteq O$) and $\mathcal{B}_O(a) = \{A \cap O | A \in \mathcal{B}(a)\}$, for any $a \in O$.

For a weakly transitive Kripke frame (X, R) with $\mathcal{B}_R(a) = \{R(a)\}$, a d-open subspace is a generated subframe. Also, for a based space $(X, \mathcal{B})$, if $\sigma = \bigcup_{a \in X} \mathcal{B}(a) \cup \{\varnothing\}$ is a topological base, then a d-open subspace is a topologically open subspace.

Proposition 4.9 *Let O be a d-open subset of a based space $\mathcal{X}$. Then $\mathcal{X} \models \varphi$ implies that $\mathcal{O} \models \varphi$, for any formula φ.*

Example 4.10 The class of all based spaces with some point $a \in X$ such that $\varnothing \in \mathcal{B}(a)$ is not modally definable. To see this consider a based space $\mathcal{X} = (\{w_0, w_1, w_2\}, \mathcal{B}_R)$ with $R = \{(w_1, w_2), (w_2, w_2)\}$ and its d-open subspace $\mathcal{Y}$ over $\{w_1, w_2\}$.

Definition 4.11 Let $(X, \mathcal{B})$ and $(X', \mathcal{B}')$ be two based spaces. The function $f : X \to X'$ is a *d-morphism* if

(i) for any $A \in \mathcal{B}(a)$, there exists $A' \in \mathcal{B}'(f(a))$ such that for any $b' \in A'$ there is $b \in A$ with $f(b) = b'$,

(ii) for any $A' \in \mathcal{B}'(f(a))$, there exists $A \in \mathcal{B}(a)$ such that for any $b \in A$ there is $b' \in A'$ with $f(b) = b'$.

If f is a surjective d-morphism, then we say that $\mathcal{X}'$ is a *d-morphic image* of $\mathcal{X}$.

It can easily be checked that a function $f : X \to X'$ is a d-morphism between two derivative spaces (X, d) and (X', d') if $f^{-1}(d'(A')) = d(f^{-1}(A'))$, for any $A' \subseteq X'$.

As usual, validity of formulas is preserved under d-morphic images.

Proposition 4.12 *If $\mathcal{X}'$ is a d-morphic image of $\mathcal{X}$, then $\mathcal{X} \models \varphi$ implies $\mathcal{X}' \models \varphi$.*

We may also use d-morphisms to identify open subspaces.

Proposition 4.13 *$\mathcal{O} = (O, d_O)$ is a d-open subspace of $\mathcal{X} = (X, d)$, iff $O \subseteq X$ and the inclusion function $i : O \to X$ is a d-morphism.*

Example 4.14 The classes of T_1 and T_2-spaces are not definable over the class of all topological derivative spaces. To see this, let f be a function from the ordinal ω^2 equipped with the interval topology to the Sierpiński space, i.e., the space $(\{0,1\}, \{\varnothing, \{1\}, \{0,1\}\})$, defined as $f(\omega.k) = 0$ for each $0 < k < \omega$ and $f(x) = 1$ for others.

In the Goldblatt-Thomason theorem for Kripke semantics, there is another important construction, namely, the ultrafilter extension, which reflects the validity of modal formulas. In [12] a similar construction is introduced for c-semantics of topological spaces, named Alexandroff extension. Now, we define an analogous notion for derivative spaces.

Let $\mathcal{X} = (X, \mathcal{B})$ be a based space and X^* be the set of all ultrafilters over X. Recall that for each $A \subseteq X$, we have $d_{\mathcal{B}}(A) = \{a \mid \forall B \in \mathcal{B}(a)\ B \cap A \neq \varnothing\}$. Define a binary relation R^* over X^* as follows:

$$\mathfrak{u} R^* \mathfrak{u}' \text{ iff } A \in \mathfrak{u}' \text{ implies } d_{\mathcal{B}}(A) \in \mathfrak{u} \text{ for any } A \subseteq X.$$

Lemma 4.15 *R^* is a weakly transitive relation over X^*.*

The proof of this lemma is given in the Appendix. Note that, in general, (X^*, R^*) is not irreflexive. To see this, consider the topological derivative space of $(\mathbb{N}, \tau)$ where $\tau = \{\varnothing, \text{ all co-finite sets}\}$. Then for any $A \subseteq \mathbb{N}$, we have

$$d(A) = \begin{cases} \varnothing & A \text{ is finite} \\ \mathbb{N} & A \text{ is infinite} \end{cases}$$

Thus, any non-principal ultrafilter in $\mathbb{N}^*$ is reflexive since each of its members is infinite. Indeed, any non-principal ultrafilter is an R^*-successor of all members of $\mathbb{N}^*$.

Definition 4.16 [d-Alexandroff extension] Let $\mathcal{X} = (X, \mathcal{B})$ be a based space. The *d-Alexandroff extension* of $\mathcal{X}$ is the space $\mathcal{X}^* = (X^*, \mathcal{B}^*)$, where $\mathcal{B}^*(\mathfrak{u}) = \{R^*(\mathfrak{u})\}$, for each $\mathfrak{u} \in X^*$.

In other words, for any derivative space $\mathcal{X} = (X, d)$, the *d-Alexandroff extension* of $\mathcal{X}$ is the space $\mathcal{X}^* = (X^*, d^*)$ where

$$d^*(A) = \{\mathfrak{u} \mid \exists \mathfrak{u}' \in A \ s.t \ \mathfrak{u} R^* \mathfrak{u}'\},$$

for any $A \subseteq X^*$.

Example 4.17 For any weakly transitive Kripke frame (W, R), its d-Alexandroff extension is equal to its ultrafilter extension (W^*, R^{ue}).

Note that the d-Alexandroff extension of a derivative space (X, d) is the derivative space corresponding to the ultrafilter frame of the wK4-algebra $(\mathcal{P}(X), d_\tau)$ (see [3]).

Example 4.18 For any topological closure space (X, τ), its d-Alexandroff extension is equal to its Alexandroff extension defined in [12].

But for any topological derivative space, its d-Alexandroff extension is not necessarily a topological derivative space. In other words, the class of all topological derivative spaces is not closed under d-Alexandroff extensions.

Proposition 4.19 *For any based space $\mathcal{X}$ we have, $\mathcal{X}^* \models \varphi$ implies $\mathcal{X} \models \varphi$.*

The proof of the proposition is given in the Appendix.

Recall that a topological space (X, τ) is Alexandroff if every point has a minimal open neighbourhood. The Alexandroff extension of any topological closure space, is an Alexandroff space, and this implies that the class of Alexandroff spaces is not definable in the c-semantics (see Corollary 41 in [12]). We can extend this notion to derivative spaces. A derivative space (X, d) is called *Alexandroff* if every point has a minimal basic d-neighbourhood.

Example 4.20 The class of all Alexandroff derivative spaces is not modally definable. For any arbitrary derivative space $\mathcal{X}$, even for non-Alexandroff one, $\mathcal{X}^*$ is Alexandroff, since $R^*(\mathfrak{u})$ is the only element of $\mathcal{B}^*(\mathfrak{u})$, for all $\mathfrak{u} \in X^*$.

5 The Goldblatt-Thomason Theorem

We are now ready to prove our main result. We begin with the following useful fact; the proof can be found in the Appendix.

Proposition 5.1 *For any based space $\mathcal{X}$, there exists a based ultrapower $\prod_{\mathcal{D}} \mathcal{X}$ with a surjective d-morphism $f : \prod_{\mathcal{D}} X \to X^*$.*

With this, we are ready to prove our main result.

Theorem 5.2 *Let K be an $\mathcal{L}_2$-elementary class of based spaces. Then, K is modally definable iff K is closed under d-sums, d-open subspaces, d-morphisms and reflects d-Alexandroff extensions.*

Proof. The left-to-right direction is obvious by the results in the above part. For the other direction, let $Log(K) = \{\varphi \mid K \models \varphi\}$. We show that for any based space $\mathcal{X}$ we have $\mathcal{X} \in K$ iff $\mathcal{X} \models Log(K)$. Clearly, $Log(K)$ is valid on any space in K. Now suppose that $\mathcal{X} \models Log(K)$. Consider a language containing a propositional variable p_A for any $A \subseteq X$, and let $\mathfrak{M} = (\mathcal{X}, [\![\cdot]\!])$ where $[\![\cdot]\!]$ is a natural valuation in this language, i.e., $[\![p_A]\!] = A$ for any $A \subseteq X$. Now take Δ as a theory containing all the formulas in the following form for any $A, B \subseteq X$:

$$\begin{aligned} P_{A^c} &\leftrightarrow \neg P_A \\ p_{A\cap B} &\leftrightarrow p_A \wedge p_B \\ p_{d(A)} &\leftrightarrow \Diamond p_A \\ p_{\tilde{d}(A)} &\leftrightarrow \Box p_A \end{aligned}$$

Then $\mathfrak{M} \models \Delta$.

Claim. For any $a \in X$ there exists a model $\mathfrak{N}_a$ based on some $\mathcal{Y}_a \in K$ and $b \in Y_a$ such that $\mathfrak{N}_a \models \Delta$ and $\mathfrak{N}_a, b \models p_{\{a\}}$.

Proof of the claim: Suppose that $a \in X$. Let

$$\Delta_a = \{\Box\varphi \wedge \varphi \mid \varphi \in \Delta\} \cup \{p_a\}.$$

Δ_a is finitely satisfiable in K, since otherwise $\neg\delta \in Log(K)$ for some finite subset δ of Δ_a, which is a contradiction with $\mathfrak{M} \models Log(K)$.

Since K is an elementary class, it is closed under ultraproducts, so we can assume that there is a d-saturated model $\mathfrak{N}$ based on some $\mathcal{Y} \in K$ and $b \in Y$ such that $\mathfrak{N}, b \models \Delta_a$. By d-saturation, b has a d-neighbourhood $U_b \in \mathcal{B}(b)$ such that φ holds throughout U_b for all $\varphi \in \Delta$. Let $\mathfrak{N}_a$ be a d-open subspace of $\mathfrak{N}$ generated by $O = U_b \cup \{b\}$, and this completes the proof of the claim.

Let $\mathfrak{N}$ be the ω-saturated ultrapower of $\biguplus_{a\in X} \mathfrak{N}_a$. By the closure of K under d-sums and d-ultraproducts, $\mathcal{Y} = \prod_{\mathcal{D}} \uplus_{a\in X} \mathcal{Y}_a \in K$.

Now, by Proposition 5.1, we know that there is a d-morphism from $\mathcal{Y}$ to $\mathcal{X}^*$. The closure of K under d-morphisms implies that $\mathcal{X}^* \in K$. So $\mathcal{X} \in K$ since K reflects d-Alexandroff extensions. $\Box$

As mentioned above, for any derivative space (X, d), the frame (X^*, R^*) is not necessarily irreflexive. Specially, for any topological derivative space, (X^*, R^*) is not necessary irreflexive, so (X^*, d^*) is not a topological derivative space. Thus, for giving a version of the Goldblatt-Thomason theorem for such classes of structures, i.e., for those that are not closed under the d-Alexandroff extension, we consider the following definition.

Definition 5.3 Let K be a class of derivative spaces. We say that K *reflects d-Alexandroff images* whenever for some derivative space $\mathcal{Y} \in K$, if $\mathcal{X}^*$ is a d-morphic image of $\mathcal{Y}$, then $\mathcal{X}$ is in K.

Theorem 5.4 *Let K be an $\mathcal{L}_2$-elementary class of derivative spaces. Then K is modally definable iff K is closed under d-sums, d-open subspaces, d-morphisms and reflects d-Alexandroff images.*

Specifically for the class of topological derivative spaces, we can also express the above theorem in another way based on the following definition.

Definition 5.5 Suppose that K is a class of topological spaces. Let K^+ be the class of all Kripke frames that are d-morphic images of some elements of

K. We say that a class K of topological spaces *reflects the weak transitive extensions* from K^+, whenever for any topological space (X, τ) if (X^*, R^*) is in K^+, then (X, τ) is in K.

Corollary 5.6 *Let K be an $\mathcal{L}_2$-elementary class of topological spaces. K is modally definable iff it is closed under disjoint unions, open subspaces, and d-morphic images and reflects weak transitive extensions from K^+.* [5]

6 Concluding Remarks

In this paper we give a version of the Goldblatt-Thomason theorem for derivative spaces. There are some lines of research that can be considered for future work.

One of the natural further directions is to investigate definability for extended languages which provide the ability to define more properties. For example one can show that the class of all topological derivative spaces is definable by the hybrid formula $@_i \neg \Diamond i$ while it is not definable in the basic language. Utilising the method used for topological c-semantics in [12], one can extend our results to extended languages such as modal logic with universal modality and hybrid logics.

First-order modal logic (FML) is another extension of modal logic. There is some work on definability in the context of FML, e.g. [17,18]. For example, [18] gives a version of the Goldblatt-Thomason FML for Kripke frames. It uses a well-known technique from classical model theory called Morleyization (or atomization). Providing a version of the Goldblatt-Thomason theorem for FML with respect to topological semantics is an interesting future work.

The other challenging extension of modal logic is the modal μ-calculus ($\mu\mathsf{ML}$)—modal logic enriched with fixed point operators. There are many obstacles for this logic, for example, $\mu\mathsf{ML}$ does not enjoy the compactness property, which plays a significant role in proving the Goldblatt-Thomason theorem. To tackle this problem, one might need to consider simpler extensions such as modal languages with tangled operator.

This also suggests a line of study of general definability characterization theorems, for classes of structures that are not necessarily elementary. There are many important examples of classes of Kripke frames or topological spaces that are definable, but not elementary. For example, the class of scattered spaces (Example 4.4) is definable by Löb's formula, but is not elementary.

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Appendix

In this Appendix we prove some results from the main text. We begin with the proof of Equation 1 in Section 2, which states that for any $A \subseteq X$ we have

$$d(A) = d_{\tau_d}(A) \cup \mathrm{ref}(A),$$

where d_{τ_d} is the Cantor operator of τ_d and $\mathrm{ref}(A) = \{a \in A \mid a \in d(\{a\})\}$.

First assume that $a \notin d_{\tau_d}(A) \cup \mathrm{ref}(A)$. Since $a \notin d_{\tau_d}(A)$, there is an open $O \in \tau_d$ such that $a \in O$ and $O \cap A - \{a\} = \varnothing$. By the second condition of d, we have $d(A) = d(A \cap O) \cup d(A \cap O^c)$. Since $a \in O = Int_d(O)$, we have $a \notin d(O^c) \subseteq O^c$, thus $a \notin d(A \cap O^c)$. Furthermore, if $a \in A$, then $A \cap O = \{a\}$ and $a \notin \mathrm{ref}(A)$ implies that $a \notin d(A \cap O)$. If $a \notin A$, then $A \cap O = \varnothing$, and thus $a \notin d(A \cap O)$. Therefore, $a \notin d(A)$.

For the other direction, first assume that $a \in \mathrm{ref}(A)$. Then $a \in A$ and $a \in d(\{a\})$ which implies that $a \in d(A)$. Now assume that $a \notin \mathrm{ref}(A)$ and $a \notin d(A)$. If $a \notin A$, then $a \in A^c \cap d^c(A) = Int_d(A^c) = O \in \tau_d$. Thus, $a \notin d_{\tau_d}(A)$, because $O \cap A - \{a\} = \varnothing$. If $a \in A$, then $a \notin d(\{a\})$. Let $O = Int_d(A^c \cup \{a\})$, this implies that $a \notin d_{\tau_d}(A)$.

Proof. [Proof of Lemma 3.2] We define Γ_{basic} to be the following $\mathcal{L}_2$-formulas, which express the axioms of basic d-neighbourhoods:

- $\forall x \exists U (U \ \nu \ x)$,
- $\forall x \forall U, V (U \ \nu \ x \wedge V \ \nu \ x \rightarrow \exists W (W \ \nu \ x \wedge \forall y (y \ \varepsilon \ W \rightarrow y \ \varepsilon \ U \wedge y \ \varepsilon \ V)))$,
- $\forall x \forall U (U \ \nu \ x \rightarrow \forall z (z \ \varepsilon \ U \rightarrow \exists W (W \ \nu \ z \wedge \forall y (y \ \varepsilon \ W \rightarrow y = x \vee y \ \varepsilon \ U)))))$.

□

Proof. [Proof of Proposition 3.10] From model theory, we know that for any $\mathcal{L}_2$-model, and thus for any based model, $\mathfrak{M}$ there is an ultrafilter $\mathcal{D}$ such that $\prod_{\mathcal{D}} \mathfrak{M}$ is ω-saturated (see [4], Theorem 6.1.8). We show that $\prod_{\mathcal{D}} \mathfrak{M}$ is also d-saturated.

(i) It immediately follows by ω-saturation of $\mathfrak{M}$.

(ii) Let a be any point of $\prod_{\mathcal{D}} X$, and $O \in \mathcal{B}_{\mathcal{D}}(a)$ be one of its basic neighbourhoods in $\prod_{\mathcal{D}} \mathfrak{M}$. By the definition, $O = \prod_{\mathcal{D}} O_i$ which is point-saturated, since for any set of $\mathcal{L}_2$-type $\Gamma(x)$ of O we can consider the $\mathcal{L}_2$-type $\Gamma'(x) = \{x \ \varepsilon \ U\} \cup \Gamma(x)$. Since $\prod_{\mathcal{D}} \mathfrak{M}$ is ω-saturated, there is $b \in \prod_{\mathcal{D}} X$ such that $\prod_{\mathcal{D}} \mathfrak{M} \models \Gamma'(b)$. This means that Γ is satisfiable in O.

(iii) Assume that $a \in \prod_{\mathcal{D}} X$ is given and $\Gamma(x)$ is the set of all $\mathcal{L}_2$-formulas $\varphi(x)$ such that $\varphi(x)$ holds throughout some basic neighbourhood of a. Now consider the following set of $\mathcal{L}_2$-formulas:

$$\Gamma(U) = \{U \ \nu \ a\} \cup \{\forall y (y \ \varepsilon \ U \rightarrow \varphi(y)) \mid \varphi \in \Gamma\}.$$

Since $\prod_{\mathcal{D}} \mathfrak{M}$ is an ω-saturated, there is $\prod_{\mathcal{D}} O_i \in \mathcal{B}(a)$ such that $\prod_{\mathcal{D}} \mathfrak{M} \models \Gamma(\prod_{\mathcal{D}} O_i)$.

□

Proof. [Proof of Lemma 4.15] Suppose $\mathfrak{u}, \mathfrak{u}', \mathfrak{u}'' \in X^*$ and $\mathfrak{u}R^*\mathfrak{u}'$ and $\mathfrak{u}'R^*\mathfrak{u}''$. We show that if $\mathfrak{u} \neq \mathfrak{u}''$, then $\mathfrak{u}R^*\mathfrak{u}''$. So, we have to show that $d_B(B) \in \mathfrak{u}$ whenever $B \in \mathfrak{u}''$ for any $B \subseteq X$. Take an arbitrary $B \in \mathfrak{u}''$. Since $\mathfrak{u} \neq \mathfrak{u}''$, there is $A \subseteq X$ such that $A \in \mathfrak{u}''$ and $A^c \in \mathfrak{u}$. Then $A \cap B \in \mathfrak{u}''$, and thus $d_B d_B(A \cap B) \in \mathfrak{u}$. This implies that $d_B(A \cap B) \cup (A \cap B) \in \mathfrak{u}$. Hence, $(d_B(A \cap B) \cap A^c) \cup (A \cap B \cap A^c) \in \mathfrak{u}$. So, $d_B(A \cap B) \cap A^c \in \mathfrak{u}$. Therefore, $d_B(A \cap B) \subseteq d_B(B) \in \mathfrak{u}$. □

Proof. [Proof of Proposition 4.19] If $\mathcal{X} \not\models \varphi$. Then there is a model $\mathfrak{M} = (\mathcal{X}, [\![\cdot]\!])$ and $a \in X$ such that $\mathfrak{M}, a \not\models \varphi$. Let $\mathfrak{M}^* = (\mathcal{X}^*, [\![\cdot]\!]^*)$, where $[\![p]\!]^* = \{\mathfrak{u} \in X^* \mid [\![p]\!] \in \mathfrak{u}\}$, for each $p \in \mathbb{P}$.

Claim. $\mathfrak{M}^*, \mathfrak{u} \models \theta$ iff $[\![\theta]\!] \in \mathfrak{u}$, for each modal formula θ.

This claim implies that $\mathfrak{M}^*, \mathfrak{u}_a \not\models \varphi$, where $\mathfrak{u}_a$ is the principal ultrafilter generated by a, and thus $\mathcal{X}^* \not\models \varphi$.

Proof of the Claim: By induction on the complexity of formula θ. By definition of $[\![\cdot]\!]^*$ and by the ultrafilter properties it is easy to see that the claim holds for atomic formulas and Boolean connectives. Now let $\theta = \Diamond\psi$. If $\mathfrak{u} \models \Diamond\psi$, then $[\![\psi]\!]^* \cap B^* \neq \varnothing$ for all $B^* \in \mathcal{B}^*(\mathfrak{u})$. Specifically, $R^*(\mathfrak{u}) \cap [\![\psi]\!]^* \neq \varnothing$. So, there is $\mathfrak{u}'$ such that $\mathfrak{u}R^*\mathfrak{u}'$ and $\mathfrak{u}' \in [\![\psi]\!]^*$. By induction hypothesis we have $[\![\psi]\!] \in \mathfrak{u}'$, which implies that $[\![\Diamond\psi]\!] \in \mathfrak{u}$. For the other direction, suppose that $[\![\Diamond\psi]\!] \in \mathfrak{u}$. Let $\mathfrak{u}_0 = \{A \subseteq X \mid \tilde{d}_{\mathcal{B}}(A) \in \mathfrak{u}\}$. Then $u_0 \cup \{[\![\psi]\!]\}$ has the finite intersection property, and thus it can be extended to an ultrafilter $\mathfrak{u}'$. Then $[\![\psi]\!] \in \mathfrak{u}'$ and $\mathfrak{u}R^*\mathfrak{u}'$. □

Proof. [Proof of Proposition 5.1] For any $A \subseteq X$, add a new unary predicate P_A to the language $\mathcal{L}_2$ and let $\mathfrak{M}$ ba an $\mathcal{L}_2$-model of $\mathcal{X}$ with the natural interpretation of new predicates, i.e., P_A is interpreted as A. Let $\mathfrak{M}_{\mathcal{D}}$ be the d-saturated ultrapower of $\mathfrak{M}$ as in Proposition 3.10.

Let T be the set of $\mathcal{L}_2$-sentences of the following forms:

(i) $\exists x\ P_A(x)$, for any non-empty $A \subseteq X$.

(ii) $\forall x(P_{A\cap B}(x) \leftrightarrow P_A(x) \wedge P_B(x))$.

(iii) $\forall x(\neg P_A(x) \leftrightarrow P_{A^c}(x))$.

(iv) $\forall x(P_{d(A)}(x) \leftrightarrow \forall U(U\ \nu\ x \rightarrow \exists y(y\ \varepsilon\ U \wedge P_A(y)))$.

(v) $\forall x(P_{\tilde{d}(A)}(x) \leftrightarrow \exists U(U\ \nu\ x \wedge \forall y(y\ \varepsilon\ U \rightarrow P_A(y))))$

Then we have $\mathfrak{M}_{\mathcal{D}} \models T$, since $\mathfrak{M} \models T$.

Now define a function $f : \prod_{\mathcal{D}} X \rightarrow X^*$ as follows:

$$f(a) = \{A \subseteq X \mid a \in (P_A)^{\mathfrak{M}_{\mathcal{D}}}\}.$$

We have to show that f is a d-morphism.

- f is well-defined, since (ii) and (iii) imply that $f(a)$ is an ultrafilter.
- To see that f is surjective, for any given $\mathfrak{u} \in X^*$, let $\Gamma_{\mathfrak{u}}(x) = \{P_A(x) \mid A \in \mathfrak{u}\}$. By (ii), $\Gamma_{\mathfrak{u}}$ is finitely satisfiable in $\mathfrak{M}_{\mathcal{D}}$. Since $\mathfrak{M}_{\mathcal{D}}$ is d-saturated, there is a point $a \in \prod_{\mathcal{D}} X$ such that $\mathfrak{M}_{\mathcal{D}} \models \Gamma_{\mathfrak{u}}(a)$.

- To show f is a d-morphism, first assume that $O \in \mathcal{B}_{\mathcal{D}}(a)$. We have to show there exists $O^* \in \mathcal{B}^*(f(a))$ such that for any $\mathfrak{u} \in O^*$ there is $b \in O$ with $f(b) = \mathfrak{u}$. Let $O^* = R^*(f(a))$. For a given $\mathfrak{u} \in O^*$, we have $f(a)R^*\mathfrak{u}$. Let $\Gamma(x) = \{P_A(x) \mid d(A) \in \mathfrak{u}\}$. Then, by (iv), $\Gamma(x)$ is finitely satisfiable in any $B \in \mathcal{B}_{\mathcal{D}}(a)$. Specifically, Γ is finitely satisfiable in O. Now d-saturation of $\mathfrak{M}_{\mathcal{D}}$ implies that O is point saturated, and thus there is $b \in O$ such that $\Gamma(b)$ is true and $f(b) = \mathfrak{u}$.

  For the other direction, assume that $O^* \in \mathcal{B}^*(f(a))$, we have to show that there is $O \in \mathcal{B}_{\mathcal{D}}(a)$ such that $f(O) \subseteq O^*$. Let O_a be the d-neighbourhood of a as defined in part 3 of the definition of d-saturation (cf. Definition 3.9). Let $\Gamma = \{P_A(x) \mid \tilde{d}(A) \in f(a)\}$. Then, by (v), for any finite subset $\Gamma' \subseteq \Gamma$, we have $\bigwedge \Gamma'$ is true throughout some $O \in \mathcal{B}_{\mathcal{D}}(a)$. So, Γ is true throughout O_a and $O_a \cap f^{-1}(A^*) \neq \varnothing$. Thus, there is $b \in O_a$ such that $f(b) \in A^*$ and $f(a)R^*f(b)$.

$\square$